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Spectral Invariance, Ellipticity, and the Fredholm Property for Pseudodifferential Operators on Weighted Sobolev Spaces

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Abstract: The pseudodifferential operators with symbols in the Grushin classes $\tilde{S}^0_{\rho,\delta}$, $0 \leq \delta < \rho \leq 1$, of slowly varying symbols are shown to form spectrally invariant unital Fréchet-*-algebras (Ψ^*-algebras) in $\mathcal{L}(L^2(\mathbb{R}^n))$ and in $\mathcal{L}(H^{st}_\gamma)$ for weighted Sobolev spaces H^{st}_γ defined via a weight function γ. In all cases, the Fredholm property of an operator can be characterized by uniform ellipticity of the symbol. This gives a converse to theorems of Grushin and Kumano-go-Taniguchi. Both, the spectrum and the Fredholm spectrum of an operator turn out to be independent of the choices of s, t and γ.

The characterization of the Fredholm property by uniform ellipticity leads to an index theorem for the Fredholm operators in these classes, extending results of Fedosov and Hörmander.
Key words: *Pseudodifferential operators, spectral invariance, Fredholm operators, ellipticity*
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Introduction

In the theory of pseudodifferential operators, Fredholm results have always been of particular interest. In fact, one of the first applications was the proof of the Atiyah-Singer Index Theorem for classical elliptic pseudodifferential operators on compact manifolds [A].

These operators naturally act on the Sobolev spaces H^s, $s \in \mathbb{R}$. Both, the Fredholm property and the index of these operators do not depend on the choice of s. If P is a classical pseudodifferential operator of order m on a compact manifold, then $P : H^m \to L^2$ is Fredholm, iff $P : H^{s+m} \to H^s$ is Fredholm for *any* s, namely iff it is elliptic, i.e. if its principal symbol does not vanish on the cosphere bundle of the manifold. Ellipticity allows the construction of a parametrix; in particular, it yields a Fredholm inverse which is a pseudodifferential operator.

On $\mathbb{R}^n$, and with non-classical symbols, for example the standard Hörmander classes, the situation gets more complicated. A parametrix construction requires more than invertibility of the symbol, and even a parametrix need not be a Fredholm inverse. This is why I focus on subclasses of the Hörmander classes, a family of symbols that Kumano-go calls "slowly varying". They were considered by Grushin [G8], Kumano-go and Taniguchi [K3], and e.g. Wong [W2], [W3].

The zero order symbols in these classes form Fréchet-*-algebras in $\mathcal{L}(L^2(\mathbb{R}^n))$ – a fact easily established from the standard calculus. Besides, they are spectrally invariant, and the Fredholm property can be characterized by uniform ellipticity (cf. Thms. 1.4 and 1.8).

Here, spectral invariance means the following: If an operator in such an algebra is bijective on $L^2(\mathbb{R}^n)$, then its inverse is in the algebra. In this context, Gramsch introduced the notion "Ψ^*-algebra": A symmetric unital Fréchet-*-subalgebra $\mathcal{A}$ of a C^*-algebra $\mathcal{B}$ is a Ψ^*-algebra in $\mathcal{B}$, if it has a finer topology and is *spectrally invariant*, i.e.

$$\mathcal{A} \cap \mathcal{B}^{-1} = \mathcal{A}^{-1},$$

cf. [G3], Def. 5.1: $\mathcal{A}$ is a 'full' subalgebra of $\mathcal{B}$. In fact, the operator algebras considered are Ψ^*-algebras in $\mathcal{L}(L^2(\mathbb{R}^n))$, cf. Thm. 1.4.

Spectral invariance is a distinguished property shared by many classes of pseudodifferential operators [B1], [B2], [C4], [G7], [L1], [S1], [S4], [S5], [S6], [U], however it already fails in slighty different situations [G2], [D], [W1], [G3], 6.2.

Once established, the Ψ^*-property yields remarkable results. In Ψ^*-algebras, there is a holomorphic functional calculus in several variables; the K-theory of the Ψ^*-algebra coincides with that of its C^*-closure (cf. [B3], Thm. 1.3.1, Thm. A.2.1). One obtains results about Fréchet manifolds in Fredholm and perturbation theory [G3], for the division problem for operator valued distributions [G4], and in differential geometry of Fréchet manifolds, especially for periodic geodesics [G6].

For a function $\gamma \in C^\infty(\mathbb{R}^n)$ with $\gamma(x) \geq c > 0$ and $D^\alpha \gamma(x) \to 0 \quad (|x| \to \infty, \alpha \neq 0)$ – an "admissible weight function" – define the weighted Sobolev space H_γ^{st} by

$$H_\gamma^{st} = \{\gamma^{-t} u : u \in H^s\}$$

in the same way in which one obtains the Sobolev space H^s from L^2. All pseudodifferential operators with symbols in $S^0_{\rho,\delta}$ are bounded on these spaces and their spectrum is independent of s, t, and γ, cf. [S1].

An even stronger result holds here: The operator algebras considered are Ψ^*-algebras in all the spaces H_γ^{st}. The spectrum of an operator is independent of s, t, and γ. On all these spaces H_γ^{st}, an operator is Fredholm iff its symbol is uniformly elliptic (Thm. 1.11). As a consequence, also the Fredholm property is independent of s, t, and γ.

This contrasts with the effects observed by Lockhart and McOwen [L2], [M] (cf. also Nirenberg and Walker [N]). They investigated the behavior of differential operators on another scale of weighted Sobolev spaces $W^{s\delta}$ and found that both, the Fredholm property and the index, depended on $\delta \in \mathbb{R}$.

Is there a way to compute the index? There is an obvious candidate for an index formula, namely Fedosov's formula [F]

$$\text{index Op } a = -(-2\pi \mathrm{i})^{-n}(n-1)!/(2n-1)! \int_{\partial B} \mathrm{Tr}(a^{-1} \mathrm{d}a)^{2n-1}, \tag{0.1}$$

proved e.g. in [H2], (Thm. 19.3.1′ plus the following remarks), for pseudodifferential operators with symbols in the class $S(1, G)$, $G = |\mathrm{d}x|^2 \langle x \rangle^{-2} + |\mathrm{d}\xi|^2 \langle \xi \rangle^{-2}$, using the Weyl calculus. In (0.1), B is an open ball in $\mathbb{R}^{2n}$ such that $a(x, \xi)^{-1}$ exists and is bounded outside B. $\mathbb{R}^{2n}$ is oriented by $\mathrm{d}x_1 \wedge \mathrm{d}\xi_1 \wedge \ldots \wedge \mathrm{d}x_n \wedge \mathrm{d}\xi_n > 0$, and the left-hand side gives the L^2-index.

The necessity of uniform ellipticity shows that the right-hand side of the index formula makes sense also for Fredholm operators in Op $\tilde{S}^0_{\rho,\delta}$, $\rho > \delta$. So it is reasonable to expect that the formula extends to these classes, and this turns out to be true, cf. Theorem 1.13.

Necessity of ellipticity for the Fredholm property of *classical* pseudodifferential operators was proven by Kohn and Nirenberg [K1] extending Gohberg's lemma [G1]. By showing the necessity of *uniform* ellipticity, Theorem 1.8 gives a converse to a theorem by Grushin [G8], Thm. 3.4. It also establishes a converse to Kumano-go and Taniguchi's more general hypoellipticity result in [K3], if the order is restricted to zero. Together with the index result this completely answers the question about the essential spectrum of pseudodifferential operators with slowly varying symbols [W3], Question 1.4, even without the additional assumptions on the symbol in [W3], Thm. 3.2.

The method used here to establish the spectral invariance for the classes $\tilde{S}^0_{\rho,\delta}$ is different from the C^∞-elements approach of Cordes [C5] or the commutator method of Beals [B1]. It was developed in order to show spectral invariance for a version of Boutet de Monvel's algebra on noncompact manifolds and proving necessary and sufficient conditions for pseudodifferential boundary value problems to have the Fredholm property [S2]. An important ingredient is a particular operator theoretical construction of Gramsch and Kaballo [G4].

1. Statement of the Results

We start with a review of three symbol classes on $\mathbb{R}^n$, $n \in \mathbb{N}$, the first being the 'standard' pseudodifferential symbols, the second and third more special and going back to Grushin [G8]. The definition follows Kumano-go [K2], Ch. 2, Def. 1.1, Ch. 3., Def. 5.11.

