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A comparative analysis of measures of social homogeneity

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Abstract. Social homogeneity refers to the degree to which the preferences of individuals in a society tend to be alike. A number of studies have been conducted to determine whether or not a relationship exists between various measures of social homogeneity and the probability that a Condorcet winner exists. In this study, it is shown that a strong general relationship of this type does not exist for measures of social homogeneity which account only for the proportions of individuals with various preference rankings. That is, for measures which account for these proportions but not for the preference rankings to which they are assigned.

Profile specific measures of homogeneity do account for the preference rankings to which the proportions of voters are assigned. A much stronger relationship exists between profile specific measures of homogeneity and the probability that a Condorcet winner exists than for non-profile specific measures. In particular, Kendall's Coefficient of Concordance is shown to dominate twenty other measures of social homogeneity in terms of the strength of its relationship to the probability that a Condorcet winner exists.

1. Introduction

Social homogeneity refers to the degree to which the preferences of individuals in a society tend to be alike. We shall develop this notion in the context of an election on a set of three candidates. Each member of the society is assumed to have some linear preference ranking on the candidates, so that no individuals have any feeling of indifference between pairs of candidates. For three candidates, $\{A, B \text{ and } C\}$, there are six linear preference rankings given by:

$A > B > C$: p_1

$A > C > B$: p_2

$B > A > C$: p_3

$B > C > A$: p_4

$C > A > B$: p_5

$C > B > A$: p_6

Here, $x > y$ denotes x is preferred to y and p_i is the probability that an individual selected at random from the society will have the associated preference ranking with $\Sigma_{i=1}^{6} p_i = 1$. In a specific voting situation, the preferences of n voters from the society are defined as a profile in terms of six n_i's. Here, n_i is the number of voters with the ith preference ranking so $\Sigma_{i=1}^{6} n_i = n$. The probability that a given combination of the n_i's results is obviously related to the p_i's.

If the society is completely homogeneous then each voter will have the same linear preference ranking on the candidates, say $n_1 = n$. If the society is completely heterogeneous then the voters are expected to be equally divided among all the possible preference rankings on the candidates, so we expect $n_i = n/6$ for all i. Generally speaking, social homogeneity increases as the preference orders of individuals in the society become more similar.

Various measures of social homogeneity will naturally be related to the p_i's. We shall distinguish between two types of measures of social homogeneity. Order specific measures shall use the p_i's along with the information of the specific linear ranking that each is associated with. Non-order specific measures will use only the p_i's without using the knowledge of the specific linear order that each is attached to. When considering a choice of a measure of social homogeneity, one might intuitively expect to have a tradeoff between precision and simplicity. Since the order specific measures use additional information, they are likely to give more precise results, at least when making relative comparisons between sets of p_i's. But the non-order specific measures should generally be easier to calculate.

A number of studies have been conducted to examine the relationship between social homogeneity and the probability that a Condorcet winner exists. For a Condorcet winner to exist for any specific profile, the n_i's must be such that some candidate defeats all remaining candidates in a series of pairwise elections. For example, with $n_1 = 2$ and $n_3 = 1$ for $n = 3$ we find A is the Condorcet winner since it beats B two to one and it beats C three to zero in pairwise comparisons. It is well known that a Condorcet winner does not always exist. For example, with $n_1 = 1$, $n_4 = 1$ and $n_5 = 1$ for $n = 3$, we find A beats B two to one, B beats C two to one, and C beats A two to one.

The probability that a Condorcet winner exists is obviously related to n and the p_i's. Gehrlein and Fishburn (1976a) have shown that the probability that there is a Condorcet winner for three candidates and odd n is given by $\mathrm{P}_n(p)$ where for the vector of p_i's given by p

$$P_n(p) = \sum^{3} \frac{n!}{m_1!m_2!m_3!m_4!} [(p_4 + p_6)^{m_1} p_3^{m_2} p_5^{m_3} (p_1 + p_2)^{m_4}$$

$$+ (p_2 + p_5)^{m_1} p_1^{m_2} p_6^{m_3} (p_3 + p_4)^{m_4} + (p_1 + p_3)^{m_1} p_4^{m_2} p_2^{m_3} (p_5 + p_6)^{m_4}].$$

