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The Use of Renormalization for Calculating Effective Permeability

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Abstract. There is a need in the numerical simulation of reservoir performance to use average permeability values for the grid blocks. The permeability distributions to be averaged over are based on samples taken from cores and from logs using correlations between permeabilities and porosities and from other sources. It is necessary to use a suitable 'effective' value determined from this sample. The effective value is a single value for an equivalent homogeneous block. Conventionally, this effective value has been determined from a simple estimate such as the geometric mean or a detailed numerical solution of the single phase flow equation.

If the permeability fluctuations are small then perturbation theory or effective medium theory (EMT) give reliable estimates of the effective permeability. However, for systems with a more severe permeability variation or for those with a finite fraction of nonreservoir rock all the simple estimates are invalid as well as EMT and perturbation theory.

This paper describes a real-space renormalization technique which leads to better estimates than the simpler methods and is able to resolve details on a much finer scale than conventional numerical solution. Conventional simulation here refers to finite difference (or element) techniques for solving the single phase pressure equation. This requires the pressure and permeability at every grid point to be stored. Hence, these methods are limited in their resolution by the amount of data that can be stored in core. Although virtual memory techniques may be used they increase computer time. The renormalization method involves averaging over small regions of the reservoir first to form a new 'averaged permeability' distribution with a lower variance than the original. This pre-averaging may be repeated until a stable estimate is found. Examples are given to show that this is in excellent agreement with computationally more expensive numerical solution but significantly different from simple estimates such as the geometric mean.

Key words. Heterogeneity, effective permeability, scaling, simulation.

1. Introduction

In the numerical simulation of reservoir performance an 'effective' property value has to be assigned to each grid block. Often the grid blocks are on the scale of hundreds of metres, whereas rock heterogeneities can occur on many scales down to the sampling size of cores; typically tens of centimetres. The size of fluctuation in absolute permeability can be severe, ranging over many orders of magnitude. This makes it particularly difficult to assign to this property a single effective value which gives the same mean flow. Many attempts have been made to address this problem, some numerical (Warren and Price, 1963; Freeze, 1975; Smith and Freeze, 1979; Smith and Brown, 1982) and others analytical (Bakr *et al.*, 1978; Gutjahr and Gelhar, 1981; Gelhar, 1974; Mizzell *et al.*, 1982; Dagan, 1981, 1982; King, 1987). The analytical methods are based on effective medium theory (EMT) or perturbation expansions. In the next section, the results of these methods are briefly reviewed. In either case the effective permeability estimates are only accurate when the permeability fluctuations are small. This is rarely the case for the systems of interest. Also many reservoirs contain significant amounts of impermeable material (or material of very low permeability). This situation is not amenable to treatment by the simple methods and estimates like the geometric mean are invalid.

To treat these more complicated cases we have developed a method of real-space renormalization to improve the effective permeability estimates. Originally due to Kadanoff (1966), the idea is to calculate the effective permeability over local regions first. This reduces the magnitude of the fluctuations to give a new ('renormalized') probability distribution. The more simple estimates applied to this distribution are then more accurate as the permeability variance has been reduced. Alternatively the renormalization procedure can be applied repeatedly to give a fixed point value for the effective permeability. This technique has been applied in conjunction with EMT before (Sahimi *et al.*, 1984) but only in the more specialized problem of percolation theory.

We describe the renormalization method in detail for the isotropic case and use it in conjunction with some simple permeability distributions where numerical and other simple estimates are known. We also apply the method to the case where a finite amount of impermeable material is present. Finally, we consider some real reservoir data where excellent agreement is found with detailed numerical simulation at a considerable saving in computer time. In these examples the permeability distributions are anisotropic and both horizontal and vertical permeability are calculated.

2. Simple Estimates

For many years there has been a strong practical requirement in the hydrology and oil recovery industries for simple estimates of effective permeability values for heterogeneous systems. The earliest attack on this problem was by Warren and Price (1963) who used numerical simulations to show that of the arithmetic, geometric and harmonic means the geometric mean usually gives the closest estimate to the effective permeability for a random, isotropic distribution. Indeed this is still taken as the standard estimate within the petroleum industry. However, there are many distributions for which this is not a good estimate. For example, the geometric mean of a distribution with a finite fraction of zero permeability is zero, yet such a system may have a nonzero overall permeability. To overcome this difficulty, a number of attempts have been made to provide theories of effective permeability. There have been two main approaches to the problem; perturbation theory and effective medium theory. We briefly review these theories here.

First we present a simple perturbation calculation of effective permeability. This result is well established in the physics literature (Landau and Lifshitz, 1960). We assume that the permeability may be written as a mean value ($\bar{K}$) plus a perturbation (δK) with zero mean. This perturbs the mean pressure field ($\bar{P}$) by an amount δP which has zero mean. Then in Darcy's Law

$$-\mathbf{v} = (\bar{K} + \delta K)\boldsymbol{\nabla}(\bar{P} + \delta P) \tag{2.1}$$

Averaging this gives

$$-\bar{\mathbf{v}} = \bar{K}\boldsymbol{\nabla}\bar{P} + \overline{\delta K\ \boldsymbol{\nabla}\delta P}. \tag{2.2}$$

The fluid is incompressible, hence the divergence of (2.1) is zero.

$$\bar{K}\nabla^2\delta P = -\boldsymbol{\nabla}\bar{\mathbf{P}}\cdot\boldsymbol{\nabla}K. \tag{2.3}$$

Taking the gradient of this equation gives:

$$\begin{aligned}\nabla^2\boldsymbol{\nabla}\,\delta P &= -\frac{1}{\bar{K}}\boldsymbol{\nabla}(\boldsymbol{\nabla}\bar{\mathbf{P}}\cdot\boldsymbol{\nabla}\,\delta K)\\ &= -\frac{1}{\bar{K}}(\boldsymbol{\nabla}\bar{\mathbf{P}}\cdot\boldsymbol{\nabla})\,\boldsymbol{\nabla}\,\delta K.\end{aligned} \tag{2.4}$$

(since the homogeneous pressure field $\bar{P}$ satisfies Laplace's equation). We now average (2.4). For an isotropic medium the average of the operator $\partial^2/\partial x_i \partial x_j$ is $(1/D)\delta_{ij}\nabla^2$, where D is the dimension of the problem.

Hence

$$\nabla^2\boldsymbol{\nabla}\,\delta P = -\frac{\boldsymbol{\nabla}\bar{P}}{D\bar{K}}\overline{\nabla^2\delta K}. \tag{2.5}$$

Thus, if the correlation function in (2.2) is ignored, the average velocity field is

$$\bar{\mathbf{v}} = -\bar{K}\boldsymbol{\nabla}\bar{P}\left\{1 - \frac{\overline{\delta K^2}}{D\bar{K}^2}\right\}. \tag{2.6}$$

Thus, the effect permeability is given by

$$K_{\text{eff}} = \bar{K}\left\{1 - \frac{\sigma_K^2}{D\bar{K}^2}\right\}. \tag{2.7}$$

Where σ_K^2 is the permeability variance.