Definition 1.1. Let $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$, $0 \le \delta \le \rho \le 1$, $\delta < 1$. Write $p^{(\alpha)}_{(\beta)}(x,\xi)$ instead of $D^\alpha_\xi D^\beta_x p(x,\xi)$, $x,\xi \in \mathbb{R}^n$.

(a) $p \in S^m_{\rho,\delta}$, if $|p^{(\alpha)}_{(\beta)}(x,\xi)| \le C_{\alpha\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}$ for all $x,\xi \in \mathbb{R}^n$.

(b) $p \in \dot{S}^m_{\rho,\delta}$, if $p \in S^m_{\rho,\delta}$ and $|p^{(\alpha)}_{(\beta)}(x,\xi)| \le C_{\alpha\beta}(x)\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}$ for all $x,\xi \in \mathbb{R}^n$, with a bounded function $C_{\alpha\beta}(x)$ such that $C_{\alpha\beta}(x) \to 0$ as $x \to \infty$.

(c) $p \in \tilde{S}^m_{\rho,\delta}$, if $p \in S^m_{\rho,\delta}$ and $p_{(\beta)}(x,\xi) \in \dot{S}^{m+\delta|\beta|}_{\rho,\delta}$ for all $\beta \ne 0$.

Kumano-go proves that, in (c), it is sufficient to ask that $p_{(\beta)}(x,\xi) \in \dot{S}^{m+\delta}_{\rho,\delta}$ for all $|\beta| = 1$. He calls the symbols in $\tilde{S}^m_{\rho,\delta}$ *slowly varying.* Write $S^{-\infty}_{\rho,\delta}, \dot{S}^{-\infty}_{\rho,\delta}, \tilde{S}^{-\infty}_{\rho,\delta}$ for the respective intersections taken over all $m \in \mathbb{R}$. They are independent of ρ and δ, and we shall omit the indices. For simplicity we have assumed the functions p to be scalar-valued. In general, p might have values in quadratic matrices over $\mathbb{C}$. All the results extend to that case.

Lemma 1.2. *The best constants $C_{\alpha\beta}$ in* 1.1(a) *define the usual Fréchet topology for $S^m_{\rho,\delta}$. The spaces $\dot{S}^m_{\rho,\delta}$ and $\tilde{S}^m_{\rho,\delta}$ are closed subspaces, $\dot{S}^m_{\rho,\delta} \subseteq \tilde{S}^m_{\rho,\delta} \subseteq S^m_{\rho,\delta}$.*

For an arbitrary symbol class $\mathcal{X}$, we will denote by Op $\mathcal{X}$ all the pseudodifferential operators with symbols in $\mathcal{X}$.

Lemma 1.3. *Op $\tilde{S}^0_{\rho,\delta}$ is a Fréchet-*-subalgebra of $\mathcal{L}(L^2(\mathbb{R}^n))$ containing the identity. Its topology is stronger than the norm topology on $\mathcal{L}(L^2(\mathbb{R}^n))$.*

Proof. Op : $S^0_{\rho,\delta} \to \mathcal{L}(L^2(\mathbb{R}^n))$ is injective and continuous by the Calderon-Vaillancourt theorem cf. [K2], Ch. 2, Prop. 1.2, Ch. 7, Thm. 1.6. Multiplication and $*$ are continuous, cf. [K2], Ch. 2, §2. By [K2], Ch.3, Lemma 5.13, Op $\tilde{S}^0_{\rho,\delta}$ is a $*$-algebra. □

The following theorem shows that even more holds. The proof will be given in Section 2.

Theorem 1.4. *Op $\tilde{S}^0_{\rho,\delta}$, $\delta < \rho$, is a Ψ^*-algebra in $\mathcal{L}(L^2(\mathbb{R}^n))$.*

The main result of §5 in Chapter 3 of Kumano-go's book [K2], Thm. 5.16, is the following theorem. It extends a result of Grushin [G8], Thm. 3.4.

Theorem 1.5. *Let $p \in \tilde{S}^0_{\rho,\delta}$, $\delta < \rho$, $m \geq 0$. Assume that for some $R > 0$, $0 \leq m' \leq m$, $C_0 > 0$:*

$$|p(x,\xi)| \geq C_0 \langle\xi\rangle^{m'}, \qquad |x| + |\xi| \geq R, \quad \text{and}$$
$$|p^{(\alpha)}_{(\beta)}(x,\xi)/p(x,\xi)| \leq C_{\alpha\beta}(x)\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|}, \quad |x| + |\xi| \geq R,$$

where $C_{\alpha\beta}$ are bounded functions such that $C_{\alpha\beta}(x) \to 0$ ($|x| \to \infty$) for $\beta \neq 0$. Then Op p has a parametrix Op q, $q \in \tilde{S}^{-m'}_{\rho,\delta}$ such that

$$\text{Op}\, p \circ \text{Op}\, q - I \in \dot{S}^{-\infty}, \quad \text{and} \quad \text{Op}\, q \circ \text{Op}\, q - I \in \dot{S}^{-\infty}.$$

In particular, Op p is a Fredholm operator on $\mathcal{L}(L^2(\mathbb{R}^n))$ in view of the following lemma:

Lemma 1.6. ([K2], Ch.3, Lemma 5.14). *Let $p \in \dot{S}^{-\varepsilon}_{\rho,\delta}$ for some $\varepsilon > 0$. Then Op $p : H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ is compact on each of the Sobolev spaces $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$.*

Definition 1.7. Call $p \in S^0_{\rho,\delta}$ uniformly elliptic, if there are constants $R, C > 0$ such that for all $|x| + |\xi| > R$, $p(x,\xi)$ is invertible and $|p(x,\xi)^{-1}| \leq C$.

Theorem 1.8. *An element $P \in$ Op $\tilde{S}^0_{\rho,\delta}$, $\delta < \rho$, is a Fredholm operator on $L^2(\mathbb{R}^n)$ if and only if its symbol is uniformly elliptic.*

Theorem 1.8 has an obvious application to differential operators $P = \sum\limits_{|\alpha|\leq m} a_\alpha(x) \mathrm{D}^\alpha$, where the a_α are bounded C^∞ functions on $\mathbb{R}^n$ with $\mathrm{D}^\beta_x a_\alpha(x) \to 0$ as $|x| \to \infty$ for all $\beta \neq 0$: Consider $P\langle \mathrm{D}\rangle^{-m}$, which has its symbol in $\tilde{S}^0_{1,0}$. Then $P : H^m \to L^2$ is Fredholm if and only if the symbol of $P\langle \mathrm{D}\rangle^{-m}$ is uniformly elliptic. This result has been established earlier by the C^*-algebra methods of Cordes, cf. [C3], [T2].

Definition 1.9. Call $\gamma \in C^\infty(\mathbb{R}^n)$ an admissible weight function, if the following holds

(i) For some $c > 0$, $\gamma(x) \geq c$

(ii) For all $\alpha \neq 0$, $\mathrm{D}^\alpha\gamma(x) \to 0$ ($|x| \to \infty$).

The condition on the weight function is a little more restrictive than in [S1]. Examples for admissible weight functions are: $\gamma(x) = \langle x\rangle^{\frac{1}{2}}$, $\gamma(x) = 1 + \ln(\langle x\rangle)$, and, writing $x = (x', x'')$, $\gamma(x) = (\langle x'\rangle + \langle x''\rangle^2)^{1/3}$, $\gamma(x) = (\langle x'\rangle + \ln(\langle x''\rangle))^{\frac{1}{2}}$, $\gamma(x) = 1 + \ln(\langle x'\rangle + \langle x''\rangle^2)$, etc. If γ is an admissible weight function, then $\gamma^{-1} \in \tilde{S}^0_{1,0}$, and $\mathrm{D}^\alpha\gamma \in \dot{S}^0_{1,0}$ for all $\alpha \neq 0$. If $0 \le t \le 1$ and γ is admissible, then the function $\tilde{\gamma}(x) = \gamma(x)^t$ is also admissible.

Definition 1.10. For an admissible weight function γ and $s, t \in \mathbb{R}$ define the weighted Sobolev space $H^{st}_\gamma = H^{st}_\gamma(\mathbb{R}^n)$ by

$$H^{st}_\gamma = \{\gamma^{-t}u : u \in H^s\}.$$

The scalar product is given canonically by $(u,v)_{s,t} = (\langle \mathrm{D}\rangle^s\gamma^t u, \langle \mathrm{D}\rangle^s\gamma^t v)_{L^2}$. Of course, the Schwartz space $\mathcal{S}$ is dense in all the spaces H^{st}_γ.