Here, $a!$ is a-factorial, $m_4 = n - m_1 - m_2 - m_3$ and Σ^3 is the triple sum over m_1, m_2 and m_3 with

$$0 \leqslant m_1 \leqslant (n - 1)/2$$
$$0 \leqslant m_2 \leqslant (n - 1)/2 - m_1$$
$$0 \leqslant m_3 \leqslant (n - 1)/2 - m_1.$$

A more complicated relation for $P_n(p)$ has been developed by DeMeyer and Plott (1970) and $P_n(p)$ relations have been obtained for special assumptions about n and the p_i's by Garman and Kamien (1968), Niemi and Weisberg (1968), Sevcki (1969), Gehrlein and Fishburn (1976b, 1979) and Fishburn and Gehrlein (1978).

A number of studies have suggested that there should tend to be a positive relationship between measures of social homogeneity and the probability of a Condorcet winner. That is, as societies become more homogeneous the probability that a Condorcet winner exists should also increase. This relationship has been found to hold up for several different profile specific measures of social homogeneity. Niemi (1969) found this relationship by measuring social homogeneity by the maximum number of voters whose preference orders are jointly single peaked. Fishburn (1973) measured the homogeneity by using the Kendall-Smith coefficient of concordance (Kendall and Smith (1939)) and found a positive relationship. Jamison and Luce (1972) and Kuga and Nagatani (1974) also found this positive relationship to hold up.

Several articles have been concerned about the relationship between the probability of a Condorcet winner and the non-profile specific measure of social homogeneity $S^1(p)$ with

$$S^1(p) = \sum_{i=1}^{6} p_i^2.$$

$S^1(p)$ is maximized with one of the p_i's equal to one with the rest equal to zero, which is complete homogeneity. $S^1(p)$ is minimized by $p_i = 1/6$ for all i which reflects a totally heterogeneous situation. The measure $S^1(p)$ was suggested by Abrams (1976) who showed that a perfect positive relationship does not exist. That is, we can define two sets of p_i's, given by p and p' where $P_n(p)$ is greater than $P_n(p')$ while $S^1(p)$ is less than $S^1(p')$. Fishburn and Gehrlein (1978) consider the $S^1(p)$ vs $P_n(p)$ relation more generally. While in fact, specific examples can be developed to show the behavior described by Abrams, we might not generally expect this to happen. This study

(Fishburn and Gehrlein 1978) was conducted in two steps. The first step was to consider the set of p vectors for which a direct positive $S^1(p)$ and $P_n(p)$ relation holds up. The second step was to look at an indirect relation between $P_3(p)$ and $S^1(p)$ over all p vectors.

When considering the p vectors for which a direct positive $S^1(p)$ and $P_n(p)$ relation exists, the restriction was made that n was infinite. The set of p vectors considered was then limited to those meeting the dual culture condition. If a p vector meets the dual culture condition it must be true that the probability attached to any linear preference order is the same as the probability attached to the dual (all preferences reversed) of that preference order. For our three alternative cases we require $p_1 = p_6$, $p_2 = p_4$ and $p_3 = p_5$. It is shown that if $S^1(p)$ is increased by changing two of p_1, p_2 and p_3 while keeping the other fixed then $P(p)$ also increases if p_4, p_5 and p_6 change accordingly to stay in the space of dual culture vectors.

The indirect relationship between $P_3(p)$ and $S^1(p)$ was done by considering a different measure for $P_3(p)$. The measure used was $Q_3(p)$ which was obtained as the sum of the $P_3(p)$ values where the sum was over the 6! permutations of the p vector. The upper and lower bounds on $Q_3(p)$ were found for fixed $S^1(p)$ and it was found that both of these bounds increase as $S^1(p)$ increases. This suggests a generally positive relation between $Q_3(P)$ and $S^1(p)$.

The purpose of the current study is to examine the general relationship between $P_n(p)$ and $S^1(p)$ over all p vectors. This relationship has only been shown to hold under specific restrictions in the previously mentioned studies. In addition, a wide range of non-profile specific and profile specific measures of social homogeneity will be developed. These measures will then be compared on the basis of how they relate to $P_n(p)$.