It can be seen from this and (2.1) that if the permeability variance is large the result is invalid. A more sophisticated approach using the methods of field theory (King, 1987, also Gutjahr *et al.*, 1978) gives the result

$$K_{\text{eff}} = K_g \exp \sigma^2\left[\frac{1}{2} - \frac{1}{D}\right]. \tag{2.8}$$

Where σ^2 is the variance in the logarithm of permeability and K_g is the geometric mean of the permeability. This approach is the same as the perturbation theory, except that it uses field theoretical methods to include higher-order terms in the perturbation expansion. This method is referred to later as ‘improved perturbation’.

Effective medium theory (Kirkpatrick, 1973) is based on the idea of replacing the inhomogeneous medium by an effective homogeneous medium of permeability K_{eff} such that if a single inclusion of a different permeability is introduced then the mean pressure fluctuation (caused by that inclusion) is zero. The distribution of pressure in the heterogeneous medium may be considered as an ‘external field’ due to the homogeneous effective medium and a fluctuating ‘local field’ whose average over any sufficiently large region will be zero.

To see how this works consider a piece of rock of permeability K embedded in the effective medium (K_{eff}). The pressure at large distances will be that due to a uniform velocity $\mathbf{v}$, which may conveniently be set along the x axis. Also for simplicity, we consider the inclusion to be spherical (radius r_0) in a D-dimensional space (Figure 1). We solve for the pressures exterior and interior to the inclusion in D-dimensional spherical coordinates. The problem has azimuthal symmetry and the far field has a cos θ dependence which must be reflected in the

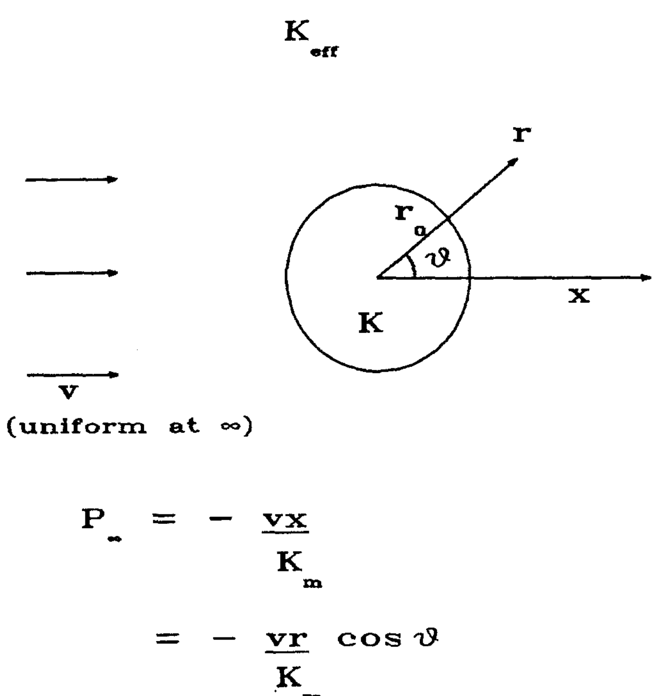


Fig. 1. Geometry for EMT calculation.

pressure solution. Hence the radial dependence of the pressure is given by

$$\left\{\frac{d^2}{dr^2}+\frac{D-1}{r}\frac{d}{dr}-\frac{D-1}{r^2}\right\}P=0. \tag{2.9}$$

The solutions of which are r and $r^{-(D-1)}$. The pressure within the inclusion must be finite so the second solution is not admissible there. The far field (P_∞) is $-vr\cos\theta/K_{\text{eff}}$. Finally, the pressure at the boundary of the inclusion must be continuous as must the normal velocity. From these conditions the pressure fields external and internal to the inclusion are:

$$P_e=\left[1+\left(\frac{r_0}{r}\right)^D\frac{K_{\text{eff}}-K}{K+(D-1)K_{\text{eff}}}\right]P_\infty, \tag{2.10}$$

$$P_i=\frac{DK_{\text{eff}}}{[K+(D-1)K_{\text{eff}}]}P_\infty. \tag{2.11}$$

Then the condition that the mean fluctuation in the pressure field due to the inclusion is zero is

$$\left\langle\frac{P_i-P_\infty}{P_\infty}\right\rangle=\left\langle\frac{K-K_{\text{eff}}}{K+(D-1)K_{\text{eff}}}\right\rangle=0. \tag{2.12}$$

Hence, the effective permeability K_{eff} is that value which satisfies the integral equation (Kirkpatrick, 1973; Koplik, 1982)

$$\left\langle\frac{1}{K+(D-1)K_{\text{eff}}}\right\rangle=\frac{1}{DK_{\text{eff}}}. \tag{2.13}$$

This method can be implemented by using an iterative procedure. If at the Nth stage the estimate of effective permeability is K_N then the next estimate, K_{N+1}, is found from

$$K_{N+1}=\frac{1}{D\langle(K+(D-1)K_N)^{-1}\rangle}, \tag{2.14}$$

Repeat this process until the difference between successive estimates is considered to be small enough. The limit of K_N as $N\to\infty$ is K_{eff}.

Both effective medium and perturbation theory break down when the permeability fluctuations become very large. The rest of this paper is devoted to a study of real-space renormalization which provides a means of estimating the effective permeability when the permeability fluctuations are large.

3. Real Space Renormalization

The assumption behind effective medium theory is that the mean fluctuations in the pressure field are negligible and average to zero. As the permeability fluctuations increase, we would expect the pressure fluctuations also to increase.

We would, therefore, expect the effective medium approximation to break down in this case. To counter this we use a position space renormalization approach. Originally, due to Kadanoff (1966), the idea of position space renormalization is first to perform a certain amount of the averaging explicitly and then to consider the properties of the pre-averaged model. It has been applied in connection with effective medium conductivity calculations before, but only for a bimodal distribution (Sahimi *et al.*, 1984).

Explicitly, the renormalization procedure is as follows. We describe the method in two dimensions, but it is readily extended to three. We imagine the permeabilities to be distributed on a grid whose scale length is representative of the original data sample size. For example, if the permeabilities are derived from samples at one-foot intervals, the original grid blocks should have dimensions of one foot. The permeability distribution is taken from the sample distribution found from the cores. We will only consider uncorrelated media here so the permeabilities are distributed at random in the examples given. This is not a restriction on the method but to treat correlated media we need statistical methods for generating large grids of correlated variables. This is a different problem and so we will restrict our discussion to uncorrelated media here. The same technique is used for correlated media once the permeability grid has been determined. The original grid blocks are grouped into blocks of four (or eight in three dimensions). The effective permeability of the four blocks is calculated and assigned to a new, coarser grid (Figure 2). In this context the effective permeability is a single value which gives the same flow across the four blocks for a given pressure drop, as would the original blocks.

