Determinacy of Abstract Objects: The Platonist's Dilemma

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1. Introduction

The age-old disagreement between the proponents and opponents of ideal or abstract objects, which Plato compared to a titanic battle, has stubbornly resisted resolution despite all the advances brought by modern philosophical methods. However, some of the issues involved have become clearer, and it is perhaps easier than it was a century ago to anticipate what kinds of considerations are likely to close the argument in favour of one side or the other. In this paper I want to present one issue which I think must play a central role in any acceptable resolution of the disagreement: the problem of the determinacy or indeterminacy of abstract objects. This poses a dilemma which a Platonist is obliged to successfully overcome if he is to carry the argument. The dilemma in a nutshell is this: if abstract objects are determinate, they are referentially and cognitively inscrutable, whereas if they are indeterminate, their status as independently existing objects is dubitable. This is only the skeleton of an account, and I shall now try to put some flesh on it.

2. Terminological clarifications

2.1 '*Object*'

Since many of the issues I shall discuss arise with special clarity in the writings of Frege, it is important to point out, first of all, that I do not use the term 'object' in the same way as Frege and his followers do. For Frege there is a basic ontological distinction between objects on the one hand and functions on the other, which corresponds exactly to the distinction between saturated and unsaturated expressions. Objects and only objects may be designated by means of saturated expressions, functions and only functions by means of unsaturated expressions. Functions cannot be signified by proper names. In contrast to this, I believe, following Husserl, that nominal expressions may be employed to designate anything whatever, including Fregean functions, Wittgensteinian states of affairs, or other entities which have been held to be incapable of being named. It is therefore appropriate to use the term 'object' in the Austrian sense,[1] as applying to anything whatever. Fregean objects I call 'individuals'; these form a proper subcollection of *particulars*, which include, besides individuals, classes as many and masses.[2]

2.2. '*Abstract*'

An object is real if it has a spatio-temporal location, or – to take account of the possibility that dualists might be right – at any rate a temporal location. An object having neither a spatial nor temporal location is abstract. Sometimes abstract objects are called 'ideal', and I have no particular objection to this. On the other hand, the term 'abstract object' is sometimes used for objects which are, in my terms, real, namely individual moments such as headaches and gestures. I prefer not to follow this usage.[3]

2.3. '*Realism*'

'Realism' does not signify a single doctrine but rather any one of many doctrines, depending on the sorts of object one is realist about. A realism is in each case first and foremost a metaphysical doctrine, whatever its connections with the considerations Dummett introduces concerning bivalence, or verification-transcendence of truth-conditions.[4] Realism with respect to a category C of objects is a two-part doctrine. It claims

R1 That there are objects of category C (Existence Claim), and

R2 That objects of category C are not in any way

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> sustained or maintained in existence by minds or by anything else (such as language) which is itself sustained by minds (Mind-Independence Claim).

Both parts of such a doctrine are necessary for it to count as realism. Dummett, for instance, believes that there are abstract mathematical objects, but denies that they are mind-independent.[5] The Mind-Independence Claim is difficult to formulate satisfactorily. If it is expressed in the usual way as meaning that there could still be Cs even if there were no minds, this has the disturbing consequence that one cannot be a realist about minds themselves or about many products of mind, such as newspapers and coffee-cups, which I find absurd. Hence the kind of mind-dependence in question is not that of minds themselves, nor their parts and processes, nor that of objects which require mental endeavour to come into existence, but may thereafter sustain themselves without minds (many artefacts satisfy this description). Rather, it is that of objects outside the mind which would cease to exist were all minds to be annihilated. This applies to an assortment of objects, such as the game of bridge, marital fidelity, and Mickey Mouse. Incidentally, my talk about minds and mind-(in)dependence is just a shorthand and does not mark adherence to dualism.

Opponents of a particular realism may be classified as extreme or as moderate, according to whether they deny the existence claim (extreme) or the mind-independence claim (moderate).

2.4. '*Platonism*'

A Platonism is a realism with respect to abstract objects. One may distinguish different kinds of Platonism, according to the kind of abstract object in question. One may be a Platonist in one respect but not another. For instance, one may deny Platonism for predicative universals, but affirm it for sets, as does Quine. I am not concerned with the connection between Platonism as understood here and the actual or traditional doctrines of Plato.

