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A LIMIT FORM OF THE EQUATIONS FOR IMMISCIBLE DISPLACEMENT IN A FRACTURED RESERVOIR

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Abstract. We study a model for simulating the flow of an immiscible displacement (waterflooding) of one incompressible fluid by another in a naturally fractured petroleum reservoir when the matrix blocks are quite small. This model is equivalent to a transformed one for immiscible flow in an unfractured reservoir with a reduced saturation and a saturation-dependent porosity. Existence and uniqueness of classical solutions are established. We present some numerical results and a comparison with a single porosity model.

Key words. Immiscible displacement, fractured reservoir, single porosity model.

1. Introduction. Naturally fractured reservoirs are formed by stress forces that occur in slow deformations of the earth's crust, with fractures frequently appearing in a reasonably regular pattern. Though the void volume in the fractures is small in comparison to that in the matrix blocks of the sedimentary formation, the very high permeability associated with the fractures requires that flow in fractured reservoirs be modelled so as to take the dual structure into account. Several models, each appropriate under different assumptions on the geometry of the fractures have been developed; see the survey article [6] for a description of a collection of models on two-sheeted coverings of the domain (reservoir). Here, we consider the case when the dimensions of the matrix blocks are very small. In this situation, we can neglect the viscous and gravity forces in the flow interior to the matrix blocks; therefore, we assume that the only forces that act on the blocks are the capillary forces and that these forces are in equilibrium across the surface between a block and the surrounding fractures. The model proposed in this paper is obtained as a limit when the size of the matrix blocks tends to zero in the two-sheeted models studied in [3], [4], [6], and [7]. The limiting process here is based on physical considerations; the justification for the resulting model is obtained through the convergence of the recovery curves for the medium and small block models of [6], [7], and [4] to that computed for the limit model of this paper. It should be noted, in particular, that the limit model is not derivable by some type of homogenization or averaging procedure beginning from the medium or small block models. The differential equations for the limiting model can be transformed to be equivalent to those for an unfractured reservoir, except that a reduced saturation must be introduced to account for the addition of a nonlinear capacitance term.

This paper is organized in the following way. In §2 we introduce the limit model and determine some of its properties. We prove existence and uniqueness of classical solutions in §3; uniqueness is proved in $\mathbf{R}^3$ and existence in $\mathbf{R}^2$. (Existence in $\mathbf{R}^3$ for a weak solution has quite recently been proved by Arbogast [2]; thus, we have made no effort to find a second proof for the existence of a weak solution in this case.) Then, §4 is devoted to the presentation of some numerical simulations; these calculations are intended to provide the

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justification of the model mentioned above and to illustrate some parametric dependencies of the model. An abbreviated derivation of the global pressure formulation of the equations describing incompressible, immiscible flow in an ordinary reservoir is given in §5. Finally, a nomenclature is given in §6.

2. The Model. We begin with the medium block model of [6], [7], and [8] that is based on three families of parallel, equispaced, planar fractures. This model, which can be derived from a microscopic description of ordinary two-phase, incompressible, immiscible flow in a porous medium, is formulated on a two-sheeted covering of the physical reservoir, with the fracture sheet Ω coinciding with the reservoir. Over each point $x \in \Omega$ a matrix block Ω_x is attached to the fracture sheet; these blocks are topologically distinct from each other. In general, these blocks have a rhomboidal shape.

Following [6] and [8], the flow in the fracture system is governed by the following set of partial differential equations:

$$\Phi \frac{\partial S_w}{\partial t} - \nabla_x \cdot [\frac{KK_{rw}}{\mu_w}(\nabla_x P_w + \rho_w g \nabla_x z)] = Q_{w,e} + Q_{w,m}, \quad x \in \Omega, t > 0, \tag{2.1}$$

$$\Phi \frac{\partial S_o}{\partial t} - \nabla_x \cdot [\frac{KK_{ro}}{\mu_o}(\nabla_x P_o + \rho_o g \nabla_x z)] = Q_{o,e} + Q_{o,m}, \quad x \in \Omega, t > 0, \tag{2.2}$$

$$S_w + S_o = 1,$$

$$P_o - P_w = P_c(S_o),$$

where $Q_{w,e}$ and $Q_{o,e}$ are the volumetric external flow rates (at wells in practice) of the water and oil phases, respectively. The terms $Q_{w,m}$ and $Q_{o,m}$ describe the volumetric flow rates of the water and oil phases resulting from flow out of the matrix blocks into the fractures. As in [4], we prefer to use a global pressure formulation [1], [9], [5] of the equations, which we derive below.