In the explicit calculations for this section, we will only treat isotropic media so we need to consider flow in only one direction (either horizontal or vertical). For anisotropic media we would have to look at the transformations in the vertical and horizontal directions separately.

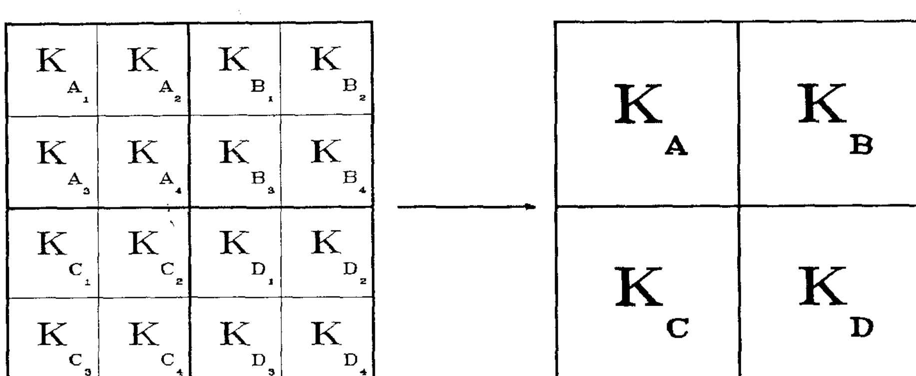


Fig. 2. Block renormalization of grid.

Blocks of more than four cells may be used, but the transformed permeability is correspondingly more complicated to calculate. Alternatively, the above transformation may be applied to the new grid and the process repeated many times until a stable result is found. This will be the effective permeability of the region. At each stage larger effective blocks will be formed whose permeability approaches that of the whole region. The variance in the permeability will be reduced, as would the correlation length in correlated media. This result shows that the grid block probability distribution to be used in a reservoir simulator will not be that of the core samples. We will examine how the renormalization rescales the permeability variance in Section 5.

Write the effective permeability of the four permeabilities as

$$\tilde{K} = f(K_1, K_2, K_3, K_4). \tag{3.1}$$

Then, if the permeability distribution on the old grid is $P(K_i)$, the probability distribution on the new grid is

$$\tilde{P}(\tilde{K}) = \int \delta(\tilde{K} - f(K_1, K_2, K_3, K_4)) \times \\ \times P(K_1)P(K_2)P(K_3)P(K_4)\, \mathrm{d}K_1\, \mathrm{d}K_2\, \mathrm{d}K_3\, \mathrm{d}K_4. \tag{3.2}$$

The integrations are performed over all possible values of the original grid block permeabilities $K_1, \ldots, K_4$ (or $K_1, \ldots, K_8$ in three dimensions).

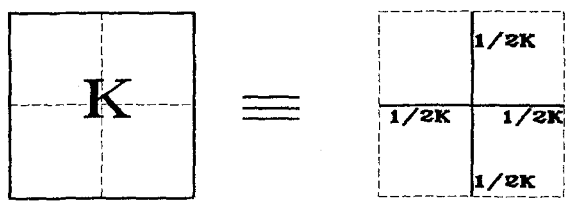
Whilst this expression is formally exact, it is not possible to proceed further analytically although we may linearize this if the permeability fluctuations are small, as shown in Section 5. Instead, we develop the probability distribution $\tilde{P}(\tilde{K})$ by Monte-Carlo sampling. First we select K_1, K_2, K_3 and K_4 from the original probability distribution. $\tilde{K}$ is calculated from (3.1). This is repeated until a satisfactory distribution $\tilde{P}(\tilde{K})$ is built up.

4. Calculation of Renormalized Permeabilities

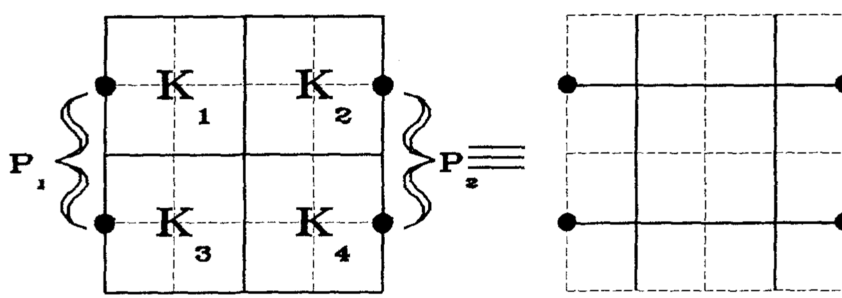
To apply the renormalization technique we first need to calculate the effective permeability of the renormalized block. We describe the method explicitly in two dimensions and give the three-dimensional result in Appendix 2.

We require an analytical result for the effective permeability of the two by two block. Unfortunately, an exact result is not available. We could, of course, calculate the effective permeability by direct numerical simulation. However, because a large number of such blocks would be required, this approach would be computationally very expensive. Instead, we adopt an approximation which is very accurate unless the permeabilities are arranged in particular configurations. These will be very rare events, however, and even in these cases the error is small.

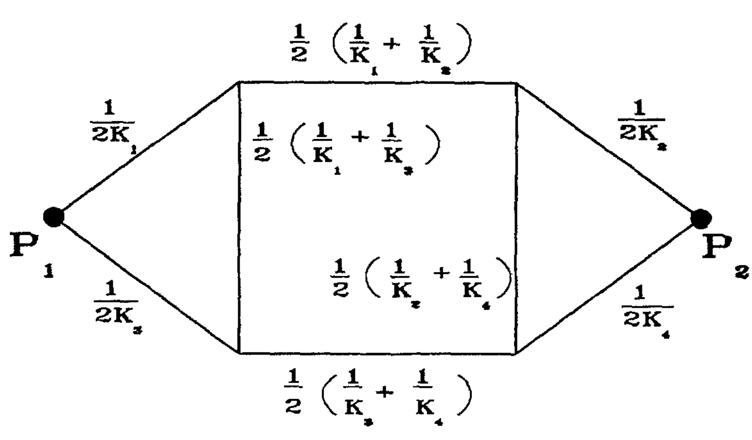
We model the block permeabilities by an equivalent resistor network. We use the boundary condition that the sides of the blocks are at uniform pressure. This is true for the external edges but not for the internal ones and is the source of the error mentioned above. Then the equivalent resistor between the midpoints of the edges is $1/K$ for a block of permeability K. This is equivalent to two resistors in series of $1/(2K)$. We can also use two resistors in the transverse direction. These will have the same resistance for an isotropic medium but would in general be different. Thus, we replace each block with a cross of resistors.