Theorem 1.11. *Op $\tilde{S}^0_{\rho,\delta}$, $\delta < \rho$, is a Ψ^*-algebra in $\mathcal{L}(H^{st}_\gamma)$ for all $s, t \in \mathbb{R}$ and every admissible weight function γ. An operator*

$$P : H^{st}_\gamma \to H^{st}_\gamma$$

is Fredholm if and only if the symbol of P is uniformly elliptic in the sense of Definition 1.7. The index is independent of the choices of s, t, and γ.

For the proof of 1.11 see Section 2. Theorem 1.11 shows that neither the spectrum nor the Fredholm spectrum of an operator in Op $\tilde{S}^0_{\rho,\delta}$, $\rho > \delta$, depends on the choice of the underlying space H^{st}_γ. In connection with Lockhart and McOwen's results [L2] note

Corollary 1.12. *Let P be a homogeneous differential operator of order $m > 0$, $P(x,\mathrm{D}) = \sum_{|\alpha|=m} a_\alpha(x)\mathrm{D}^\alpha$ with coefficients $a_\alpha \in C^\infty(\mathbb{R}^n)$ satisfying $a_\alpha(x) = O(1)$, $\mathrm{D}^\beta a_\alpha(x) = o(1)$, $\beta \neq 0$, as $x \to \infty$. Then there is* no *choice of $s, t \in \mathbb{R}$ and an admissible weight function γ such that $P : H^{s+m,t}_\gamma \to H^{s,t}_\gamma$ is a Fredholm operator.*

Proof. Consider $P\langle \mathrm{D}\rangle^{-m}$. Its symbol $\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha\langle\xi\rangle^{-m} \in \tilde{S}^0_{1,0}$ is *not* uniformly elliptic. Therefore $P\langle \mathrm{D}\rangle^{-m}$ cannot be Fredholm on $H^{s,t}_\gamma$ by Theorem 1.12. Hence $P : H^{s+m,t}_\gamma \to H^{s,t}_\gamma$ is not Fredholm. □

Hörmander's class $S(1,G)$ with $G = |dx|^2\langle x\rangle^{-2} + |d\xi|^2\langle\xi\rangle^{-2}$ is a subclass of $\tilde{S}^0_{1,0}$, thus of all the $\tilde{S}^0_{\rho,\delta}$. From Theorem 1.11 and [H2], (Thm. 19.3.1′ plus the following remarks) in connection with the above results one obtains

Theorem 1.13. *Suppose $A \in$ Op $\tilde{S}^0_{\rho,\delta}$, $\rho > \delta$, is a Fredholm operator in $\mathcal{L}(H^{st}_\gamma)$. Then the index of A is given by Fedosov's formula* (0.1).

The proof of Theorem 1.13 will be given in Section 3.

Remark 1.14. Let $a \in \tilde{S}^0_{\rho,\delta}$. The Weyl operator $\mathrm{Op}^\omega a$ is the operator with the 'double' symbol $b(x,y,\xi) = a(\frac{x+y}{2},\xi)$. By Taylor's formula, the symbol of $\mathrm{Op}^\omega a$ − Op a is

$$c(x,y,\xi) = \sum_{k=1}^{n} \int_0^1 (\partial_{y_k} \mathrm{D}_{\xi_k} b)(x, x + t(y-x), \xi)\, dt.$$

Converting this to a 'single' symbol, cf. [K2], Ch. 2, Thm. 2.5, and applying an analysis like that in [K2], Ch. 2, Lemma 2.4, one notices that $\mathrm{Op}^w a - \mathrm{Op}\ a$ is compact whenever $\rho > \delta$. Therefore the statement of Theorem 1.13 also holds in the case of the Weyl calculus.

2. Proofs of the Spectral Invariance and Ellipticity Results

The following proposition is based on results by Hörmander [H1] and Grushin [G8]:

Proposition 2.1. *Let $p \in \tilde{S}^0_{\rho,\delta}$, $\delta < \rho$, and let $d = \limsup_{(x,\xi)\to\infty} |p(x,\xi)|$. Then $\|\mathrm{Op}\ p\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \geq d$. In particular: if $\|\mathrm{Op}\ p\| < \varepsilon$ for some $\varepsilon > 0$, then there is an $M > 0$ such that $|p(x,\xi)| < 2\varepsilon$ for all $|x| + |\xi| \geq M$.*

Proof. Supposing there is a sequence (x^ν, ξ^ν) such that $|p(x^\nu,\xi^\nu)| \to d$ and $\xi^\nu \to \infty$, the argument of Hörmander in the proof of [H1], Thm. 3.3, shows that $d \leq \|\mathrm{Op}\ p\|$. On the other hand, if there is a sequence (x^ν, ξ^ν) such $x^\nu \to \infty$, ξ^ν is bounded, and $|p(x^\nu,\xi^\nu)| \to d$, then we may assume that $\xi^\nu \to \xi^0$. Now Grushin's proof of [G8], Prop. 3.3, shows that $\|\mathrm{Op}\ p\| \geq \mathrm{d}$. □

Corollary 2.2. *Suppose $p \in \tilde{S}^0_{\rho,\delta}$, $\delta < \rho$, and $\|\mathrm{Op}\ p\| < 1/4$. Then there is an $M > 0$ such that $|p(x,\xi)| < 1/2$ for all $|x| + |\xi| \geq M$, thus $|(1 + p(x,\xi))^{-1}| \leq 2$ for all $|x| + |\xi| \geq M$. Theorem 1.5 implies that there is a $q \in \tilde{S}^0_{\rho,\delta}$, such that*

$$(1+P)Q - I = R_1 \in \mathrm{Op}\ \dot{S}^{-\infty} \quad and \quad Q(1+P) - I = R_2 \in \mathrm{Op}\ \dot{S}^{-\infty},$$

with $P = \mathrm{Op}\ p$, $Q = \mathrm{Op}\ q$.

Lemma 2.3. *$\mathbb{C}I + \mathrm{Op}\ \dot{S}^{-\infty}$ is a Ψ^*-algebra in $\mathcal{L}(L^2(\mathbb{R}^n))$.*

Proof. $\mathrm{Op}\ \dot{S}^{-\infty}$ is a proper ideal in $\mathrm{Op}\ S^0_{\rho,\delta}$: It is the intersection of the ideals $\mathrm{Op}\ \dot{S}^{-m}_{\rho,\delta}$, $m > 0$, cf. [K2], Ch. 3, Lemma 5.12. It is proper, since all its operators are compact on $L^2(\mathbb{R}^n)$. Therefore, $\mathbb{C}I + \mathrm{Op}\ \dot{S}^{-\infty}$ is a Fréchet-*-subalgebra of $\mathcal{L}(L^2(\mathbb{R}^n))$ with a finer topology. Suppose an element of $\mathbb{C}I + \mathrm{Op}\ \dot{S}^{-\infty}$ is invertible on $\mathcal{L}(L^2(\mathbb{R}^n))$. Then it is of the form $\lambda I + S$ for some $S \in \mathrm{Op}\ \dot{S}^{-\infty}$, $\lambda \neq 0$, and $(\lambda I + S)^{-1} \in \mathrm{Op}\ S^0_{\rho,\delta}$ by the classical spectral invariance result [B1], Thm. 3.2, cf. [U], Satz 4.3. Hence

$$(\lambda I + S)^{-1} = \lambda^{-1}(I - S(\lambda I + S)^{-1}) \in \mathbb{C}I + \mathrm{Op}\ \dot{S}^{-\infty},$$

due to the ideal property. □

Remark 2.4. Recall the following facts from operator theory. For a proof, consult Taylor's book [T1], Thm. 5.41-G, Thm. 5.5-E, Thm. 5.8-A.