Let

$$S = S_o = 1 - S_w. \tag{2.3}$$

Define the total mobility to be

$$\Lambda(S) = \frac{K_{ro}(S)}{\mu_o} + \frac{K_{rw}(S)}{\mu_w} \tag{2.4}$$

and the phase mobilities to be

$$\Lambda_\theta(S) = \frac{K_{r\theta}(S)}{\mu_\theta \Lambda(S)}, \quad \theta = o, w. \tag{2.5}$$

Define the global pressure to be

$$P = \frac{1}{2}\{P_o + P_w + \int_0^{P_c} (\Lambda_o - \Lambda_w)(P_c^{-1}(\xi))d\xi\}, \tag{2.6}$$

the global density to be

$$\rho_f(S) = \Lambda_o(S)\rho_o + \Lambda_w(S)\rho_w, \tag{2.7}$$

and the total fluid velocity to be

$$U = -K\Lambda(\nabla_x P + \rho_f g \nabla_x z). \tag{2.8}$$

It is shown in §5 (and in [5], [9]) that (2.1)-(2.8) lead to the global pressure equation

$$\nabla_x \cdot U = Q_e = Q_{w,e} + Q_{o,e} \tag{2.9}$$

and, assuming that $Q_{o,e} = -\Lambda_o Q_e^+ = -\Lambda_o max(0, Q_e)$, a saturation equation

$$\Phi \frac{\partial S}{\partial t} + \Lambda_o' U \cdot \nabla_x S - \nabla_x \cdot (K\Lambda\Lambda_o\Lambda_w P_c' \nabla_x S) - \nabla_x \cdot (K\Lambda\Lambda_o\Lambda_w(\rho_o - \rho_w)g\nabla_x z) = -\Lambda_o Q_e^+ + Q_{o,m}. \tag{2.10}$$

The initial saturation $S(x,0)$ and boundary conditions must be specified. For simplicity, assume that no flow of either phase occurs across the external boundary $\partial\Omega$. It follows that

$$U \cdot \nu_{\partial\Omega} = 0. \tag{2.11}$$

An additional boundary condition will be specified below in (2.33).

Denote the coordinates on the matrix block Ω_x by (x, y, t), where y is the local spatial coordinate on the block. The flow on Ω_x is described by the following differential equations:

$$u = -k\lambda(\nabla_y p + \rho_m g \nabla_y z), \quad y \in \Omega_x, t > 0, \tag{2.12}$$

$$\nabla_y \cdot u = 0, \quad y \in \Omega_x, t > 0, \tag{2.13}$$

and

$$\phi \frac{\partial s}{\partial t} + \lambda_o' u \cdot \nabla_y s - \nabla_y \cdot (k\lambda\lambda_o\lambda_w p_c' \nabla_y s) - \nabla_y \cdot (k\lambda\lambda_o\lambda_w(\rho_o - \rho_w)g\nabla_y z) = 0, \tag{2.14}$$

where

$$s = s_o = 1 - s_w, \tag{2.15}$$

$$\lambda(s) = \frac{k_{ro}(s)}{\mu_o} + \frac{k_{rw}(s)}{\mu_w}, \tag{2.16}$$

$$\lambda_\theta(s) = \frac{k_{r\theta}(s)}{\mu_\theta \lambda(s)}, \quad \theta = o, w, \tag{2.17}$$

$$p_o - p_w = p_c(s_o), \tag{2.18}$$

$$p = \frac{1}{2}\{p_o + p_w + \int_0^{p_c} (\lambda_o - \lambda_w)(p_c^{-1}(\xi))d\xi\}, \tag{2.19}$$

$$\rho_m(s) = \lambda_o(s)\rho_o + \lambda_w(s)\rho_w. \tag{2.20}$$

The initial saturation in each matrix block is assumed to be consistent with capillary equilibrium between the fractures and the block; that is,

$$s(x,y,0) = (p_c^{-1} \circ P_c)(S(x,0)), \quad x \in \Omega, y \in \Omega_x. \tag{2.21}$$

At the boundary of the blocks, the oil saturations satisfy the relation

$$s(x,y,t) = p_c^{-1}(P_c(S(x,t))) = (p_c^{-1} \circ P_c)(S(x,t)), \quad x \in \Omega, y \in \partial\Omega_x, t > 0, \tag{2.22}$$

and

$$p(x,y,t) = P(x,t), \quad x \in \Omega, y \in \partial\Omega_x, t \geq 0. \tag{2.23}$$

It has been shown in [8] that gravity can be neglected in the blocks for small blocks. The small block model described in [6] results from ignoring exactly these gravitational effects, with a consequent simplification in the differential system on the blocks. When the gravity term in (2.12) is omitted, (2.13) reduces to a homogeneous elliptic equation for p on Ω_x; since the boundary values given by (2.23) for p are constant for any fixed t,

$$p(x,y,t) = P(x,t), \qquad y \in \Omega_x. \tag{2.24}$$

Then, it follows that

$$u(x,y,t) = 0, \qquad y \in \Omega_x. \tag{2.25}$$

Thus, pressure differences are neglected inside each matrix block, and the only relevant mechanism on a block is the effect of the capillary forces. The equations (2.12)-(2.14) governing the flow in the matrix blocks are then reduced to the single equation

$$\phi \frac{\partial s}{\partial t} - \nabla_y \cdot (k\lambda\lambda_o\lambda_w p_c' \nabla_y s) = 0, \tag{2.26}$$

subject to the boundary condition (2.22). The fracture equations (2.8)-(2.10) are coupled to (2.26) via the term

$$Q_{o,m}(x,t) = -\frac{1}{|\Omega_x|} \int_{\partial\Omega_x} k\lambda\lambda_o \frac{\partial p_o}{\partial \nu} d\eta, \quad x \in \Omega, \quad t > 0. \tag{2.27}$$

Using Green's identity and conservation of mass in the oil phase, equation (2.27) can be rewritten as

$$Q_{o,m}(x,t) = -\frac{1}{|\Omega_x|} \int_{\Omega_x} \phi \frac{\partial s}{\partial t} dy, \quad x \in \Omega_x, \quad t > 0. \tag{2.28}$$