2.5. '*Indeterminacy*'

There are a number of ways in which this word may be understood, all of which are relevant to the ontology of abstract objects. For one thing, indeterminacy can mean that there are predicates which cannot be meaningfully predicated of the object, although they can be meaningfully predicated of objects of other kinds. Some hold that a sentence like, 'The number 5 is in my top pocket' (understanding 'the number 5' in its usual arithmetical meaning) is not false but absurd. However, if this applies to abstract objects, there is no reason to deny that it applies to concrete ones as well: 'My briefcase is divisible by 3' might be an example. So this kind of indeterminacy will be left out of account here.

Another kind of indeterminacy applies in cases where a predicate is meaningfully predicable of an object, but leaves a truth-value gap when predicated of it. This is obviously fairly similar to the previous notion of indeterminacy, and again there is every reason to suppose that concrete objects are at least as prone to such breaches of bivalence as abstract objects, in particular because of the existence of vague predicates. I shall not consider this notion of indeterminacy on its own account in what follows except to the extent that it enters into the third kind.

The third and most important concept of indeterminacy concerns questions of identity. This concern was raised explicitly in connection with abstract objects by Frege on several occasions, most notably in his *Die Grundlagen der Arithmetik* (*GLA*)[6] and *Grundgesetze der Arithmetik* (*GGA*)[7]. It has made its reputation in analytical philosophy under Quine's slogan, "No entity without identity", and informs many attempts to give necessary and sufficient conditions of identity for this or that kind of entity.

Indeterminacy with respect to identity itself comes in various kinds. In each case, it is assumed that we are talking about a category C of objects. As is often expedient when discussing delicate issues of identity, we ascend to the formal mode.[8]

(1) *Practical Limitation*: While for all expressions a and b purporting to denote objects in C, 'a = b' has a determinate truth-value, and while it is possible in principle to give general conditions under which we can know whether 'a = b' is true or not, we are unable to apply these principles.

(2) *Epistemological Limitation*: While for all expressions a and b purporting to denote objects in C, 'a = b' has a determinate truth-value, it is not possible in principle to give general conditions under which we can know whether 'a = b' is true or not.

(3) *Weak Inscrutability*: While for all expressions a and b purporting to denote objects in C, 'a = b' has a determinate truth-value, there are expressions a and b

purporting to denote objects in C such that we have in principle no way of knowing whether 'a = b' is true or not.

(4) *Strong Inscrutability*: While for all expressions a and b purporting to denote objects in C, 'a = b' has a determinate truth-value, for any expression a purporting to denote an object in C, there is an expression b likewise purporting to denote an object in C such that we have in principle no way of knowing whether 'a = b' is true or not.

(5) *Weak Truth-Value Indeterminacy*: There are expressions a and b purporting to denote objects in C such that 'a = b' has no determinate truth-value.

(6) *Strong Truth-Value Indeterminacy*: For any expression a purporting to denote an object in C, there is an expression b likewise purporting to denote an object in C such that 'a = b' has no determinate truth-value.

My concern will be with the last four concepts: two of inscrutability and two of truth-value indeterminacy. The latter two alone will from now on assume the title 'indeterminacy' to the exclusion of other meanings. The Platonist's Dilemma may now be given a somewhat more precise formulation: *he is forced to choose between scrutability and determinacy, but he cannot have both* (whether weak or strong will be something we need to discuss).

3. Retaining Determinacy

In what follows, I shall concentrate on mathematical objects to the exclusion of other putative kinds of abstract object. There are three reasons for this choice. First, I shall illustrate the Platonist's dilemma using the example of Frege, who concentrated chiefly — though not exclusively — on mathematical objects. Second, most of the other literature I draw on for my discussion is concerned exclusively with mathematics. Third, and most importantly, mathematics presents the Platonist with his strongest case. Contrast the debate about predication. There really is no accepted body of theory about predication, and in this respect other philosophical debates are, if anything, worse off. I happen to think the Platonist has the worse position on predication, but that is not important here. There is, on the other hand, a huge body of propositions of pure mathematics which are generally accepted, and for which to date no anti-Platonist account has come close to accounting for.[9]

Central to traditional Platonism is the belief that abstract objects have determinate identity. Indeed Plato, following the Pythagoreans, probably regarded abstract objects in this respect as better off than spatio-temporal objects. Certainly the prevalence of deductive methods in pure mathematics lends certainty to many statements of identity and difference, and may encourage the feeling that this certainty is characteristic of pure mathematics in particular and the realm of the abstract in general.