If $B, C \in \mathcal{L}(E)$, E a Banach space, and $BC = I + S$, S compact, then there is an $r \in \mathbb{N}$ with

$$\mathcal{N}((BC)^r) = \mathcal{N}((BC)^{r+1}), \mathcal{R}((BC)^r) = \mathcal{R}((BC)^{r+1}),$$
$$E = \mathcal{N}((BC)^r) \oplus \mathcal{R}((BC)^r),$$
$$BC : \mathcal{R}((BC)^r) \to \mathcal{R}((BC)^r) \quad \text{bijective.}$$

The spectrum of BC is discrete with only accumulation point $\lambda = 1$, for S is compact. In $\lambda = 0$, the resolvent $(\lambda I - BC)^{-1}$ has a pole of finite order (unless there is no singularity at all). Letting P be the projector defined by

$$P = \frac{1}{2\pi i}\int_{\Gamma'}(\lambda I - BC)^{-1}d\lambda = \frac{1}{2\pi i}\int_{\Gamma}(\lambda I - S)^{-1}d\lambda,$$

Γ' and Γ denoting small circles about $\lambda = 0$ and $\lambda_0 = -1$,

$$\mathcal{N}((BC)^r) = \mathcal{R}(P), \quad \text{and} \quad \mathcal{R}((BC)^r) = \mathcal{N}(P).$$

Proposition 2.5. *Let $A \in \text{Op}\ \tilde{S}^0_{\rho,\delta}$, $\rho > \delta$, with $\|A\| < \frac{1}{4}$. Then $(I + A)^{-1} \in \text{Op}\ \tilde{S}^0_{\rho,\delta}$.*

Proof. Step 1. By Corollary 2.2 there is an operator $A' \in \text{Op}\ \tilde{S}^0_{\rho,\delta}$ such that $(I + A')(I + A) = I + S$, $S \in \text{Op}\ \dot{S}^{-\infty}$. Let $B = I + A'$, $C = I + A$, and apply the results of 2.4. By Lemma 2.3, it follows from the integral representation of the spectral projection P that $P \in \mathbb{C}I + \text{Op}\ \dot{S}^{-\infty}$.

Step 2. $\mathcal{R}(P)$ is finite dimensional, and $P \in \text{Op}\ \dot{S}^{-\infty}$.

Proof. $\mathcal{R}(P) = \mathcal{N}((BC)^r) = \mathcal{N}(I + S')$ for some $S' \in \text{Op}\ \dot{S}^{-\infty}$. S' is compact, thus the range of P is finite dimensional. We already know that $P = \lambda I + S''$ for some $\lambda \in \mathbb{C}$ and $S'' \in \text{Op}\ \dot{S}^{-\infty}$. The fact that the range is finite dimensional implies $\lambda = 0$.

Step 3. There is a relative inverse F to CP in $\text{Op}\ \dot{S}^{-\infty}$, i.e., there is an $F \in \text{Op}\ \dot{S}^{-\infty}$ such that

$$F\,CP\,F = F \quad \text{and} \quad CP\,F\,CP = CP.$$

If F is any relative inverse with these properties we can – and will – replace F by PF.

Proof. It is sufficient to show that there is such an F in $\text{Op}\ S^0_{\rho,\delta}$, since we can replace F by PF, and $P \in \text{Op}\ \dot{S}^{-\infty}$. So choose a basis $\{e_1, \ldots, e_k\}$ of $\mathcal{R}(P)$, and define $f_j = CPe_j = Ce_j$. The functions f_j will be linearly independent in view of the fact that C is invertible. Define

$$\begin{aligned} &F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \quad \text{by} \\ &Ff_j = e_j, j = 1, \ldots, k, \quad \text{on} \quad \text{span}\,\{f_1, \ldots, f_k\}, \\ &F \equiv 0 \quad \text{on} \quad [\text{span}\,\{f_1, \ldots, f_k\}]^{\perp}. \end{aligned}$$

F is a relative inverse to CP in $\mathcal{L}(L^2(\mathbb{R}^n))$. By [G3], Bemerkung 5.7, the fact that $\text{Op}\ S^0_{\rho,\delta}$ is a Ψ^*-algebra in $\mathcal{L}(L^2(\mathbb{R}^n))$ ([B1], Thm. 3.2, cf. [U], Satz 4.3) implies that there exists a relative inverse already in $\text{Op}\ S^0_{\rho,\delta}$.

Step 4. Denote by F the relative inverse of type PF of Step 3 and let $D = (BC)^{r-1}B$. Then $(D + F)C$ is invertible in $\mathbb{C}I + \text{Op}\ \dot{S}^{-\infty}$.

Proof. First observe that $DC = (BC)^r = I + S'$ is Fredholm of index zero, and so is $(D + F)C$, since F is compact. So it is sufficient to prove injectivity. Let $h \in L^2$, $h = h_n + h_r$ with h_n in the nullspace, h_r in the range of P, such that $(D + F)Ch = 0$.

Then $DCh + FCh = 0$. Now $DCh = (BC)^r h \in \mathcal{R}((BC)^r)$, $FCh = PFCh \in \mathcal{R}(P) = \mathcal{N}((BC)^r)$, hence $DCh = FCh = 0$. Since $DCh = DCh_n + DCh_r$ and $DCh_r = (BC)^r h_r = 0$, we have $DCh_n = (BC)^r h_n = 0$. But $h_n \in \mathcal{R}((BC)^r)$, and BC is bijective on $\mathcal{R}((BC)^r)$, so $h_n = 0$. Therefore, $0 = FCh = PFCh_r = PFCPh_r$ (since $h_r \in \mathcal{R}(P)$) $= C^{-1}(CPFCP)h_r = C^{-1}CPh_r = h_r$, and $h = h_n + h_r = 0$. Finally, $(D+F)C \in \mathbb{C}I + \mathrm{Op}\, \dot{S}^{-\infty}$, for $DC = (BC)^r = I + S'$, $S' \in \mathrm{Op}\, \dot{S}^{\infty}$, and $FC = PFC \in \mathrm{Op}\, \dot{S}^{-\infty}$.

Conclusion. The inverse of $(D+F)C$ is in $\mathbb{C}I + \mathrm{Op}\, \dot{S}^{-\infty}$ by Lemma 2.3. The operator $[(D+F)C]^{-1}(D+F)$ is *the* inverse to $C = I + A$. It belongs to $\mathrm{Op}\, \tilde{S}^0_{\rho,\delta}$. □

Corollary 2.6. $\mathrm{Op}\, \tilde{S}^0_{\rho,\delta}$, *$\rho > \delta$, is a Ψ^*-algebra in $\mathcal{L}(L^2(\mathbb{R}^n))$.*

Proof. Proposition 2.5 says that in $\mathcal{L}(L^2(\mathbb{R}^n))$,

$$(1) \qquad \{(I+A)^{-1} : \|A\| < \tfrac{1}{4}, \quad A \in Op\, \tilde{S}^0_{\rho,\delta}\} \subseteq Op\, \tilde{S}^0_{\rho,\delta}.$$

Denote by $\mathcal{B}$ the C^*-closure of $\mathrm{Op}\, \tilde{S}^0_{\rho,\delta}$. Then $\mathrm{Op}\, \tilde{S}^0_{\rho,\delta}$ is dense in $\mathcal{B}$. Suppose $A \in \mathrm{Op}\, \tilde{S}^0_{\rho,\delta} \cap \mathcal{L}(L^2(\mathbb{R}^n))^{-1}$. Let $B = A^{-1}$. Since $\mathcal{B}$ is a C^*-subalgebra of $\mathcal{L}(L^2(\mathbb{R}^n))$, $B \in \mathcal{B}$. Choose in $\mathrm{Op}\, \tilde{S}^0_{\rho,\delta}$ a sequence $B_j \to B$, converging in $\mathcal{L}(L^2)$-norm. Then $B_j A = I + C_j$ with $C_j \to 0$ in $\mathcal{B}$, $C_j \in \mathrm{Op}\, \tilde{S}^0_{\rho,\delta}$. Hence (1) implies that $(I + C_j)^{-1} \in \mathrm{Op}\, \tilde{S}^0_{\rho,\delta}$ for sufficiently large j, and $(I + C_j)^{-1} B_j$ is a left inverse for A in $\mathrm{Op}\, \tilde{S}^0_{\rho,\delta}$. □

The next topic is the equivalence of uniform ellipticity and the Fredholm property. As a preparation we need the following lemma.

Lemma 2.7. *Suppose that $P \in Op\, S^0_{\rho,\delta}$ has finite dimensional range. Then there are functions f_j, $g_j \in \mathcal{S}$, $j = 1, \dots, J$, with $Pf = \sum_{j=1}^{J} (f, f_j) g_j$. Here $(\cdot,\cdot)$ is the scalar product in L^2.*
In particular, P is an integral operator with a kernel in $\mathcal{S} \otimes_{\mathrm{alg}} \mathcal{S}$, and $P \in \mathrm{Op}\, \dot{S}^{-\infty}$.