This model should be applicable for a small, but not necessarily vanishingly small, block. The object now is to find a limit form for the model as the blocks tend to zero size; that is, as $diam(\Omega_x) \to 0$, uniformly for all $x \in \Omega$. We assume that the void space remains distributed in a fixed way; i.e., the porosities ϕ of the matrix blocks and Φ of the

fractures are independent of the limit process. Similarly, assume that the relative and total permeabilities in the fractures remain fixed, as well as the capillary pressure relations in both the fractures and the matrix blocks. The relaxation time for the saturation equation (2.12) in each of these small blocks will be much smaller than that for the fractures; this is a statement of physical fact, though it can be verified in a semimathematical fashion by computing the first eigenvalue in a separation of variables solution for the differential systems that would result from setting the coefficients in (2.8)-(2.10) and (2.12)-(2.14) to their average values and noticing the very large difference in the expected extents of the fracture domain and a typical matrix block. As a consequence, we can assume that the delay in response in the saturation of a block can be ignored, and the solution of (2.25), (2.21) can be approximated very well by the relation

$$s(x,y,t) = (p_c^{-1} \circ P_c)(S(x,t)), \quad y \in \Omega_x, x \in \Omega; \tag{2.29}$$

note that this is the statement of the physical reality that the time scale in the blocks is sufficiently faster than that in the fractures that equilibrium is reached in a block in a time increment that is less than an increment of physical interest in the fractures. Under this hypothesis, it follows that

$$Q_{o,m}(x,t) \to -\phi \frac{\partial}{\partial t}[p_c^{-1} \circ P_c(S(x,t))], \quad x \in \Omega. \tag{2.30}$$

Substitution of the limit for $Q_{o,m}(x,t)$ into (2.10) gives the modified saturation equation

$$\begin{aligned} &\frac{\partial}{\partial t}[\Phi S + \phi p_c^{-1} \circ P_c(S)] + \Lambda_o' U \cdot \nabla_x S - \nabla_x \cdot (K\Lambda\Lambda_o\Lambda_w P_c' \nabla_x S) \\ &\quad - \nabla_x \cdot (K\Lambda\Lambda_o\Lambda_w(\rho_o - \rho_w)g\nabla_x z) = -\Lambda_o Q_e^+. \end{aligned} \tag{2.31}$$

The limit model is given by the pressure and saturation equations (i.e., by the system (2.8), (2.9) and (2.31)), subject to an initial condition

$$S(x,0) = S^0(x) \tag{2.32}$$

and the (typical) boundary conditions (2.11) and

$$K\Lambda\Lambda_o\Lambda_w[P_c'\nabla_x S + (\rho_o - \rho_w)g\nabla_x z] \cdot \nu_{\partial\Omega} = 0. \tag{2.33}$$

Equation (2.33) results from differencing weighted multiples of the Darcy velocities for oil and water and requiring the flow across $\partial\Omega$ to vanish for this combination.

While this model was derived from models that impose restrictions of local periodicity on the families of fractures defining the blocks, the limit itself can be given a very reasonable interpretation under the sole constraint that the blocks, of whatever shape, have small diameter. The absolute permeability tensor in the fractures can be derived from the micropermeability properties of the fractures and the blocks and the geometry of the fracture families in the periodic case (see [3], [6]); it would have to be determined entirely experimentally in the general case.

3. Existence, Uniqueness and Regularity. In this section we assume that there exists a small positive constant ϵ such that

$$\begin{aligned} &min\, K_{ro}, \quad min\, K_{rw} \geq \epsilon > 0, \\ &max\, |P_c'| \leq 1/\epsilon. \end{aligned}$$

This is not an inherent property of the model, but it will permit us to look at a nondegenerate mathematical problem. These conditions hold if the saturation remains in an interval bounded away from the residual saturations for both phases.

Set

$$\varphi(S) = \Phi S + \phi(p_c^{-1} \circ P_c)(S), \quad 0 \leq S \leq 1. \tag{3.1}$$

Rewrite the pressure equation (2.8)–(2.9) in the form

$$-\nabla_x \cdot (\gamma \nabla_x P + \delta \nabla_x z) = Q_e, \tag{3.2}$$

where

$$\begin{aligned} \gamma &= \gamma(x, S) = K\Lambda, \\ \delta &= \delta(x, S) = K\Lambda\rho_f g. \end{aligned}$$

The boundary condition becomes

$$(\gamma \nabla_x P + \delta \nabla_x z) \cdot \nu_\Omega = 0, \quad x \in \partial\Omega. \tag{3.3}$$

The saturation equation (2.31) becomes

$$\frac{\partial}{\partial t}\varphi(S) - \nabla_x \cdot [\alpha \nabla_x S + \beta U + \zeta] = Q_{o,e}, \tag{3.4}$$

where

$$\begin{aligned} \alpha &= \alpha(x, S) = K\Lambda\Lambda_o\Lambda_w P_c', \\ \beta &= \beta(S) = -\Lambda_o(S), \\ \zeta &= \zeta(x, S) = K\Lambda\Lambda_o\Lambda_w(\rho_o - \rho_w) g \nabla_x z; \end{aligned}$$

the boundary condition becomes

$$(\alpha \nabla_x S + \zeta) \cdot \nu = 0, \quad x \in \partial\Omega. \tag{3.5}$$

Now, consider semi-classical solutions in the sense defined by Kružkov and Sukorjanskiĭ [9]; $(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$. This definition is applicable for either two-dimensional or three-dimensional domains Ω and is equivalent to the one given in [9].