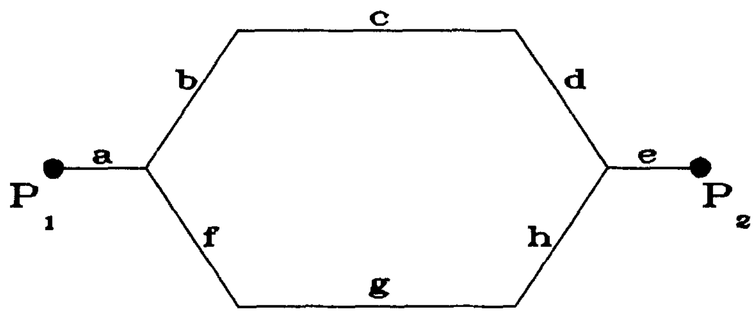
Because we are only considering isotropic media here, we calculate the effective permeability in only one direction, horizontal. For anisotropic media we must consider the transformation in orthogonal directions. We, therefore, set the end edges to a uniform pressure.



We trim off the dead end branches and join together those nodes at the same pressure to give the equivalent resistor network.



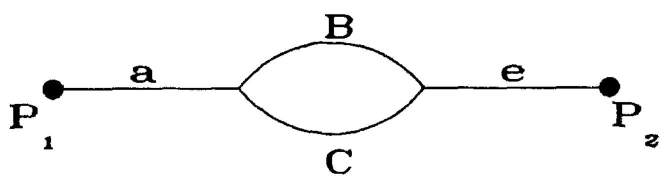
This network may be simplified by use of the star-triangle transformation (Appendix 1) to give a circuit consisting of resistors in series and parallel.



$$a=\frac{1}{4(K_1+K_3)},\quad b=\frac{1}{4K_1},\quad c=\frac{1}{2}\left(\frac{1}{K_1}+\frac{1}{K_2}\right),\quad d=\frac{1}{4K_2},$$

$$e=\frac{1}{4(K_2+K_4)},\quad f=\frac{1}{4K_3},\quad g=\frac{1}{2}\left(\frac{1}{K_3}+\frac{1}{K_4}\right),\quad h=\frac{1}{4K_4}$$

This circuit is equivalent to



$$B=\frac{3}{4}\frac{K_1+K_2}{K_1K_2},\qquad C=\frac{3}{4}\frac{K_3+K_4}{K_3K_4}.$$

This may be reduced to a single resistance whose conductance is

$$\begin{aligned} f(K_1,K_2,K_3,K_4) &= 4(K_1+K_3)(K_2+K_4)[K_2K_4(K_1+K_3)+K_1K_3(K_2+K_4)]\times \\ &\quad\times\{[K_2K_4(K_1+K_3)+K_1K_3(K_2+K_4)][K_1+K_2+K_3+K_4]+ \\ &\quad+3(K_1+K_2)(K_3+K_4)(K_1+K_3)(K_2+K_4)\}^{-1}. \end{aligned}$$

We must be careful at this point. Permeability is an intensive property, whereas conductance is extensive; thus we have to take into account the change in dimensions of the renormalized block. Consider a renormalization whereby β small blocks in each direction are combined ($\beta=2$ in the case above). Then the cross-sectional area to flow scales as β^{d-1} (in d dimensions) and the length scales as β. Hence, for fixed volumetric flow rate and pressure drop, the ratio of flow to pressure change scales as β^{2-d}. This is the conductance scaling required. Thus, the permeability is the conductance calculated above multiplied by β^{d-2}. So Equation (4.1) also gives the effective permeability in two dimensions. The three-dimensional conductance must be divided by β (=2 here) to give the effective permeability.

This result (and the equivalent three-dimensional result) was compared against direct numerical simulation for a number of test cases. The maximum error found was around 7% for an extreme case where flow through the block was highly convoluted. This is precisely the situation encountered with shales in Section 8. This could be avoided by using a direct numerical simulation of the renormalization gridblock or by using a larger unit cell. For example a 3×3 unit cell would better resolve cross flow.

This then is the effective permeability to be used to find the renormalized probability distribution in Equation (3.2). The equivalent set of transformations for the three-dimensional case is considerably more complicated and does not yield a simple closed form result (like (4.1)). However, the various steps may be implemented numerically. The transformations (just repeated application of the star-triangle transformation) are outlined in Appendix 2. It should be noted that (4.1) is only valid for isotropic media. In the presence of anisotropy the resistances in the vertical direction will be different from the horizontal ones. This does not lead to a close form formula like (4.1), but the solution may still be achieved numerically.

5. Scaling of Renormalization Equations

Although the renormalization equation (3.2) cannot be solved exactly for general distributions it is possible to study its behaviour after a large number of renormalizations have been made. This gives information about how the variance of the renormalized permeability distribution decreases at each renormalization. The idea is to consider the fixed point probability distribution after a large number of renormalizations. The result of repeated renormalization is to give a single value, that is the probability distribution reduces to a delta function. This is the limit of a Gaussian distribution for a small variance. This is independent of the initial permeability distribution all of which will eventually tend towards the Gaussian. Hence, in equation (3.2) we consider $P(K_i)$ to be a Gaussian. We take K_i to be normally distributed with a small variance and so only look at first-order perturbations to the renormalized permeability $f(\{K_i\})$ given by equation (4.1). That is we write $K_i = K_n(1 + \Delta_i)$ for the permeability of the ith block after the nth renormalization and expand (4.1) up to first order in the Δ_i. For convenience we write $\Delta = \Sigma_i \Delta_i$. The first-order expansion gives

$$f(K_1, K_2, K_3, K_4) = K_n(1 + \Delta/4) + O(\Delta^2). \tag{5.1}$$

We now write the probability distribution of the K's after the nth renormalization (n is large to justify the Gaussian although, in practice, n of order three or four gives a good approximation to the Gaussian) as

$$P_n(K_i) \sim \exp[-(K_i - K_n)^2/2\sigma_n^2] \sim \exp[-K_n^2\Delta_i^2/2\sigma_n^2]. \tag{5.2}$$

We can now integrate Equation (3.2) exactly by completing squares within the exponentials and using the property that integrals of Gaussians yield Gaussians. Omitting the tedious algebra gives the probability distribution at the $n+1$th renormalization to be

$$P_{n+1}(\tilde{K}) \sim \exp[-2(\tilde{K} - K_n)^2/\sigma_n^2]. \tag{5.3}$$

Now for this to be a fixed point distribution P_{n+1} must be of the same form as P_n. That is

$$P_{n+1} \sim \exp[-(\tilde{K} - K_{n+1})^2/2\sigma_{n+1}^2]. \tag{5.4}$$

This implies that the distribution parameters behave in the following way under renormalization.