Proof. Choose an orthonormal basis $\{g_1, \dots, g_J\}$ of the range of P. Then $Pf = \sum c_j(f) g_j$ with continuous linear $c_j : L^2 \to \mathbb{C}$. By Riesz's theorem, there are $f_j \in L^2 - \{0\}$ with $Pf = \sum (f, f_j) g_j$. The g_j are functions in $\mathcal{S}$: $\mathcal{S}$ is dense in L^2, and P is continuous. So, $P\mathcal{S}$ is dense in the range of P. Since the latter is finite dimensional, both are equal.
So we know that P is an integral operator with an L^2-kernel $k(x,y) = \sum \overline{f_j(y)} g_j(x)$. Then P^* is the integral operator with the kernel $k'(x,y) = \overline{k(y,x)}$. Since $P^* \in \mathrm{Op}\, S^0_{\rho,\delta}$, one concludes as before that the f_j are in $\mathcal{S}$. □

We also need the following result [K2], Ch. 3, Lemma 5.13.

Lemma 2.8. *Let $p \in \tilde{S}^m_{\rho,\delta}$, $q \in \tilde{S}^{m'}_{\rho,\delta}$, $0 \le \delta \le \rho \le 1$, $\delta < 1$. Then*

$$\mathrm{Op}\, p \circ \mathrm{Op}\, q - \mathrm{Op}\, (pq) = \mathrm{Op}\, r,$$

where $r \in \dot{S}^{m+m'-\rho+\delta}_{\rho,\delta}$. For $\rho > \delta$ and scalar-valued symbols, $\mathrm{Op}\, \tilde{S}^0_{\rho,\delta}$ *thus is an algebra with compact commutators.*

Theorem 2.9. *Let $P \in \mathrm{Op}\ \tilde{S}^0_{\rho,\delta}$, $\rho > \delta$. Then $P : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is Fredholm if and only if P is uniformly elliptic.*

Proof. It follows from Theorem 1.5 that uniform ellipticity implies the Fredholm property. Thus assume that P is Fredholm. Since $\mathrm{Op}\ \tilde{S}^0_{\rho,\delta}$ is a Ψ^*-algebra, there exists a Fredholm inverse B in $\mathrm{Op}\ \tilde{S}^0_{\rho,\delta}$ such that

$$PB - I \in \mathcal{F} \cap \mathrm{Op}\ \tilde{S}^0_{\rho,\delta},$$

where $\mathcal{F}$ denotes the ideal of finite dimensional operators. This is a consequence of the fact that P has closed range and thus the projection onto the finite dimensional kernel can be given as a resolvent integral, cf. [C2], §7, which is in $\mathrm{Op}\ \tilde{S}^0_{\rho,\delta}$, since this is a Ψ^*-algebra, cf. [G3], Bem. 5.7. Lemma 2.7 shows that therefore $PB - I \in \mathrm{Op}\ \dot{S}^{-\infty}$.

Denote by p, b the symbols of P, B, respectively. Then $pb - 1 \in \dot{S}^{-\rho+\delta}_{\rho,\delta}$ by Lemma 2.8. It tends to zero as $|x|+|\xi| \to \infty$. For all sufficiently large $|x|+|\xi|$, $|p(x,\xi)b(x,\xi)-1| < 1/2$, hence $|p(x,\xi)^{-1}| \le 2\|b\|_{\sup}$. □

Theorem 2.10. *Let $P \in \mathrm{Op}\ \tilde{S}^m_{\rho,\delta}$, $0 \le \delta \le \rho \le 1$, $\delta < 1$, and suppose that γ is an admissible weight function. Then the commutator $[P, \gamma]$ is in $\mathrm{Op}\ \dot{S}^{m-\rho}_{\rho,\delta}$.*

The proof is lengthy but straightforward, cf. the proof of Lemma 2.4 in [S1].

Corollary 2.11. *Suppose γ is an admissible weight function, and $P \in \mathrm{Op}\ \tilde{S}^0_{\rho,\delta}$, $\rho > \delta$. Then $\gamma^{-t}P\gamma^t \in \mathrm{Op}\ \tilde{S}^0_{\rho,\delta}$ for all t, and the algebra $\mathrm{Op}\ \tilde{S}^0_{\rho,\delta}$ is adjoint-invariant in $\mathcal{L}(H^{st}_\gamma)$.*

Proof. Suppose first that $t \ge 0$. Choose a positive integer k such that $t/k < 1$. Then the weight function $\tilde{\gamma} = \gamma^{t/k}$ is also admissible. From Theorem 2.10 and the identity

$$(1) \qquad \gamma^{-t}P\gamma^t = \tilde{\gamma}^{-k}P\tilde{\gamma}^k = \sum_{j=0}^{k} \binom{k}{j}(-\tilde{\gamma})^{-j}\mathrm{ad}^j\tilde{\gamma}(P),$$

one obtains the statement. In case $t < 0$, choose also k such that $-t/k < 1$. Let $\tilde{\gamma} = \gamma^{-t/k}$. Then use also Theorem 2.10 plus the identity

$$(2) \qquad \gamma^{-t}P\gamma^t = \tilde{\gamma}^kP\tilde{\gamma}^{-k} = \sum_{j=0}^{k} \binom{k}{j}\ \mathrm{ad}^j\tilde{\gamma}(P)(\tilde{\gamma})^{-j}. \qquad \square$$

In order to show the adjoint invariance in $\mathcal{L}(H^{st}_\gamma)$ one has to check that for a given $P \in \mathrm{Op}\ \tilde{S}^0_{\rho,\delta}$, $\gamma^{-t}\langle D\rangle^{-2s}\gamma^{-t}P^*\gamma^t\langle D\rangle^{2s}\gamma^t \in \mathrm{Op}\ \tilde{S}^0_{\rho,\delta}$. By [K2], Ch. 3, Lemma 5.13, $P^* \in \mathrm{Op}\ \tilde{S}^0_{\rho,\delta}$. From what was just proven, conjugation with γ^t leaves the class invariant, and so does conjugation with $\langle D\rangle^{2s}$, since $\langle\xi\rangle^{2s} \in \tilde{S}^{2s}_{1,0}$, in view of [K2], Ch. 3, Lemma 5.13.

Corollary 2.12. *Suppose that γ is an admissible weight function and that $P \in \mathrm{Op}\ \tilde{S}^0_{\rho,\delta}$, $\rho > \delta$. For all $s, t \in \mathbb{R}$, the operator $\gamma^{-t}P\gamma^t - P$ is compact on H^{st}_γ.*

Proof. Since $\gamma^t : H^{st}_\gamma \to H^{s0}_\gamma = H^s$ is a topological isomorphism, it is sufficient to show that $P - \gamma^tP\gamma^{-t} = \gamma^t[\gamma^{-t}P\gamma^t - P]\gamma^{-t}$ is a compact operator on H^s. Equations

(1) and (2) in the proof of 2.11 together with Theorem 2.10 show that $P - \gamma^t P \gamma^{-t}$ is an operator with a symbol in $\dot{S}^{-\rho}_{\rho,\delta}$. By Lemma 1.6 it is compact on H^s. □

Theorem 2.13. *Suppose γ is an admissible weight function. Then* Op $\tilde{S}^0_{\rho,\delta}$, $\rho > \delta$, *is a Ψ^*-algebra in $\mathcal{L}(H^{st}_\gamma)$ for all $s,t \in \mathbb{R}$, and an operator $P \in$* Op $\tilde{S}^0_{\rho,\delta}$ *is a Fredholm operator on H^{st}_γ if and only if its symbol is uniformly elliptic.*

Whenever this is the case, the index is independent of the choice of s, t, and γ.

Proof. Let us first show the Ψ^*-property: By [S1], Thm. 1.7, the operators with symbols in $S^0_{\rho,\delta}$ are continuous on H^{st}_γ, and the topology of $S^0_{\rho,\delta}$ is stronger than that of $\mathcal{L}(H^{st}_\gamma)$. So Op $\tilde{S}^0_{\rho,\delta}$ is a Fréchet-algebra in $\mathcal{L}(H^{st}_\gamma)$. By 2.11 it is adjoint invariant. It remains to show spectral invariance: Suppose $P \in$ Op $\tilde{S}^0_{\rho,\delta}$ is invertible on H^{st}_γ. By [S1], Cor. 1.9, P is also invertible on L^2. Theorem 1.4 then shows that its L^2-inverse is in Op $\tilde{S}^0_{\rho,\delta}$. This operator also inverts P on H^{st}_γ.