DEFINITION. *A pair of functions* $\{P(x,t), S(x,t)\}$, *bounded and measurable in* $\overline{\Omega \times J}$, $J = [0,T]$, *is called a semi-classical solution of the problem (3.2)–(3.5) if*

i) P *and* S *are Lipschitz-continuous in* $\Omega \times J$;

ii) $(\gamma \nabla_x P + \delta \nabla_x z, \nabla_x q) = (Q_e, q),\ q \in H^1(\Omega), t \in J$;

iii) $(\dfrac{\partial \varphi(S)}{\partial t}, v) + (\alpha \nabla_x S + \beta U + \zeta, \nabla_x v) = (Q_{o,e}(S), v),\ v \in H^1(\Omega),\ t \in J.$

Note that $Q_e = Q_e(x,t)$ is given and it does not depend on the solution, whereas $Q_{o,e}$ does.

Let us consider uniqueness; the theorem below is valid for Ω a bounded domain in either $\mathbf{R}^2$ or $\mathbf{R}^3$.

THEOREM 1. *There exists at most one semi-classical solution of the problem (3.2)–(3.5) satisfying a given initial condition (2.32).*

PROOF: Let $\{P_1, S_1\}$ and $\{P_2, S_2\}$ be solutions for the same data $S^0(x)$ and $Q_e(x,t)$. Set

$$\Pi = P_1 - P_2 \quad \text{and} \quad \Sigma = S_1 - S_2.$$

Then, from *ii)* in the definition above,

$$(\gamma_1 \nabla_x \Pi, \nabla_x q) + ((\delta_1 - \delta_2)\nabla_x z, \nabla_x q) + ((\gamma_1 - \gamma_2)\nabla_x P_2, \nabla_x q) = 0,\ q \in H^1(\Omega).$$

Note that, since Λ is a continuously differentiable function of S, there exists $b \in L^\infty(\Omega \times J)$ such that

$$(\gamma_1 - \gamma_2)\nabla_x P_2 + (\delta_1 - \delta_2)\nabla_x z = -b(x,t)\Sigma.$$

Thus,

$$(\gamma_1 \nabla_x \Pi, \nabla_x q) = (b\Sigma, \nabla_x q), \quad q \in H^1. \tag{3.6}$$

A useful equation for the difference in saturations will require more effort. First,

$$\begin{aligned}(\frac{\partial \varphi(S_1)}{\partial t} - \frac{\partial \varphi(S_2)}{\partial t}, v) + (\alpha_1 \nabla_x \Sigma + \beta_1(U_1 - U_2) + (\zeta_1 - \zeta_2) \\ + (\alpha_1 - \alpha_2)\nabla_x S_2 + (\beta_1 - \beta_2)U_2, \nabla_x v) = (Q_{o,e}(S_1) - Q_{o,e}(S_2), v)\end{aligned}$$

Note that there exist functions c, d, and e in $L^\infty(\Omega \times J)$ such that

$$\begin{aligned}(\alpha_1 - \alpha_2)\nabla_x S_2 + \beta_1(U_1 - U_2) + (\beta_1 - \beta_2)U_2 + (\zeta_1 - \zeta_2) = -c(x,t)\Sigma - d(x,t)\nabla_x \Pi, \\ Q_{o,e}(S_1) - Q_{o,e}(S_2) = e(x,t)\Sigma.\end{aligned}$$

Thus,

$$(\frac{\partial \varphi(S_1)}{\partial t} - \frac{\partial \varphi(S_2)}{\partial z}, v) + (\alpha_1 \nabla_x \Sigma, \nabla_x v) = (c\Sigma + d\nabla_x \Pi, \nabla_x v) + (e\Sigma, v),\ v \in H^1. \tag{3.7}$$

Now, we wish to choose $q = \Pi$ and $v = \Sigma$. The first term in (3.7) can be handled by use of a transformation due to Dupont and Wheeler [**10**], §3.4. Let $R : \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$R(S, \xi) = \int_0^{\xi} \frac{\partial \varphi}{\partial S}(S - \mu)\mu d\mu.$$

Note first that

$$\int_\Omega R(S, \xi)dx = \int_\Omega \int_0^{\xi} \frac{\partial \varphi}{\partial S}(S - \mu)\mu d\mu\, dx \geq \int_\Omega \varphi'_{\min} \frac{1}{2}\xi^2 dx \geq \frac{1}{2}\varphi'_{\min}\|\xi\|_0^2.$$