$$K_{n+1} = K_n, \tag{5.5a}$$

$$\sigma_{n+1}^2 = \sigma_n^2/4. \tag{5.5b}$$

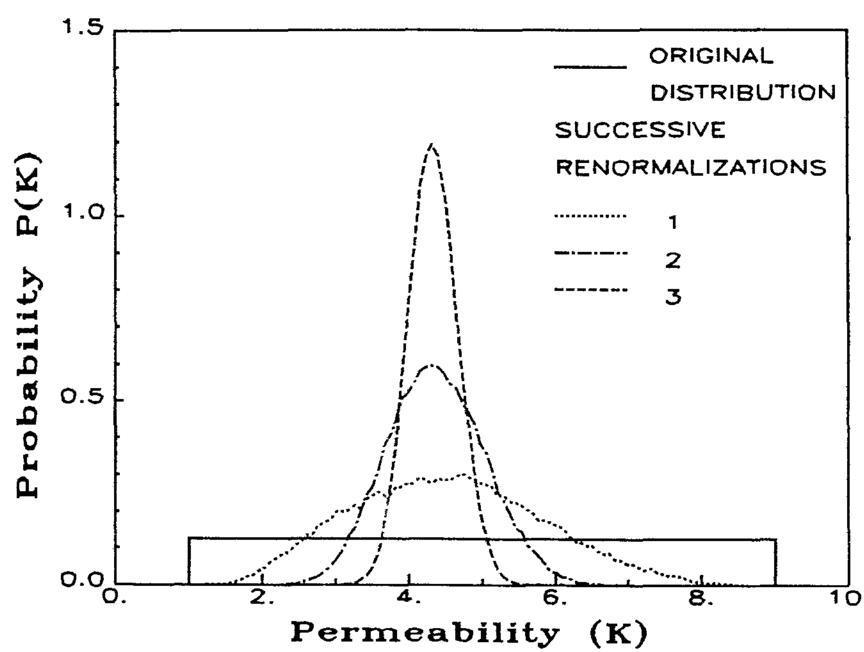


Fig. 3. Renormalization of probability distributions.

The equivalent results in three dimensions are

$$K_{n+1} = K_n, \tag{5.6a}$$

$$\sigma^2_{n+1} = \sigma^2_n/8. \tag{5.6b}$$

That is the mean of the distribution is unchanged and the variance is reduced by a factor of four (2D) or eight (3D) at each renormalization. The effect of repeated renormalization on a uniform probability distribution is shown in Figure 3. The scaling of the mean and variance is shown in Figures 4 (2D) and 5 (3D), which can be seen to be in agreement with Equations (5.5) and (5.6).

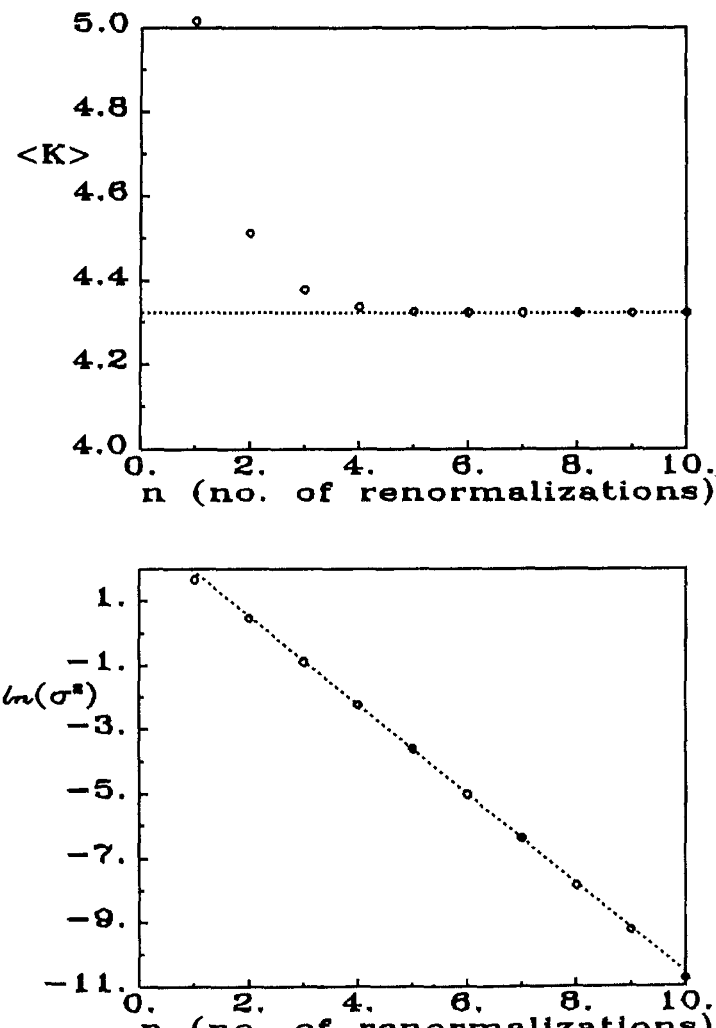


Fig. 4. Scaling of probability parameters in two dimensions. The circles are from numerical experiment and the dashed lines from Equations (5.5).

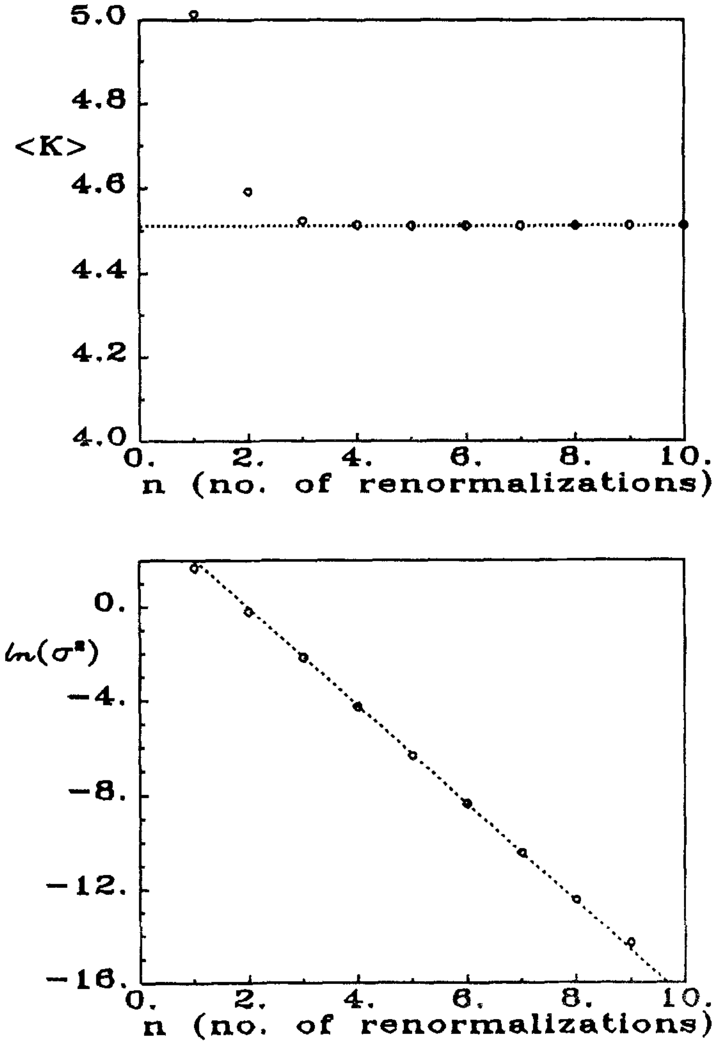


Fig. 5. Scaling of probability parameters in three dimensions. The circles are from numerical experiment and the dashed lines from Equations (5.6).