Now suppose $P \in$ Op $\tilde{S}^0_{\rho,\delta}$ is Fredholm on H^{st}_γ. It follows from the Ψ^*-property that there is a Fredholm inverse in Op $\tilde{S}^0_{\rho,\delta}$ modulo finite rank operators in Op $\tilde{S}^0_{\rho,\delta}$. As in the proof of Theorem 2.9, this implies the uniform ellipticity of the symbol.

On the other hand, if the symbol of P is uniformly elliptic, then Theorem 1.5 shows that there exists a parametrix to P modulo Op $\dot{S}^{-\infty}$ in Op $\tilde{S}^0_{\rho,\delta}$. By Lemma 1.6 P is a Fredholm operator on H^s for every $s \in \mathbb{R}$. By definition, $\gamma^t : H^{st}_\gamma \to H^{s0}_\gamma = H^s$ is an isomorphism. Thus $\gamma^{-t} P \gamma^t$ is a Fredholm operator on H^{st}_γ. Corollary 2.12 shows that $\gamma^{-t} P \gamma^t - P$ is compact, and so P is Fredholm on H^{st}_γ.
That the index is independent of the choice of s, t, and γ is due to the fact that, by a similar argument as in the proof of 2.12, the operator $\langle D \rangle^{-s} \gamma^{-t} P \gamma^t \langle D \rangle^s - P$ is compact on L^2. □

3. Proof of the Index Formula

Since the index is independent of the choice of the space by Theorem 1.11, it is sufficient to show the formula on $L^2(\mathbb{R}^n)$. The following proof is based on [H2], Thms. 19.3.1, 19.3.1′, and a deformation argument. We will use the notation of Theorem 1.13: a is the symbol of the Fredholm operator A, $a \in \tilde{S}^0_{\rho,\delta}$. For $m = (m_1, m_2)$ let us introduce the symbol classes

$$SG^m = \{a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) : D^\alpha_\xi D^\beta_x a(x,\xi) = O(\langle \xi \rangle^{m_1 - |\alpha|} \langle x \rangle^{m_2 - |\beta|})\}.$$

For the metric $G = |dx|^2 \langle x \rangle^{-2} + |d\xi|^2 \langle \xi \rangle^{-2}$ in [H2], (19.3.11),

$$SG^m = S(\langle x \rangle^{m_2} \langle \xi \rangle^{m_1}, G).$$

Definition 3.1. For $0 \le t \le 1$ let $a_t(x,\xi) = a(x/\langle x \rangle^t, \xi/\langle \xi \rangle^t)$.

Lemma 3.2. *For arbitrary multi-indices α, β, the function $D^\alpha_\xi D^\beta_x a_t(x,\xi)$ can be written as a linear combination of terms of the form*

$$(D^\mu_\xi D^\nu_x a)(x/\langle x \rangle^t, \xi/\langle \xi \rangle^t) \langle \xi \rangle^{-t|\mu|} \langle x \rangle^{-t|\nu|} m(x,\xi),$$

where $\mu \le \alpha$, $\nu \le \beta$ and

- *the coefficients of the linear combination are polynomials in t of degree $\le |\alpha| + |\beta|$,*
- *$m = m_{\alpha,\beta,\mu,\nu} \in SG^{-(|\alpha-\mu|, |\beta-\nu|)}$ is independent of t.*

Proof. By induction on $|\alpha| + |\beta|$. □

Definition 3.3. A family $\{b_t : t \in J \subset \mathbb{R}\}$ of symbols in $\tilde{S}^m_{\rho,\delta}$ is *uniformly bounded* in $\tilde{S}^m_{\rho,\delta}$, if

$$(1) \qquad |D^\alpha_\xi D^\beta_x b_t(x,\xi)| \le C_{\alpha\beta}(x)\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|},$$

for bounded functions $C_{\alpha\beta}$, independent of t, with $C_{\alpha\beta} \to 0$, $|x| \to \infty$, if $\beta \ne 0$. Call $\{b_t\} \subset \dot{S}^m_{\rho,\delta}$ uniformly bounded in $\dot{S}^m_{\rho,\delta}$ if (1) holds with all $C_{\alpha\beta}$ tending to zero.

Corollary 3.4. *The family $\{a_t : 0 \le t \le 1\}$ is uniformly bounded in $\tilde{S}^0_{\rho,\delta}$.*

Proof. For $0 \le t \le 1$, the coefficients in the linear combination in Lemma 3.2 can be estimated by a uniform constant. Therefore,

$$|D^\alpha_\xi D^\beta_x a_t(x,\xi)| \le C \max_{\mu\le\alpha,\nu\le\beta}\{C_{\mu\nu}(x/\langle x\rangle^t)\langle\xi/\langle\xi\rangle^t\rangle^{-\rho|\mu|+\delta|\nu|}\langle\xi\rangle^{-t|\mu|-|\alpha-\mu|}\langle x\rangle^{-t|\nu|-|\beta-\nu|}\}.$$

Now a simple observation: For $1 \le c \le \langle v\rangle, c \in \mathbb{R}, v \in \mathbb{R}^n : \langle v/c\rangle^2 = c^{-2}(c^2+|v|^2) \le 2c^{-2}\langle v\rangle^2$ and $\langle v/c\rangle^2 \ge c^{-2}\langle v\rangle^2$. For $0 \le t \le 1$, $1 \le \langle\xi\rangle^t \le \langle\xi\rangle$, thus the estimate can be continued by

$$(1) \qquad \le C \max_{\mu\le\alpha,\nu\le\beta}\{C_{\mu\nu}(x/\langle x\rangle^t)\langle x\rangle^{-t|\nu|-|\beta-\nu|} \times\langle\xi\rangle^{-\rho|\mu|+\delta|\nu|+t\rho|\mu|-t\delta|\nu|-t|\mu|-|\alpha-\mu|}\}$$

$$(2) \qquad \le C \max_{\mu\le\alpha,\nu\le\beta}\{C_{\mu\nu}(x/\langle x\rangle^t)\langle x\rangle^{-t|\nu|-|\beta-\nu|}\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|}\}.$$

We may suppose without loss of generality that the functions $C_{\alpha\beta}$ are non-increasing functions of $|x|$, tending to zero as $|x| \to \infty$ whenever $\beta \ne 0$. In this case,

$$C_{\mu\nu}(x/\langle x\rangle^t)\langle x\rangle^{-t|\nu|-|\beta-\nu|} \le \|C_{\mu\nu}\|_{\sup} \quad \text{for} \quad \beta = 0.$$

For $\beta \ne 0$ and $\nu = 0$

$$C_{\mu\nu}(x/\langle x\rangle^t)\langle x\rangle^{-t|\nu|-|\beta-\nu|} \le \|C_{\mu\nu}\|_{\sup}\langle x\rangle^{-1},$$

whereas for β, $\nu \ne 0$

$$C_{\mu\nu}(x/\langle x\rangle^t)\langle x\rangle^{-t|\nu|-|\beta-\nu|} \le C_{\mu\nu}(x/\langle x\rangle^t)\langle x\rangle^{-t} \le \begin{cases} \|C_{\mu\nu}\|_{\sup}\langle x\rangle^{-1/2} & t \ge 1/2 \\ C_{\mu\nu}(x/\langle x\rangle^{1/2}) & t < 1/2 \end{cases}.$$

So the $\tilde{S}^0_{\rho,\delta}$-seminorms can be estimated independently of t in view of the estimate (2). □

Corollary 3.5. *Breaking off the estimate in the proof of Corollary* 3.4 *at the inequality* (1), *we obtain that* $a_1 \in SG^0$:

$$|D^\alpha_\xi D^\beta_x a_1(x,\xi)| \le C \max_{\mu\le\alpha,\nu\le\beta}\{\|C_{\mu\nu}\|_{\sup}\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-|\beta|}\}.$$

Observation 3.6. Suppose J is a compact interval in $\mathbb{R}_+$ and $a \in \tilde{S}^0_{\rho,\delta}$. Let $d_t(x,\xi) = a(tx, t\xi)$. Then $\{d_t : t \in J\}$ is uniformly bounded in $\tilde{S}^0_{\rho,\delta}$.