Next, consider $\frac{d}{dt}\int_\Omega R(S_1, S_1 - S_2)dx$:

$$\begin{aligned}\frac{d}{dt}\int_\Omega R(S_1, S_1 - S_2)dx &= \int_\Omega \varphi'(S_2)\frac{\partial \Sigma}{\partial t}\Sigma dx + \int_\Omega \int_0^{\Sigma} \varphi''(S_1 - \mu)\frac{\partial S_1}{\partial t}\mu d\mu dx \\ &= \int_\Omega \left[\varphi'(S_1)\frac{\partial S_1}{\partial t} - \varphi'(S_2)\frac{\partial S_2}{\partial t}\right]\Sigma dx \\ &\quad + \int_\Omega [\varphi'(S_2) - \varphi'(S_1)]\frac{\partial S_1}{\partial t}\Sigma dx + \int_\Omega \int_0^{\Sigma} \varphi''(S_1 - \mu)\frac{\partial S_1}{\partial t}\mu d\mu dx.\end{aligned}$$

Hence,

$$\begin{aligned}\int_\Omega \left[\frac{\partial \varphi(S_1)}{\partial t} - \frac{\partial \varphi(S_2)}{\partial t}\right]\Sigma dx &= \frac{d}{dt}\int_\Omega R(S_1, \Sigma)dx + \int_\Omega [\varphi'(S_1) - \varphi'(S_2)]\frac{\partial S_1}{\partial t}\Sigma dx \\ &\quad - \int_\Omega \int_0^{\Sigma} \varphi''(S_1 - \mu)\frac{\partial S_1}{\partial t}\mu d\mu dx.\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{d}{dt}\int_\Omega R(S_1, \Sigma)dx + (\alpha_1 \nabla_x \Sigma, \nabla_x \Sigma) \\ &\quad = (c\Sigma + d\nabla_x \Pi, \nabla_x \Sigma) + (e\Sigma, \Sigma) + \int_\Omega \int_0^{\Sigma} \varphi''(S_1 - \mu)\frac{\partial S_1}{\partial t}\mu d\mu dx \\ &\qquad - \int_\Omega [\varphi'(S_1) - \varphi'(S_2)]\frac{\partial S_1}{\partial t}\Sigma dx \\ &\quad \leq M_1\|\Sigma\|_0^2 + M_2\|\nabla_x \Pi\|_0^2 + \frac{1}{2}\alpha_{min}\|\nabla_x \Sigma\|_0^2.\end{aligned}$$

Since it follows from (3.6) that

$$\|\nabla_x \Pi\|_0 \leq M_3\|\Sigma\|,$$

then

$$\frac{d}{dt}\int_\Omega R(S_1, \Sigma)dx + \frac{1}{2}(\alpha_1 \nabla_x \Sigma, \nabla_x \Sigma) \leq M_4\|\Sigma\|_0^2.$$

The inequality $\int_\Omega R(S_1,\Sigma)dx \geq \frac{1}{2}\varphi'_{min}\|\Sigma\|_0^2$ and a Gronwall argument show that

$$\|\Sigma(t)\|_0 + \|\Sigma\|_{L^2(0,t;H^1)} + \|\Pi\|_{L^2(0,t;H^1)} \leq M(t)\|\Sigma(0)\|_0 = 0,$$

and uniqueness of the semi-classical solution has been established.

A trivial modification in the argument above gives continuous dependence on the data for semi-classical solutions. With Σ and Π the differences defined above, it follows that

$$\begin{aligned}\|\Sigma\|_{L^\infty(J;L^2(\Omega))} + \|\Sigma\|_{L^2(J;H^1(\Omega))} + \|\Pi\|_{L^2(J;H^1(\Omega))} \\ \leq M(T)\Big[\|S_1^0 - S_2^0\|_0 + \|Q_{1,e} - Q_{2,e}\|_{L^2(J;L^2(\Omega))}\Big].\end{aligned} \tag{3.8}$$

The existence and regularity of the solution of the system given by (2.8), (2.9), and (2.31) will be demonstrated only in the case of a *two-dimensional* domain; see [2] for a proof of the existence of a particular type of weak solution in the three-dimensional case. Here, let us assume that the function φ of (3.1) is smooth (i.e., $\varphi \in C^2([0,1])$, at least). Since $\varphi' \geq \Phi$, φ is invertible. Let

$$C = \varphi(S), \tag{3.9}$$

so that (3.4) can be rewritten as

$$\frac{\partial C}{\partial t} - \nabla_x \cdot (\tilde{\alpha}\nabla_x C + \tilde{\beta}U + \tilde{\zeta}) = -\tilde{\Lambda}_0 Q_e^+, \tag{3.10}$$

where

$$\begin{aligned}\tilde{\alpha} &= \tilde{\alpha}(x,C) = \alpha(x,S)/\varphi'(S),\\ \tilde{\beta} &= \tilde{\beta}(C) = \beta(S),\\ \tilde{\zeta} &= \tilde{\zeta}(x,C) = \zeta(x,S),\\ \tilde{\Lambda}_i &= \tilde{\Lambda}_i(x,C) = \Lambda_i(x,S);\end{aligned}$$

if either of the porosities is spatially dependent, then the functions $\tilde{\alpha}$ and $\tilde{\zeta}$ will have slightly more complicated dependencies on x than indicated above, but for porosities bounded away from zero the form of (3.10) will remain the same.