The most strenuous attempts to formulate the principle of truth-value determinacy for abstract objects and to ensure us that the objects of mathematics satisfy this principle were undertaken by Frege in the course of his crusade for logicism. It is instructive to examine the fate of Frege's efforts in this regard, as it is symptomatic of the Platonist's problem.

Frege was led by linguistic considerations to the view that numbers are abstract individuals (in his terminology: objects),[10] and that they must therefore have a determinate identity.[11] This led to two connected moves in *GLA*: rejection of contextual introduction of abstract individuals,[12] and identification of numbers as concept-extensions.[13] Contextual introduction of numbers would work as follows: given a statement of equinumerosity, like

(1) There are as many knives on this table as there are forks on this table

one can, as Frege suggestively puts it, "recarve" the content of the sentence, retaining the logical properties of the equivalence relation in the logical relation of identity, and distributing the specific content to a functor occurring twice, once on either side of the identity, to yield

(2) The number of knives on this table = the number of forks on this table

This is, to use Frege's own metaphor, an equation in one unknown, which we can therefore solve. So we come to understand what 'the number of' means. As Frege's illustration of the introduction of directions via the relation of being parallel indicates,[14] the same procedure can be applied to any equivalence relation. For the theory of numbers to have any interest, a fair number of predicates predicable of (in Frege's terms) concepts have to be invariant with respect to equinumerosity, and we can carry these over to yield new predicates (often using the same words as the old) having application in the theory of numbers. This account is based on the work of Lorenzen,[15] but the basic idea was already familiar and not unpopular long before. In particular it informs Peano's conception of *definizione per astrazione*,[16] and was also embraced somewhat later by Weyl.[17]

Although Frege was himself one of the staunchest opponents of this view, he was clearly attracted to it for a while. This emerges not only in the care with which he expounds it in *GLA*, but also in the fact that, even after he had published *GLA*, he still entertained the suggestion of a cursory footnote in that work and considered deriving arithmetic on the basis of concepts rather than the extensions of concepts.[18]

However, Frege rejected the approach precisely because it did not give us answers to all identity questions. The way in which number terms are contextually introduced leaves us essentially in the dark — according to Frege — about the truth-value of such sentences as "the number of knives on this table = Julius Caesar".[19] It is not clear from Frege's remarks whether he intends this as a matter of epistemological limitation, inscrutability or indeterminacy. But the point seems to be that a theory of numbers which makes them out to be individuals must be embedded in a bivalent logic for identity statements about all individuals, and that the falsity of all identities of the form "a = b", where "a" names an abstract individual and "b" names a concrete individual, should follow from the manner of introduction of numerical terms. For this reason, Frege goes over to an explicit definition of numbers as concept extensions, which set in motion the fateful chain of events leading to Russell's antinomy.

Unfortunately for Frege, while his decision to take numbers as extensions was intended to rule out inscrutability of indeterminacy, it simply exposed the same problems at a deeper level. This emerges in the permutation argument in *GGA* I, §10, where Frege shows that his basic laws do not allow us to uniquely identify the value-courses, nor to say whether or not either of the truth-values is a value-course, and it ends with Frege's conventional stipulations for the identity of the truth-values.[20] While Peter Schroeder-Heister has shown that there are limits to the permutations admissible under this scheme,[21] Terence Parsons has further shown that these limitations are rather weak.[22] Compatible with the consistency of the first-order fragment of Frege's system in *GGA*, a large number of permutations are permissible. Frege now appears to be caught between two positions, neither of which is prima facie very congenial to him: on the one hand, if he insists on scrutability and determinacy, he is forced into a conventionalism which sits unhappily with his realism. On the other hand, if he rejects the conventional identifications of numbers, value-courses and truth-values with certain particular individuals, he is back to the position rejected in *GLA*: either inscrutability or (more strongly) indeterminacy holds for abstract individuals.