Lemma 3.7. *Suppose J is a compact interval in $\mathbb{R}$ and $\{d_t : t \in J\}$ is a uniformly bounded family in $\tilde{S}^0_{\rho,\delta}$ such that $d_{t+h}(x,\xi) - d_t(x,\xi) \to 0$, as $h \to 0$, for all fixed (x,ξ).*

Then the mapping $t \mapsto D_t := \mathrm{Op}\, d_t$ is strongly continuous from J to $\mathcal{L}(L^2(\mathbb{R}^n))$.

Proof. First choose $f \in \mathcal{S}$, γ an arbitrary multi-index. Then

$$(1) \qquad x^\gamma D_t f(x) = (2\pi)^{-n/2} \int e^{ix\xi} \mathrm{D}^\gamma_\xi (d_t(x,\xi)\hat{f}(\xi)) \mathrm{d}\xi = O(1),$$

uniformly in t by Leibniz' rule. Moreover,

$$(D_{t+h} - D_t) f(x) = (2\pi)^{-n/2} \int e^{ix\xi} (d_{t+h} - d_t)(x,\xi) \hat{f}(\xi) \mathrm{d}\xi.$$

Together with the assumptions, Lebesgue's theorem on dominated convergence shows that for each fixed x, $(D_{t+h} - D_t)f(x) \to 0$ as $h \to 0$. Now (1) implies that

$$|(D_{t+h} - D_t)f(x)| \le C_N \langle x \rangle^{-N},$$

with a constant depending on $N > n/2$, but independent of t, h and x. Again we can apply Lebesgue's theorem, and we get

$$\|(D_{t+h} - D_t)f\|^2_{L^2} = \int |(D_{t+h} - D_t)f(x)|^2 \mathrm{d}x \to 0, \text{ as } h \to 0.$$

Now let $f \in L^2$ and $\{f_\nu\} \subset \mathcal{S}$ with $f_\nu \to f$ in L^2. Then

$$\|(D_{t+h} - D_t)f\|_{L^2} \le \|(D_{t+h} - D_t)f_\nu\|_{L^2} + \|D_{t+h} - D_t\|_{\mathcal{L}(L^2)} \|f - f_\nu\|_{L^2}.$$

The boundedness of $\{d_t\}$ in $S^0_{\rho,\delta}$ implies a uniform bound on the operator norms, showing that the right-hand side tends to zero as $h \to 0$, $\nu \to \infty$. □

Proposition 3.8. *Let J be an interval and $\{a_t, b_t : t \in J\}$ a uniformly bounded subset of $\tilde{S}^0_{\rho,\delta}$. Assume that there is an $R > 0$ such that $a_t(x,\xi) b_t(x,\xi) = 1$ for all $|x| + |\xi| \ge R$. Then*

$$\mathrm{Op}\, a_t\, \mathrm{Op}\, b_t - I \in \mathrm{Op}\, \dot{S}^{\delta-\rho}_{\rho,\delta},$$

uniformly in t, i.e., if $\mathrm{Op}\, a_t\, \mathrm{Op}\, b_t - I = \mathrm{Op}\, c_t$, then $\{c_t : t \in J\}$ is uniformly bounded in $\dot{S}^{\delta-\rho}_{\rho,\delta}$.

Proof. Let $M_{\alpha\beta} = \max\limits_{x,\xi,t} \{|\mathrm{D}^\alpha_\xi \mathrm{D}^\beta_x (a_t(x,\xi) b_t(x,\xi) - 1)|\}$. In view of the fact that $a_t b_t = 1$ on $\{|x| + |\xi| \ge R\}$ one has for arbitrary $N \in \mathbb{N}$

$$\begin{aligned} |\mathrm{D}^\alpha_\xi \mathrm{D}^\beta_x \{a_t(x,\xi) b_t(x,\xi) - 1\}| &\le M_{\alpha\beta} 1_{\{|x| \le R\}} 1_{\{|\xi| \le R\}} \\ &\le M_{\alpha\beta} 1_{\{|x| \le R\}} \langle R \rangle^N \langle \xi \rangle^{-N}. \end{aligned}$$

Hence $a_t b_t - 1 \in \dot{S}^{-N}_{1,0}$, uniformly in t. So it remains to show that

$$(1) \qquad \mathrm{Op}\, a_t\, \mathrm{Op}\, b_t - \mathrm{Op}\,(a_t b_t) \in \dot{S}^{\delta-\rho}_{\rho,\delta}, \quad \text{uniformly in } t.$$

It is well known that for $a, b \in \tilde{S}^0_{\rho,\delta}$, Op a Op b − Op $ab \in \dot{S}^{\delta-\rho}_{\rho,\delta}$, cf. [K2], Ch. 3, Lemma 5.13, Ch. 2, Lemma 2.4. An analysis of the proof shows that the boundedness of $\{a_t, b_t\}$ implies in fact the uniform boundedness in (1). □

For our considerations we will need the following lemma.

Lemma 3.9. *Suppose C_j, $j = 1, 2, \ldots$ is a sequence of bounded C^∞-functions on $\overline{\mathbb{R}}_+$ with $C_j(r) \to 0$, $r \to \infty$, monotonely in r, for every fixed j. Then there is a C^∞-function C on $\overline{\mathbb{R}}_+$ with*
(i) $0 < C(r) \to 0$, $r \to \infty$,
(ii) $C_j(r)/C(r) = O(1)$ *for every* j,
(iii) $\partial_r^\beta C^{-1}(r) = O(1)$ *for every* $\beta \neq 0$.

Proof. Let $R_0 = 0$. For $k = 1, 2, \ldots$ choose $R_{k+1} > R_k + (k+1)^2$ such that $C_j(r) < \frac{1}{k}$ for all $j \leq k$, $r \geq R_k$. Now choose a monotonely decreasing function $h \in C^\infty(\mathbb{R})$ with $0 \leq h \leq 1$ and $h(r) = 1$ for $r < 1/3$, $h(r) = 0$ for $r > 2/3$. For $k \in \mathbb{N}$ let

$$f_k(r) = \left(\frac{1}{k} - \frac{1}{k+1}\right) h((r - R_k)/(R_{k+1} - R_k))$$

and

$$C(r) = \sum_{k=1}^{\infty} f_k(r).$$

By construction, (i) and (ii) are satisfied, and it remains to check (iii). The functions f_k are constant outside the interval $\{r : \frac{1}{3} \leq (r - R_k)/(R_{k+1} - R_k) \leq \frac{2}{3}\} \subset\subset]R_k, R_{k+1}[$. In particular, the supports of the derivatives of the functions f_k are all disjoint. We can therefore basically concentrate on one of the f_k.

First make the following observation. For a positive C^∞-function f, the derivative $\partial^\beta f^{-1}$, $\beta \neq 0$, is a linear combination (with universal coefficients) of terms of the form

$$f^{(\nu_1)} \cdot \ldots \cdot f^{(\nu_s)} f^{-s-1}$$

with $0 < \nu_1, \ldots, \nu_s$, $s \leq \beta$, and $\nu_1 + \ldots + \nu_s = \beta$.

On $]R_k, R_{k+1}[$, $\partial^\beta C^{-1}$ thus is a linear combination of terms of the form

$$f_k^{(\nu_1)} \cdot \ldots \cdot f_k^{(\nu_s)} C^{-s-1}. \tag{1}$$

On the other hand, the value of $C(r)$ there is $\geq \frac{1}{k+1}$, whence $C^{-s-1} \leq (k+1)^{s+1}$. Moreover, $f_k^{(m)}(r) = (\frac{1}{k} - \frac{1}{k+1})(R_{k+1} - R_k)^{-m} \cdot h^{(m)}((r - R_k)/(R_{k+1} - R_k))$. Letting $M_\beta = \max\{|h^{(\gamma)}(t)| : 0 \leq t \leq 1, 0 < \gamma \leq \beta\}$, one concludes that the terms (1) can be estimated by

$$(R_{k+1} - R_k)^{-\beta} M_\beta^\beta (k+1)^{s+1} \leq M_\beta^\beta,$$

independent of k, in view of the facts that $R_{k+1} - R_k \geq (k+1)^2$ and $\beta \geq 1$. Since the coefficients in the linear combination are also independent of k, this concludes the proof. □

Proposition 3.10. *Suppose J is a compact interval and $\{d_t : t \in J\}$ is a uniformly bounded family in $\tilde{S}^0_{\rho,\delta}$ such that*
(i) $d_{t+h}(x,\xi) - d_t(x,\xi) \to 0$ *as* $h \to 0$ *for all fixed* (x,ξ), *and*
(ii) *there is an open ball* $B \subseteq \mathbb{R}^{2n}$ *such that* $d_t(x,\xi)^{-1}$ *exists and* $|d_t(x,\xi)^{-1}| \leq C$ *for all* $t \in J$ *and* (x,ξ) *outside* B.
Then all the operators $D_t = \text{Op } d_t$ *are Fredholm, and* $\operatorname{index} D_t \equiv \text{const.}$ *on* J.