Equations (2.8) and (2.9) can be written analogously in terms of C in place of S:

$$U = -\tilde{\gamma}\nabla_x P - \tilde{\delta}\nabla_x z, \tag{3.11}$$

$$\nabla_x \cdot U = Q_e, \tag{3.12}$$

where

$$\begin{aligned}\tilde{\gamma} &= \tilde{\gamma}(x,C) = \gamma(x,S),\\ \tilde{\delta} &= \tilde{\delta}(x,C) = \delta(x,S).\end{aligned}$$

The resulting system is of exactly the same form as that for a standard, unfractured immiscible displacement, except that C belongs to a different, still compact, interval. If initial values

$$C(x,0) = C^0(x), \ x \in \Omega, \tag{3.13}$$

and the boundary conditions

$$U \cdot \nu = -(\tilde{\alpha}\nabla_x C + \tilde{\zeta}) \cdot \nu = 0, \ x \in \partial\Omega, \tag{3.14}$$

are imposed, then the existence and regularity results of Kružkov and Sukorjanskiĭ [9] can be applied directly to show that (3.10)-(3.14) has a unique classical solution for smooth, consistent data.

Finally, invert C to find a function S that is a classical solution of the original problem. Since it is clearly a semi–classical solution as well, it is the unique solution.

4. Some Numerical Experiments. While a recovery curve, the plot of the volume of the oil produced versus the volume of the water injected, gives only a gross indication of the flow of fluids within a reservoir, it is sufficient to illustrate important features of flows in naturally fractured reservoirs. Consider a rectangular reservoir Ω with height 10 meters and length 300 meters. For computational simplicity, assume that the reservoir is uniform in the other direction, so that the fracture simulation can be considered to be two-dimensional, though the matrix simulation remains three-dimensional for each block Ω_x.

Let the capillary pressure functions be assumed in the forms

$$P_c(S) = (1-S)\{\gamma(S^{-1}-1)+\theta\}, \tag{4.1a}$$

$$p_c(s) = \alpha(\{s_0 - s + \beta\}^{\frac{1}{2}} - \beta^{\frac{1}{2}})(s - s_{rw})^{-\delta}$$

$$s_0 = 1 - s_{ro}, \quad \beta = s_{ro}^2(s_0 - s_{rw})^{-2}, \tag{4.1b}$$

and let the relative permeabilities be taken in the forms

$$K_{ro}(S) = 1 - S, \qquad K_{rw}(S) = S, \tag{4.2a}$$

$$k_{ro}(s) = s_0^{-2}(s_0 - s)^2, \quad k_{rw}(s) = (s - s_{rw})^2(1 - s_{rw})^{-2}. \tag{4.2b}$$

Let the base case be described by the following data:

Fluid Properties:		
Viscosity	$\mu_w = .5\ cP$	$\mu_0 = 2\ cP$
Density	$\rho_w = 1\ g/cm^3$	$\rho_0 = .7\ g/cm^3$
Absolute Permeabilities:	$K = 1\ darcy$	$k = 0.05\ darcy$
Porosities:	$\Phi = .01$	$\phi = .2$
Residual Saturations (matrix):	$s_{ro} = .15$	$s_{rw} = .2$
Capillary Pressure Parameters:		
	$\gamma = 20{,}000\ dynes/cm^2$	
	$\theta = 100\ dynes/cm^2$	
	$\alpha = 1{,}500\ dynes/cm^2$	

The residual saturations in the fractures are zero. Assume that the reservoir contains 76% oil and 24% water initially, in capillary equilibrium. For the base case, assume that the angle of inclination ϑ of the reservoir is zero. The water injection rate is $.2pv/yr$, with uniform injection along the left edge and production at the upper right corner. The numerical experiments reported below used a finite difference method that has been described in [6]. Tests run earlier [6], [7] showed that a 40x10 grid suffices to give accurate recovery curves. A variable time step was used, beginning with one day and ending with twenty days.

Other tests reported in [8] indicated quite clearly that the solution of both the medium and small block models produced recovery curves that converged to the corresponding recovery curve associated with the limit model of this paper. It was seen in [8] that, as the block size tends to zero, the recovery curves for the medium block model are very close to those of the limit model for block sizes below 50 cm and become almost indistinguishable from that for the limit model at about 10 cm. The recovery curves for the small block model fall in between the corresponding curves for the medium block model and the curve for the limit model. These convergences are the validations of the limit model, for the object of the model is to produce a mathematical model of greater computational simplicity than either the medium or small block models while retaining adequate engineering accuracy for very small block size. The conclusions of some of the experimental results in [8] differ from those reported earlier in [6] and [7]; there was a data representation error in the earlier calculations.

Figure 1 indicates that a higher absolute permeability in the fractures leads to earlier and greater water production, a conclusion that appears to be correct physically, as it is easier for water to bypass the matrix blocks. Similarly, increasing the capillary pressure function in the fractures slows oil production, as it should; see Figure 2. Correspondingly,