Another and more subtle way in which the same problem emerges in Frege's philosophy is connected with the logicistic account of real numbers, which he embarked upon but did not carry to completion in the second volume of *GGA*.[23] Frege offers reasons for refusing to identify the natural numbers used in counting with a subset of the real numbers, and the value-courses which he would almost certainly have identified as the real numbers are indeed such that the real whole numbers and the natural numbers are quite different abstract individuals. Frege's reason is connected with his decision to view natural numbers as concept extensions. It is by means of this identification that he secures the link between mathematical theory and counting practice upon which he laid such stress. Likewise, because Frege lays stress on the non-contingent link between real numbers and the measurement of magnitudes, he is led to associate real numbers with value-courses associated with classes of relations forming what he calls a domain of magnitudes. Measuring and counting being, in his view, quite distinct activities involving quite distinct kinds of functions, he is constrained to reject the identity, and reflects the difference even in his notation. Frege's account of real numbers differs also from most others in that he does not first construct or obtain the rational numbers from the integers, and then go on to construct the reals, but rather obtains the reals all in one go. Indeed, he criticizes Russell's construction of the reals for just this reason, although he admits that Russell's alternative is logically unobjectionable.[24] However, Frege's insistence on tying mathematical entities to their applications has as a consequence that he is forced to follow alternative applications where they lead. One hint which he drops in the second volume of *GGA* is that one may build up a theory of non-negative *rational* numbers on the basis of a theory of the powers of relations.[25] While he does not do so, the construction would not be difficult to carry out, and if we were indeed to accept some kind of application for such powers of relations, he would be forced to accept as "rational numbers" another kind of object, value-courses different from those which "are" the rational numbers in the theory of reals. Another question mark hangs over the complex numbers, about which Frege has very little to say in his middle period, but which he clearly regarded as the numbers par excellence in his early and very late writings.[26] To summarize the outcome very briefly: because of his insistence that numbers have a determinate identity as particular value-courses, Frege reduplicates mathematical structure in different places, and if we follow his precepts, we are forced to multiply such entities more radically. The question remains whether such a multiplication is desirable. Each time we reduplicate a structure, we raise the question which of the embodiments of this structure is the proper embodiment. And since in any case the whole identification takes place within the framework of Frege's theory of value-courses, even settling such questions within this theory leaves the external question untouched as to which individuals the value-courses indeed are.

The problem of multiple representations of mathematical structures was raised much later by Benacerraf as being an argument against mathematical Platonism.[27] Benacerraf's chosen example was the multiple representability of natural number theory within set theory. Frege's work, despite his efforts to minimize multiple representability, and despite some of his strictures on unbridled conventionalism, exemplifies in a poignant form precisely the point Benacerraf makes, poignant because Frege was as free as possible from the pragmatist stance of Quine, whose attitude may be summed up by saying that the formal structure is all that matters, and that given the structure, the applications will follow of themselves.[28] Since formal structure is usually multiply representable, since formal theories determine their models at best up to isomorphism, Quine's attitude represents a retreat from the realism of objects advocated by Frege and traditional Platonists.

4. Retaining Scrutability

So it seems that, as long as we attempt to retain truth-value determinacy for abstract objects, we are forced to conclude that, for all our theories tell us, we have no idea which of perhaps infinitely many individuals any singular term of such a theory signifies.

This has motivated an alternative Platonist approach to mathematical objects: a Platonism of structures. On this view, mathematical theories neither aspire to nor can they succeed in enabling us to refer to determinate abstract individuals. The principal objects of mathematical theories are not the mathematical individuals such as the number 5 or the number π, but rather the structures in which such individuals occur, not as independent entities accidentally occurring together in structures, but as structureless points or moments whose identity is determined precisely by their position and role within these structures. Thus the number 5 is not an object with an essence or identity apart from its relations to the other numbers in the mathematical structure determined by the (second-order) Dedekind-Peano axioms. The consequences of this position for the ontology and epistemology of abstract individuals has been worked out most fully to date by Michael Resnik.[29]

This kind of mathematical structuralism has its attractions, and Resnik presents it as the only hope for rescuing Platonism from the difficulties we have mentioned above in connection with the attempt to retain truth-value determinacy. In particular it rejects the pure pragmatism of Quine, while seeking to take account of the reasons Quine offers for his pragmatism. On the structuralist view, there is no longer any issue as to the scrutability of mathematical objects: there is no question to be answered as to the real nature of mathematical objects beyond the structural properties the objects possess in virtue of modelling a set of axioms. So, to take Benacerraf's example,[30] the question whether the number 3 is a member of the number 17 (as it is in von Neumann's rendering of the numbers and it is not in Zermelo's rendering) is simply out of place within the theory of natural numbers: it is at least void of truth-value, and possibly void of sense. On the other hand the question whether 3 is a divisor of 17 is perfectly legitimate and has a determinate answer.