Proof. Choose a function $\psi \in SG^0$ with $\psi(x,\xi) \equiv 1$ for large $|(x,\xi)|$, and $\psi(x,\xi) \equiv 0$ for $(x,\xi) \in B$. Let $e_t(x,\xi) = \psi(x,\xi)d_t^{-1}(x,\xi)$ for $(x,\xi) \notin B$, $\equiv 0$ otherwise. The uniform boundedness of $\{d_t\}$ together with the quotient rule for differentiation implies that $\{e_t : t \in J\}$ is bounded in $\tilde{S}^0_{\rho,\delta}$. Property (i) shows that also

$$e_{t+h}(x,\xi) - e_t(x,\xi) \to 0 \quad \text{as} \quad h \to 0$$

for fixed (x,ξ). By Lemma 3.7, both $\{D_t : t \in J\}$ and $\{E_t = \text{Op } e_t : t \in J\}$ are strongly continuous maps from J to $\mathcal{L}(L^2)$. By 3.8, $D_tE_t - I \in \text{Op } \dot{S}^{\delta-\rho}_{\rho,\delta}$ uniformly in t, i.e., if c_t is the symbol of $D_tE_t - I$, then there are bounded functions $C_{\alpha\beta}(x)$, tending to zeros as $|x| \to \infty$ such that

$$|D_\xi^\alpha D_x^\beta c_t(x,\xi)| \leq C_{\alpha\beta}(x)\langle\xi\rangle^{-\rho(|\alpha|+1)+\delta(|\beta|+1)}.$$

Without loss of generality we may assume that all the $C_{\alpha\beta}$ are C^∞-functions of $|x|$ only.

An application of Lemma 3.9 shows that there is a C^∞-function $C(x) = C(|x|)$ with
(i) $0 < C(x) \to 0$, $|x| \to \infty$,
(ii) $C_{\alpha\beta}(x)/C(x) = O(1)$ for all α, β fixed, and
(iii) $\partial^\beta C^{-1}(x) = O(1)$ for all $\beta \neq 0$ fixed.

Now write

$$\text{Op } c_t = \text{Op } (C(x)\langle\xi\rangle^{\delta-\rho})\{\text{Op}\langle\xi\rangle^{\rho-\delta}\text{Op } (C^{-1}(x)c_t(x,\xi))\}.$$

In view of (i), $\text{Op } C(x)\langle\xi\rangle^{\delta-\rho}$ is compact. On the other hand, $D_\xi^\alpha D_x^\beta(C^{-1}(x)c_t(x,\xi))$ is a linear combination of terms of the form $D_x^{\beta_1}C^{-1}(x)D_x^{\beta_2}D_\xi^\alpha c_t(x,\xi)$. It can be estimated by $O(1)C_{\alpha\beta_2}(x)\langle\xi\rangle^{-\rho(|\alpha|+1)+\delta(|\beta_2|+1)}$ for $\beta_1 \neq 0$, and by $C^{-1}(x)\, C_{\alpha\beta_2}(x)$ $\langle\xi\rangle^{-\rho(|\alpha|+1)+\delta(|\beta_2|+1)}$ for $\beta_1 = 0$. Therefore $C^{-1}(x)c_t(x,\xi) \in S^{\delta-\rho}_{\rho,\delta}$, uniformly in t, and the operator norms of $T_t = \text{Op}\langle\xi\rangle^{\rho-\delta}\text{Op } C^{-1}(x)c_t(x,\xi)$ are uniformly bounded. Letting $K = \text{Op } C(x)\langle\xi\rangle^{\delta-\rho}$, one concludes that

$$\begin{aligned} M_1 &= \{\text{Op } c_t f : t \in J,\ \|f\| \leq 1\} = \{KT_t f : t \in J,\ \|f\| \leq 1\} \\ &\subseteq \{Kf : \|f\| \leq \sup_{t\in J} \|T_t\|\} \end{aligned}$$

is precompact in $L^2(\mathbb{R}^n)$.

The same argument holds for $E_tD_t - I$ and the corresponding set M_2. We know already that D_t and E_t are strongly continuous mappings from J to $\mathcal{L}(L^2)$. Hence $D_tE_t - I$ and $E_tD_t - I$ are uniformly compact in the sense of [H2], Thm. 19.1.10, D_t and E_t are Fredholm operators for all t, and $\operatorname{index}(D_t) = -\operatorname{index}(E_t)$ is constant on J. This concludes the proof. □

Corollary 3.11. *Suppose a is a uniformly elliptic symbol in $\tilde{S}^0_{\rho,\delta}$, i.e., for all $|(x,\xi)| \geq R$, $a^{-1}(x,\xi)$ exists and $|a^{-1}(x,\xi)| \leq C$. Let $\tilde{a}(x,\xi) = a(2Rx, 2R\xi)$. Then $\tilde{a}(x,\xi)$ is invertible with inverse bounded by C for all (x,ξ) with $|(x,\xi)| \geq 1/2$. Moreover,* 3.6 *and Proposition* 3.10 *imply that* index(Op $\tilde{a}$) = index(Op a). *Without loss of generality we may therefore assume that the symbol a is invertible (with bounded inverse) for all $|(x,\xi)| \geq 1/2$.*

It is easy to see the lemma, below.

Lemma 3.12. *Suppose $a \in \tilde{S}^0_{\rho,\delta}$ is invertible for $|(x,\xi)| \geq 1/2$, and $a_t(x,\xi) = a(x/\langle x\rangle^t, \xi/\langle\xi\rangle^t)$, $0 \leq t \leq 1$. Then a_t is invertible for $|(x,\xi)| \geq 2$, and*

$$\sup\{|a_t(x,\xi)^{-1}| : |(x,\xi)| \geq 2\} \leq \sup\{|a(x,\xi)^{-1}| : |(x,\xi)| \geq 1/2\}.$$

Conclusion 3.13. Proof of Theorem 1.13.
As before, let $a_t(x,\xi) = a(x/\langle x\rangle^t, \xi/\langle\xi\rangle^t)$. The family $\{a_t : 0 \leq t \leq 1\}$ is uniformly bounded in $\tilde{S}^0_{\rho,\delta}$ by 3.4. For fixed (x,ξ), $a_{t+h}(x,\xi) - a_t(x,\xi) \to 0$ as $h \to 0$. By Lemma 3.12, $a_t(x,\xi)$ is invertible for $|(x,\xi)| \geq 2$ with bounded inverse. By Proposition 3.10, the indices of the operators A_t = Op a_t are all constant on $[0,1]$. Now, $A_0 = A$, and A_1 = Op a_1 is an operator with a symbol in SG^0, satisfying the assumptions of [H2], Thm. 19.3.1′. Thus, the index formula [H2], (19.3.1), holds for A_1, if we choose $B = B(0,2) = \{|(x,\xi)| \leq 2\}$.

The number $(-2\pi i)^{-n}\frac{(n-1)!}{(2n-1)!}\int_{\partial B} \mathrm{Tr}\,(a_t^{-1}da_t)^{2n-1}$ is always an integer. Clearly, it is constant in t. Therefore

$$\text{Index } A = \text{index } A_1 = \text{index } A_0 = -(-2\pi i)^{-n}\frac{(n-1)!}{(2n-1)!}\int_{\partial B} \mathrm{Tr}(a^{-1}da)^{2n-1}.$$

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$$\mathcal{B}^0 = \{a \in C_b^\infty(I\!R \times I\!R) : a^{(\alpha)}_{(\beta)}(x,\xi) \to 0 \text{ for } |x| \to \infty \text{ if } \beta \neq 0 \text{ or } |\xi| \to \infty \text{ if } \alpha \neq 0\}.$$

He showed e.g. that an operator with a symbol in $\mathcal{B}^0$ has an inverse modulo the class $\mathcal{K}$ of integral operators with kernels in $\mathcal{S}(I\!R^n \times I\!R^n)$ iff it is uniformly elliptic. The research for this paper was partially supported by the Deutsche Forschungsgemeinschaft.

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