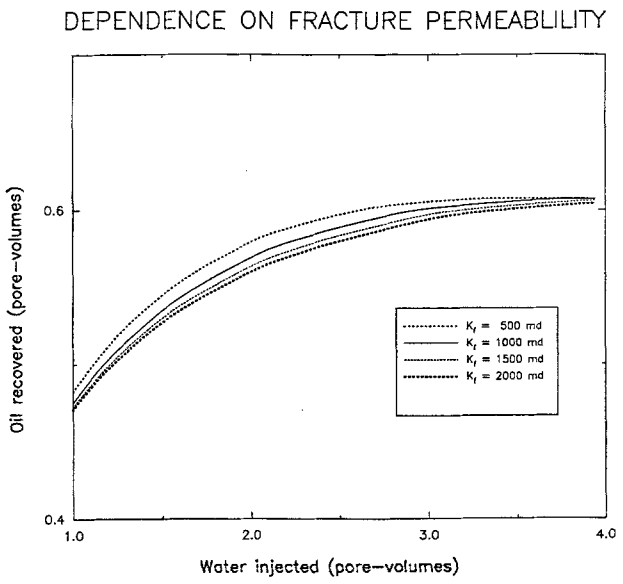

Figure 1

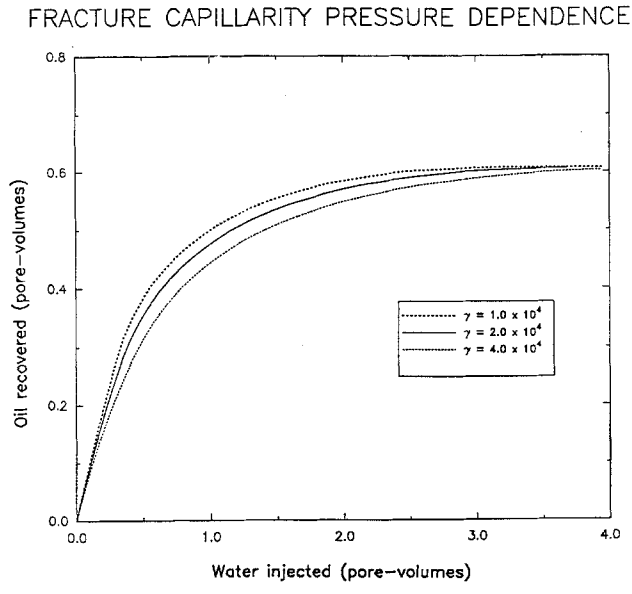

Figure 2

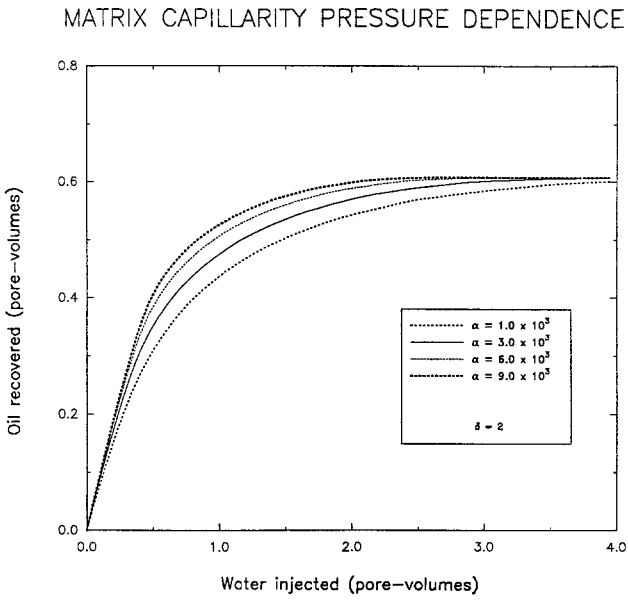

Figure 3

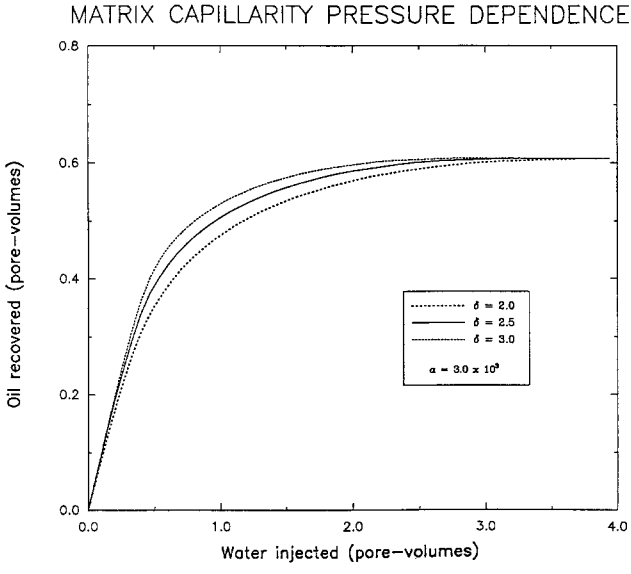

Figure 4

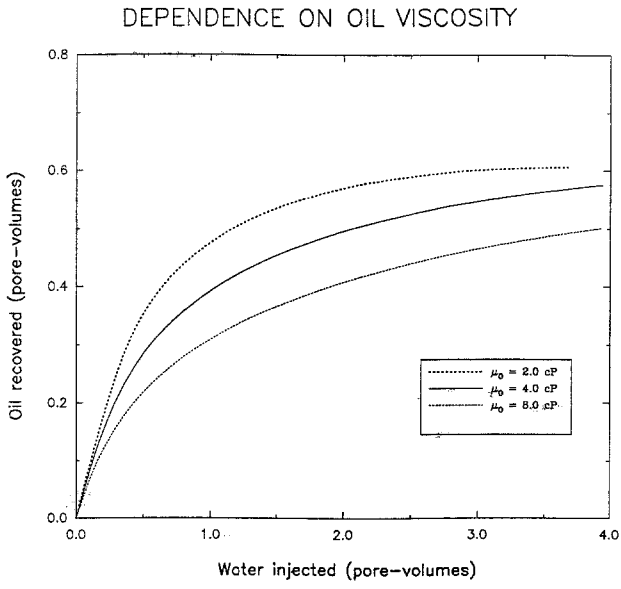

Figure 5

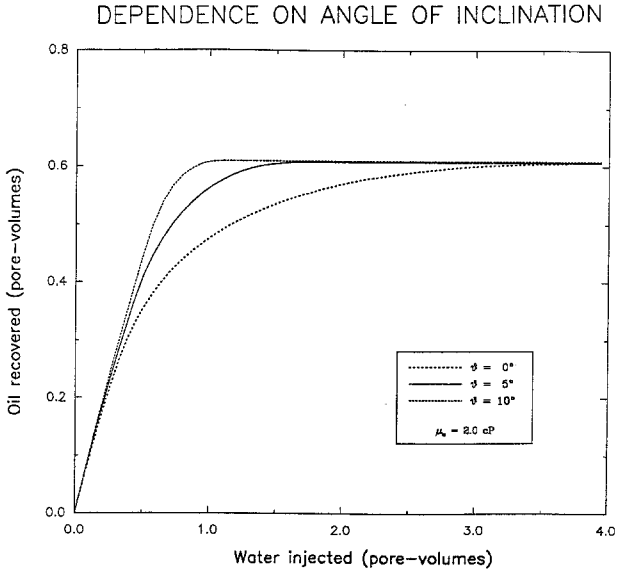

Figure 6

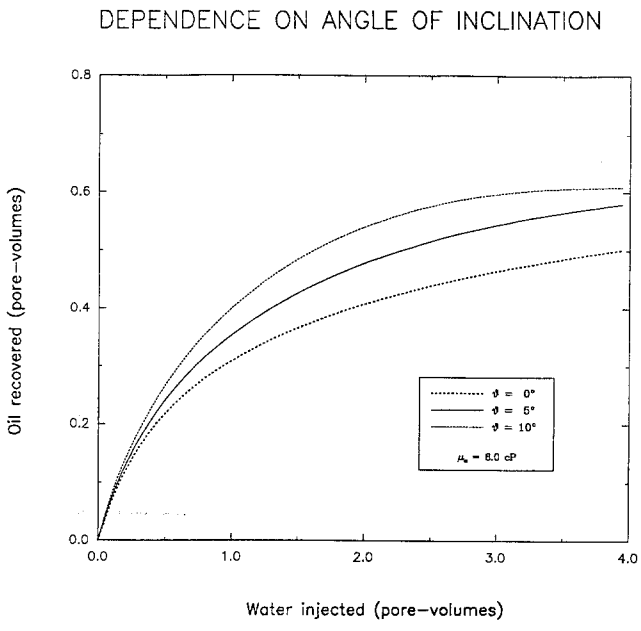

Figure 7

increasing the capillary pressure function in the matrix blocks slows the production of water and improves oil recovery; see Figures 3 and 4.

Figure 5 shows the physically reasonable behavior that oil recovery is slowed by increasing oil viscosity. If the reservoir is inclined by an angle ϑ ($\vartheta > 0$ indicating that the production corner is above the injection side of the reservoir), oil recovery is improved, the more so for more viscous oil. See Figures 6 and 7.

Additional experiments indicated a greater sensitivity to the matrix porosity when the fracture porosity was .001 than when it was .01, with more rapid oil recovery (again for *pv*–injected vs. *pv*–recovered) for a higher matrix porosity.

5. Global Pressure Formulation. The derivation of the global pressure formulation of (2.1) and (2.2) is outlined here. See also [1], [9], [5]. It follows easily from (2.6) that