To take account of the fact that the same structure may be multiply realized, Resnik introduces the notion of one structure's occurring within another.[31] This is equivalent with there being a homomorphism of the first structure into the second. This enables the issue of multiple realizations to be taken as nearly as possible at face value: the structure of the natural number systems occurs within some structures and not within others. Where it occurs within a structure, it occurs infinitely often in different ways. For example the structure of the natural numbers as determined by the Dedekind-Peano axioms occurs within itself (the even numbers, the multiples of 3, the prime numbers . . .), within the rationals, the reals, the complex numbers and so on.

The price for this is truth-value indeterminacy. Take for example the natural numbers as embedded within the cumulative set-theoretical hierarchy in Zermelo's and in von Neumann's fashion. Within the cumulative hierarchy, it is clear that the two realizations are different. Hence the Zermelo number 3 is not identical with the von Neumann number 3. But to the question which, if either, is *the* number 3 there is no answer. Likewise there is no answer to the question whether the one realization or the other (or neither) is *the* natural number structure. One might perhaps have expected a negative answer: since the structure within which the two series are embedded is more highly structured than the natural numbers themselves, it appears that, by Leibniz's Law, we can infer that neither the Zermelo 3 nor the von Neumann 3 is identical with the number 3, since for each of these, but not for the number 3, the question whether 3 is a member of 17 has a determinate answer. But Resnik specifically denies the applicability of Leibniz's Law to such cases.[32] Identity questions for abstract objects cannot be given determinate answers except within a particular structure as context. Since the question which number 3 is *the* number 3 is a trans-structural identity question, it has no true or false answer. Similarly there is no true or false answer to the question which realization, if any, of the natural number structure is identical with or different from the natural numbers *per se*. Resnik does not admit any trans-structural identity questions to have determinate answers, even in cases where it would appear to me to be perfectly harmless to do so, e.g. when two structures have different cardinalities. The only trans-structural questions he admits to have determinate answers are those concerning the occurrence of one structure within another and the inclusion of one structure within another, which is a special case of occurrence.[33]

Before drawing a provisional lesson from this, I should like to make one historical remark. It appears to me that one important facet of Resnik's approach was anticipated by Meinong with his theory of implection. According to Meinong, an object is implected in another when all the properties of the one object are properties of the other.[34] This is part of Meinong's solution to the traditional problem of participation: the horse as such is implected in each individual horse. But the horse is further implected in other non-existent objects, such as the white horse, the cart-horse, and so on. Such relations connect not only abstract objects like the horse with concrete ones like individual horses, but also some abstract objects with others. This is not unlike the more specific relation of occurring-in which Resnik describes. Meinong's incomplete objects also share another aspect of Resnik's structures: they violate one form of bivalence, namely property bivalence. However, there is no firm indication that they are thereby constrained to violate propositional bivalence, and it appears that Meinong applies Leibniz's Law just as much to incomplete as to complete objects, at least when it comes to distinguishing them. I do not see why this much cannot be admitted by Resnik, too.

Whatever the merits of this programme — and I think it has several — it is certainly a far cry from traditional Platonism. It may indeed be the price one has to pay for Platonism, since the alternative view runs into the problem of inscrutability. However, by biting the bullet of truth-value indeterminacy for identity statements, it raises the question whether it is appropriate to consider the result a form of Platonism at all. For Platonism holds that abstract objects exist and are not mind-dependent. Now the fact that identity statements are not always decidable or determinate or even meaningful may not after all be an insuperable barrier to our accepting the objects concerned. Apart from the issues raised by vagueness, it may indeed be the case that many or all physical objects are not immune to truth-value indeterminacy because of quantum phenomena at the micro-level.[35] This does not have to entail that physical objects do not exist independently of mind, as the Copenhagen interpretation would have us believe. We have already become accustomed to the difficulties of applying Leibniz's Law in view of intensionality and vagueness. While the radical truth-value indeterminacy of abstract objects on the structuralist view probably goes beyond these difficulties in its scope, it is at any rate not unprecedented.