The significance of this result is that the variability in permeability observed from small length scale samples such as cores is not necessarily that which should be used at the reservoir simulator grid block scale. Whilst this is a well known result, Equations (5.5b) and (5.6b) give an estimate of how the variance should be reduced between the two scales.

6. Renormalization of Probability Distributions

The block renormalized permeability $f(\{K\})$ should be used in (3.2) to determine the renormalized probability for the permeability. Clearly, this is not going to be amenable to analysis even for simple probability distributions. However, we may sample $\{K\}$ from the initial distribution, calculate $\tilde{K}$ and from this build up the new distribution. We do this for two distributions; uniform and log-normal, in two and three dimensions.

For the uniform distribution defined on the interval $[a, b]$ (with $a, b \geqslant 0$) the perturbation parameter $\sigma_K^2/(dK^2)$ is bounded by $1/3d$. Hence, we would expect the EMT result to be a good approximation for this distribution. In Table I we present the results of EMT, renormalization, first order perturbation, the improved perturbation theory of King (1987) and direct numerical simulation of the effective permeability for two uniform distributions; [1, 9] and [1, 99]. These results are given for two and three dimensions. For comparison the arithmetic, geometric and harmonic means of these distributions are also presented. The renormalization result is the fixed point after several transformations. That is four (or eight in three dimensions) permeabilities were chosen from the initial permeability probability distribution. The renormalization transformation was applied to these. This was repeated many times (about 10 000 in these studies) to give a new, or renormalized, permeability probability distribution. This process is repeated until the distribution is so sharp that it is defined by a single value, this is the fixed point. Usually around four or five renormalizations were required to get the fixed point although one transformation was often adequate to get a result within a few per cent.

The first point to notice is that all these estimates of the effective permeability are very close. The maximum deviation (away from the numerical simulation which is taken to be the best estimate) is about 8%. The agreement between the renormalization and simulation estimates is within 1%. The main difference is in computational effort. The direct numerical simulation (Begg and King, 1985)

Table I. Summary of results for uniform distribution

Permeability range		[1, 9]	[1, 99]
Arithmetic mean (K_A)		5.0	50.0
Geometric mean (K_G)		4.357	38.168
Harmonic mean (K_H)		3.641	21.327
Coefficient of variation (σ^2/K_A^2)		0.213	0.333
Simulation	2D	4.25	38.0
Result	3D	4.46	41.2
EMT	2D	4.4	40.5
	3D	4.6	44.2
1st order perturbation	2D	4.47	41.75
(Eq. 2.1)	3D	4.64	44.5
Improved perturbation	2D	4.36	38.17
(King, 1987; Eq. (2.2)	3D	4.6	43.48
Renormalization	2D	4.28	37.9
	3D	4.48	41.5

solves the pressure equation for single phase fluid flow in a porous medium explicitly and, hence, has to store the pressure at each grid block as well as the permeabilities plus working space for a large matrix inversion. This limits the size of problem that can be solved to $42 \times 42 \times 42$ on a Cray X-MP/12. This took about 35 s which is about 500 μs per grid block. The computer time for this method scales as $N \log N$, where N is the number of grid blocks. The renormalization method, however, does not need to store all of the information about every grid block simultaneously and so can be applied, in principle, to an indefinitely large problem. It takes about 50 μs per grid block including the generation of the permeabilities. The time for this method scales like N. The largest problem to which it has been applied so far was nearly 540 million grid blocks, this will be described in more detail in Section 9.

The second point is that in both two and three dimensions, the geometric mean is a very good estimate of the effective permeability. In three dimensions, the improved perturbation result gives a reasonable estimate. Thus, for the uniform distribution where the variance in permeability is not high, there are several techniques available for estimating the effective permeability, all of which agree to within 10%.

However, for the other distribution which has been considered the log-normal distribution, the coefficient of variation can become unbounded so all the perturbation techniques break down. Table II summarizes the results for two different log-normal distributions. It can be seen that for a large logarithmic variance, the first-order perturbation theory is meaningless. Also direct application of EMT is in error by as much as 30% for the worst case considered here and the improved perturbation result is not a good estimate in this case. However, the renormalization technique described in this paper gives excellent

Table II. Summary of results for log normal distribution

Distribution Parameters (K_G, $\sigma^2_{\ln}$)		(2, 0.5)	(2, 10)
Arithmetic mean (K_A)		2.568	296.83
Geometric mean (K_G)		2.0	2.0
Harmonic mean (K_H)		1.558	0.01348
Coefficient of variation (σ^2/K_A^2)		0.6487	22, 025
Simulation	2D	1.95	1.31
Result	3D	2.02	3.31
EMT	2D	2.0	2.0
	3D	2.14	4.7
Ist order perturbation	2D	1.735	-3.3×10^6
(Eq. 2.1)	3D	2.013	-2.2×10^6
Improved perturbation	2D	2.0	2.0
(King, 1987; Eq. (2.2)	3D	2.17	10.59
Renormalization	2D	1.85	1.29
	3D	2.06	3.2

agreement to within 3%. Again the computer time required is in favour of the renormalization method. Another point to note is that the geometric mean is not always a good estimate of the effective permeability in two dimensions with an error of up to 50% for the cases considered here. Hence, when the permeability variance is large, it is necessary to use the more sophisticated technique described here, or direct numerical simulation.

7. Systems with Zero Permeability

In all the systems considered so far the probability of having zero permeability has been zero. However, it is quite common for a reservoir to have a finite amount of impermeable material; the volume fraction of permeable rock is called the net to gross ratio. Conventional simple estimates are now quite inadequate as the geometric mean of such a permeability distribution is zero, yet provided the net to gross is above the percolation threshold, the effective permeability is greater than zero. Similarly EMT and perturbation theory fail in this case. However, renormalization provides a good alternative to numerical simulation and is computationally quicker.

We tested this by considering a uniform distribution on $[a, b]$ $(a > 0)$ with a fraction p of zero permeability.

$$P(K) = p\delta(K) + \frac{(1-p)}{(b-a)}\,\theta(K-a)\theta(b-K). \tag{7.1}$$

θ is the Heaviside step function.

This random, isotropic distribution is probably atypical of a real reservoir but provides a good test case. In reality the impermeable material is usually distributed in a more systematic way, for example in shale beds or in faults. This will be dealt with in the next section where real reservoir data are used.

If the fraction p is above the percolation threshold (about 0.407 in 2D and 0.689 in 3D, these are the site percolation thresholds on square and cubic lattices) then successive renormalizations will drive the impermeable fraction up until the overall effective permeability is zero. If, on the other hand, p is less than this fraction then successive renormalizations will reduce the impermeable fraction to zero and the effective permeability will be finite. However, in this case both the geometric and harmonic means are zero so these are not good estimates. We took the example given in Table I of a uniform distribution on $[1, 9]$ with different initial impermeable fractions and compared the results using numerical simulation and renormalization. Above the threshold, both renormalization and conventional simulation gave zero effective permeability. Below the threshold, the results were that in two dimensions at $p = 0.25$ conventional simulation gave 1.26 and renormalization gave 1.52. In three dimensions at $p = 0.5$, conventional simulation gave 0.58 and renormalization 0.52. Once again the agreement between the method is very good, with the computational advantage in favour of the renormalization method. There is a clear difference between the behaviour above and below the percolation threshold.