$$\nabla_x P = \Lambda_o \nabla_x P_o + \Lambda_w \nabla_x P_w.$$

With U being given by (2.8), the pressure equation (2.9) is immediate upon addition of (2.1) and (2.2).

In order to obtain (2.10), first note that

$$\nabla_x P_o = \nabla_x P + \Lambda_w \nabla_x P_c,$$

as $\Lambda_o + \Lambda_w = 1$. Then,

$$K\Lambda\Lambda_o\nabla_x P_o = -\Lambda_o(U + K\Lambda g\rho_f\nabla_x z) + K\Lambda\Lambda_o\Lambda_w\nabla_x P_c;$$

(2.10) follows by substitution of this relation into (2.2) and using the assumption that

$$Q_{o,e} = -\Lambda_o Q_e^+.$$

6. Nomenclature.

θ - phase = o (oil) or w (water)

μ_θ - viscosity of phase θ

Φ - fracture porosity

K - absolute permeability in the fractures

$K_{r\theta}$ - relative permeability of phase θ in the fractures

P_θ - pressure of phase θ in the fractures

P_c - capillary pressure in the fractures ($= P_o - P_w$)

S_θ - saturation of phase θ in the fractures

ϕ - matrix porosity

k - absolute permeability in the matrix

$k_{r\theta}$ - relative permeability of phase θ in the matrix

p_θ - pressure of phase θ in the matrix

p_c - capillary pressure in the matrix ($= p_o - p_w$)

s_θ - saturation of phase θ in the matrix

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