Why then should truth-value indeterminacy cast doubt on the independent existence of abstract objects? I suggest it is because it invites us to look back to the issue which our considerations took as their point of departure, namely the contextual introduction of abstracta toyed with and then dropped by Frege. The lesson of subsequent history seems to have been that in demanding truth-value determinacy for abstract objects, Frege was asking too much. But relaxing the demand of determinacy allows the abstractionist view more room to breathe. If abstract objects are conceived as fictions introduced on the back of functional expressions, enjoying a limited vocabulary of significant predicates because of the restrictions imposed by the requirement of invariance, then it is not to be expected that the variety of contextual backgrounds from which such abstracta arise will allow itself to be so neatly ordered that the resulting fictions can be neatly slotted into a single, overarching context within which each has its fixed place. The abstractionist view *predicts* lacunae among predicates and among identity questions raised concerning abstracta which do not arise from the same conceptual background. This retains the contextualism of Resnik's theory at the cost of his Platonism, since abstracta are the products of inquiry raised on a bed of previous cognition, sometimes itself also such a product, and they are sustained by the continued presence of contextually-introduced names for them, names which require minds for their interpretation. This is no longer Platonism, but some sort of conceptualism. Its advantages are its truthfulness to much of our practice and its evasion of the Platonist dilemma of inscrutability vs. indeterminacy. I also consider it an advantage that it does not turn on pragmatic considerations. On the other hand, it faces many difficulties of its own. The first is that there is simply no definitive statement of the position.[36] Another is the difficulty of making straightforward sense of the idea of bringing a new abstract object into being, and of giving an intuitive semantic account of the singular terms which purport to denote such objects. Most importantly, it must be seen whether such a theory can succeed in accounting for the acceptance of the results of pure mathematics, rather than copying constructivists in placing unacceptable normative restrictions on what is to count as mathematics. Until these challenges are met, mathematical Platonism, despite the dilemma it faces, must be seriously entertained.

Notes

[1] I call this the "Austrian" sense because it is clearly found in the work of such Austrians as Bolzano, who used the word "*Etwas*" (something) in this sense, Husserl ("*Gegenständlichkeit*"), and Meinong ("*Gegenstand*").

[2] See Simons, 1987a, ch. 4.

[3] For the distinction between the two senses of "abstract" see Husserl, 1984 (1970), Inv. II, ch. 6.

[4] Cf. Dummett, 1978, chs. 10, 21. In using "realism" in this way I am returning to its original use as describing a kind of metaphysical doctrine (with an epistemological dimension). Cf. Devitt, 1984, p. 11.

[5] Cf. "the picture of a mathematical reality not already in existence but as it were coming into being as we probe. Our investigations bring into existence what was not there before, but what they bring into existence is not of our own making." Dummett, 1959, 162 (1978, 18).

[6] *GLA*, §§62—69.

[7] *GGA*, §10.

[8] Quotes are used temporarily here like Quine's corners (quasi-quotes).

[9] Despite the efforts of Field, 1980 and others.

[10] *GLA*, §§55—61.

[11] *GLA*, §56, §§66—68.

[12] *GLA*, §§66—68.

[13] *GLA*, §68.

[14] *GLA*, §64.

[15] Lorenzen, 1962.

[16] Peano, 1915.

[17] Weyl, 1949, pp. 9—13. Cf. also Angelelli, 1985.

[18] A manuscript of 1884 or 1885 that has not survived attempts to define "the number of Fs" without using the extensions of concepts (Manuscript N 47*: cf. Veraart, 1976, p. 95; cf. also Burge, 1984, pp. 12—15).

[19] Cf. *GLA*, §56, §66.

[20] *GGA*, §10.

[21] Schroeder-Heister, 1987.

[22] Parsons, 1987.

[23] Cf. Simons, 1987b.

[24] Cf. *WB*, p. 239; *PMC*, p. 156.

[25] Cf. *GGA* II, §173.

[26] Frege's dissertation and a number of early papers are concerned with the complex numbers, as is his last attempt to give a geometric definition of number. Cf. *NS*, p. 300, *PW*, p. 280.

[27] Benacerraf, 1965.

[28] Cf. Quine, 1969, p. 81, p. 135.

[29] Resnik, 1981.

[30] Benacerraf, 1965, p. 54.

[31] Resnik, 1981, p. 533.

[32] Resnik, 1981, pp. 537—8.

[33] Resnik, 1981, p. 533.

[34] Cf. Meinong, 1915, §29.

[35] That this may be seen as an application of Meinong's idea is put forward by Lambert, 1989.

[36] Peano, 1915 stands only at the beginning of the development and does not face many of the problems; Weyl, 1949 is very brief; Lorenzen, 1962 wobbles uneasily between use and mention and is difficult to make use of; Simons, 1981 is no more than a sketch.

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