8. Systems with Shales

A common feature in many reservoirs is the presence of shales. There are large laterally extensive permeability barriers which greatly impede vertical flow but have little affect on horizontal flow. A number of techniques have been developed specifically to model the effects of shales (Begg and King, 1985). There are two problems in using renormalization for this problem. The first is that the representation of shales as very low permeabilities embedded in a high permeability background leads to configurations within the renormalization block which are badly represented by the resistor network model. This is because the resistor network analogy gives poor resolution of the flow around the edges of the shales. As mentioned in Section 4, this could be resolved by using a direct numerical solution of the unit renormalization grid cell. The other is that enough grid blocks have to be specified to avoid the renormalization amalgamating the shales and thus further reducing the vertical flow. These considerations mean that renormalization is not the best technique to use for the shale problem. However, it is still instructive to try the renormalization method of this system.

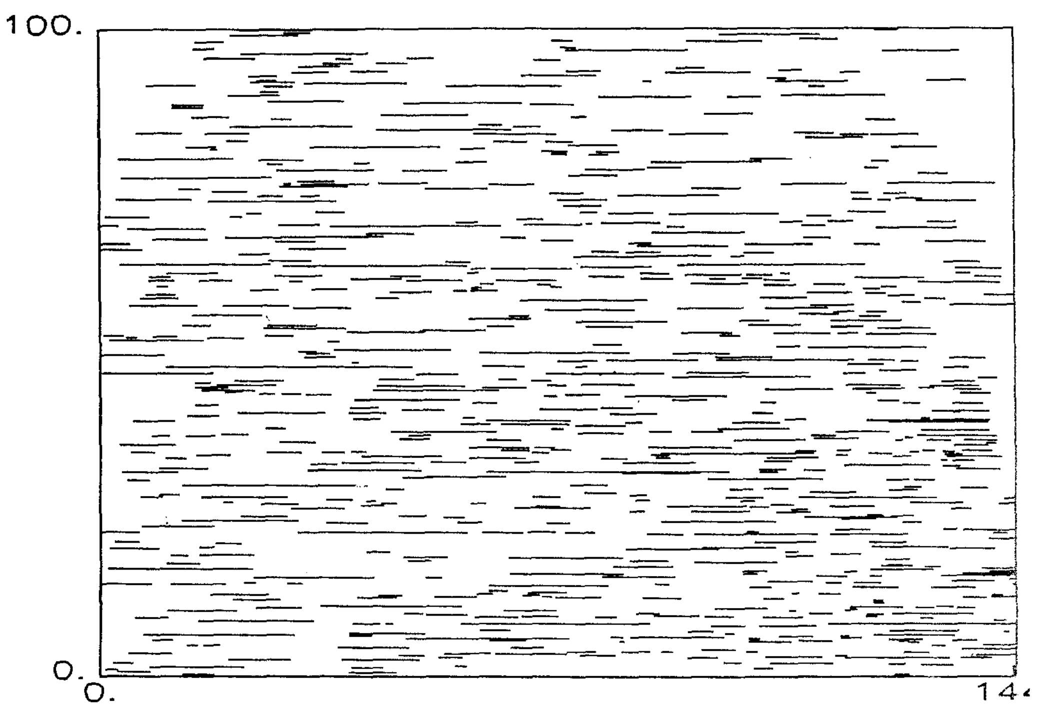


Fig. 6. Assakao cliff showing location of the shales.

The test problem uses data from Dupuy and LeFebvre (1968) who analyzed a precise mapping of the Assakao cliff in the Tassili region of the Central Sahara. Figure 6 is a scaled map of a 145×100 meter section of the cliff showing the positions of the shales. Dupuy and LeFebvre determined the permeability anisotropy by reproducing Figure 6 on conducting paper and cutting slits at the locations of the shales. They then measured the vertical and horizontal resistivities. They found the vertical to horizontal anisotropy to be 0.203. Begg *et al.* (1985) applied the methods of Begg and King (1985) to this problem and found the anisotropy to be between 0.202 and 0.216 depending on the method. We used a fine scale grid of 2048×2048 for the renormalization technique. The anisotropy was found to be 0.102. In view of the above comments this is a fair approximation, although inferior to those methods specifically developed for shales, in particular the streamline method of Begg and King (1985).

9. Application to Real Field Data

As a final example, we apply renormalization with EMT to some real field data. We use the Sherwood Reservoir where conventional numerical calculations have been made (Dranfield *et al.*, 1987) of the effective permeability. This is a very complicated reservoir containing three different rock types each with highly variable permeability combined with shales. Because of the anisotropy of the rock permeabilities and the presence of the shales the effective permeability is also anisotropic. Both vertical and horizontal permeabilities were calculated. Only three-dimensional simulations were performed.

The reservoir contains three rock types: clean sand, muddy sand and muddy silt, each of which have very different permeability distributions. Each of these rock types may be considered to be deposited randomly in large rectangular sand bodies whose length (and width) are about 100 times their thickness. Each layer of the reservoir contains different proportions of the different rock types.

The conventional estimation of the effective permeability (Dranfield *et al.*, 1987) was done by first estimating the effective permeability of each rock type on a 42^3 grid. The moments of the distributions along with the estimates of the effective permeability are given in Table III. This is a very good example of how poor the usual simple estimates can be. The renormalization procedure, however, gives an excellent result.

Then the effective permeability of each layer was calculated by representing a whole sand body by a single grid block with its permeability generated from the first stage using average rock type permeabilities. This was done because the conventional method had insufficient resolution to do the whole calculation in a single step. However, the multiple-scale structure of the problem makes it ideal for analysis by the renormalization method. To this end, each sand body was represented by a slab of varying rock type permeabilities on a grid of size $128 \times 128 \times 1$. A grid of 32^3 of these was made for each layer. This gives a

Table III. Sherwood data (H – horizontal values, V – vertical values)

Sand		K_G	Simple perturbation	Perturbation (Eqn. 2.2)	EMT	Numerical	Renormalization
Clean	H	91	168	273	223	160	163
	V	40	6.6	164	121	80	89
Muddy sand	H	2.15	−12.2	4.63	3.7	2.4	2.7
	V	0.26	−11.1	0.58	0.31	0.4	0.3
Muddy silt	H	0.165	0.19	0.22	0.21	0.19	0.18
	V	0.06	0.063	0.064	0.064	0.063	0.062

representation of the layer with nearly 537 million grid blocks. The results of this are given in Table IV. For comparison a renormalization calculation corresponding to the second stage of the conventional approach is included (this is the column marked 'average'). One disadvantage of the conventional simulation is that a small sample of permeabilities is used, in this case around one thousand. This means that there will be a distribution of effective permeabilities calculated. To determine this several realizations were made and the standard deviation calculated. In the case of the small sample size used in the direct numerical calculation this was typically about 10 to 30% of the mean value. For the renormalization it was at most 5% because of the much larger sample size. The errors shown in Table IV represent an estimate of one standard deviation. These results show the advantage of the renormalization method in the much higher level of resolution available. The final stage incorporating the shales was not attempted by renormalization because of the reasons stated in the previous section. The streamline method would appear to be the most appropriate for a three dimensional problem of this complexity.

Table IV. Sherwood data (H – horizontal values, V – vertical values)

Layer		Numerical	Renormalization	
			Average	Distribution
Upper A	H	76.6 ± 14.7	84.3 ± 1.4	70.5 ± 1.8
	V	0.17 ± 0.04	0.18 ± 0.008	0.19 ± 0.02
Upper B	H	69.4 ± 18.1	67.2 ± 2.9	54.9 ± 2.0
	V	0.3 ± 0.18	0.22 ± 0.004	0.32 ± 0.10
Upper C	H	72.3 ± 16.3	68.4 ± 3.5	53.93 ± 3.6
	V	0.29 ± 0.18	0.22 ± 0.007	0.24 ± 0.02
Middle	H	82.4 ± 8.2	84.8 ± 3.6	70.8 ± 3.5
	V	0.64 ± 0.11	0.60 ± 0.063	0.92 ± 0.07
Lower	H	43.4 ± 15.7	53.7 ± 2.1	39.4 ± 2.1
	V	0.24 ± 0.10	0.23 ± 0.011	0.24 ± 0.03

10. Conclusions

We have described a technique (real-space renormalization) for calculating the effective permeability of a heterogeneous medium when the permeability fluctuations are too large for other, simpler, methods to be applicable. It has been shown that this method is accurate in comparison with direct numerical simulation but is computationally much cheaper.

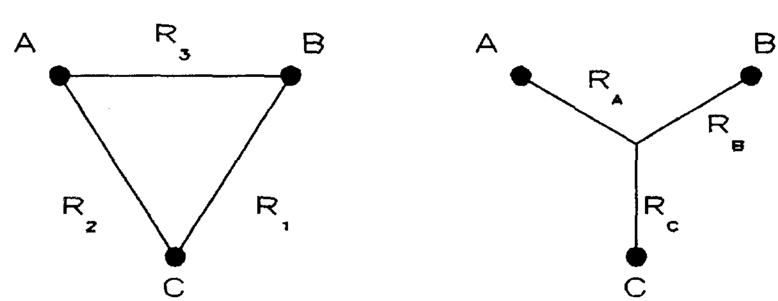
Another advantage of the method is that since explicit realizations are not required to be held in computer core and no costly solution of differential equations is performed the procedure is able to treat much larger problems than conventional simulation. This enables a much finer resolution of sand bodies allowing for a more accurate estimate of effective permeability with more economy in computer time. The maximum size of problem that can be solved using the renormalization method is limited by the amount of computer time that is available rather than the amount of computer storage.

However, the method as described has some drawbacks. If the flow paths a very contorted then the resistor network used does not give a good representation. In such cases the estimate of effective permeability is not good. Such cases will be common when there is a high contrast between neighbouring permeabilities, for example in the shaly reservoirs. Also the renormalization approach will not give a direct realization of the flow paths. If this is required then a direct numerical simulation is necessary which may be done for small models.

In conclusion we have described a powerful new method of calculating effective permeability which is useful in a large number of reservoir engineering situations. It is a simple numerical method which only requires a permeability probability distribution to be input. As seen in the previous section this may be determined from real field data and input digitally.

11. Appendix 1: The Star-Triangle Transformation

This transformation proves very useful for reducing resistor networks to a more simple form. It provides an equivalent circuit for three nodes joined by three resistors. The transformation is



where

$$R_A = \frac{R_2 R_3}{R_1 + R_2 + R_3}$$

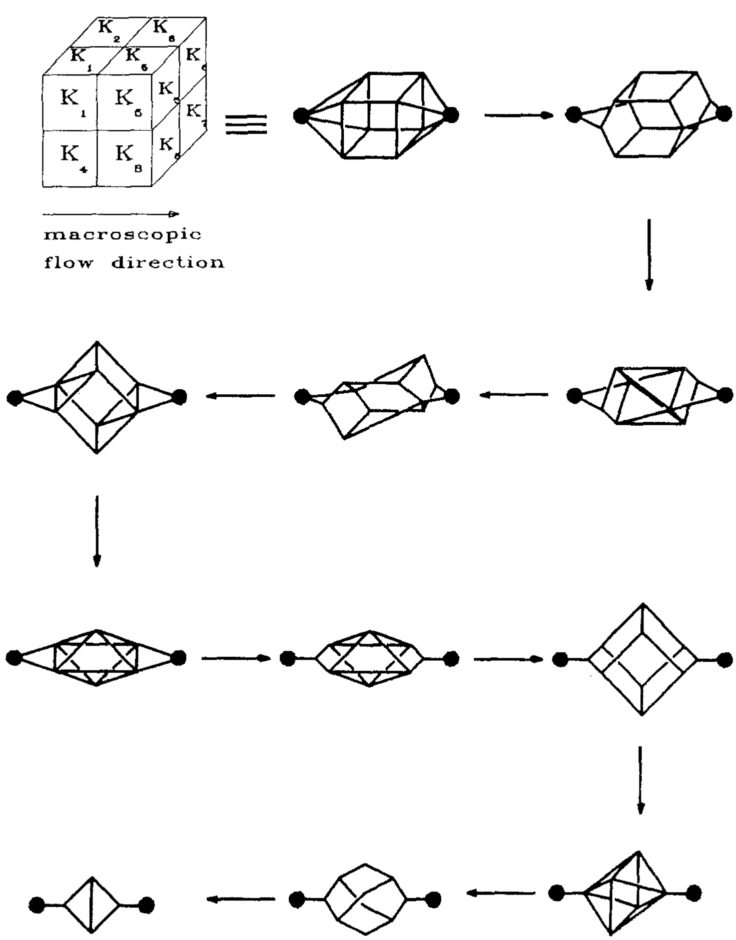
and

$$R_1 = \frac{R_B R_C + R_C R_A + R_A R_B}{R_A}$$

with similar expressions for the other resistors.

12. Appendix 2: Three-Dimensional Renormalization Procedure

A $2 \times 2 \times 2$ block is written as an equivalent resistor network which is reduced, through a sequence of star-triangle transformations, to a single resistance. This sequence is given here without any commentary.



Whence, the rest is derived relatively easily. The conductance of this equivalent network must be divided by two as explained in Section 3. Whilst this procedure does not lead to a simple closed form expression for the renormalized permeability, as it does in two dimensions, it may be readily implemented numerically. The new resistances in each stage above are calculated explicitly.

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