2. Non-profile specific measures of social homogeneity

$S^1(p)$ is the non-profile specific measure of social homogeneity that has received the most attention in the literature. There are many more non-profile specific measures of social homogeneity that might be considered. The measures considered in this study are defined on a probability vector p by

$$S^2(p) = \sum_{i=1}^{6} p_i^3$$

$$S^3(p) = \sum_{i=1}^{6} p_i^4$$

$$S^4(p) = \text{Max}_i \{p_i\}$$

$$S^5(p) = \text{Min}_i \{p_i\}$$

$$S^6(p) = S^3(p) - S^4(p)$$

$$S^7(p) = \prod_{i=1}^{6} p_i$$

$$S^8(p) = \sum_{i=1}^{6} |p_i - \tfrac{1}{6}|.$$

By these definitions we see that $S^2(p)$ is the sum of the cubes of the p_i's. This measure is a natural extension of $S^1(p)$. Similarly, $S^3(p)$ is the sum of the fourth powers of the p_i's. $S^4(p)$ is simply the maximum p_i component in p. $S^5(p)$ is the minimum p_i in p. $S^6(p)$ is the range of the p_i components. $S^7(p)$ is the product of the six p_i components. $S^8(p)$ is the sum of this absolute deviations of the p_i components from their mean, where $\frac{1}{6}$ is the mean p_i for the three candidate case. All eight non-profile specific measures of social homogeneity are fairly common measures of dispersion or variation among a set of numbers. The most common measure of variation in a set of numbers is variance and this is not considered since it is directly related to $S^1(p)$.

As with $S^1(p)$, all of these measures, except $S^5(p)$ and $S^7(p)$, are expected to increase as social homogeneity increases. Both $S^5(p)$ and $S^7(p)$ should tend to decrease as homogeneity increases. In order to determine whether or not a positive relationship exists between the social homogeneity and $P_n(p)$, simulation analysis was employed.

3. Simulation format

In order to begin our analysis it is first necessary to generate p vectors. This was done by generating a random number on the interval [0, 1] for each of the six p_i's. Since we require $\Sigma_{i=1}^{6} p_i = 1$, each p_i was normalized by dividing by the sum of the p_i's. Once a p vector was obtained, $S^i(p)$ was determined for $i = 1, 2, \ldots, 8$ and $P_n(p)$ was calculated. For each n value of

3, 5, 7 . . . , 21; this process was repeated 2500 times. Having these results, the $S^i(p)$ vs $P_n(p)$ relationships can be considered.

4. Simulation results on non-profile specific measures

The attempt to find a general relationship between $S^i(p)$ and $P_n(p)$ takes place in two steps. In the first step, a relation is considered by using the coefficient of correlation between $P_n(p)$ and each of the $S^i(p)$ measures. The results, which are presented in Table 1, give almost no support to the statement that a positive relationship exists between social homogeneity and $P_n(p)$ for any i. The negative correlations for $S^5(p)$ and $S^7(p)$ exist since, by their definition, increases in these two measures suggest a decrease in homogeneity. Trends in the data suggest that the weak relationship that does exist tends to decrease for all $S^i(p)$ as n increases.

These results could come from two sources. Either there simply is no relationship between $S^i(p)$ and $P_n(p)$ or if there is a relationship, it is a non-linear relationship. To test the notion that there is a stronger relation than that suggested by correlation calculations, another measurement can be used. In this stage the 2500 $S^i(p)$ and $P_n(p)$ values were checked to see the proportion of times that each changed in the same direction. That is, start with the first and second p vectors, namely p^1 and p^2. If $S^i(p^1) > S^i(p^2)$ and $P_n(p^1) > P_n(p^2)$ then each changed in the same direction. The same is true if $S^i(p^1) < S^i(p^2)$ and $P_n(p^1) < P_n(p^2)$. By going through all consecutive p vectors in the 2500 observations, we can find the proportion of times that $P_n(p)$ and $S^i(p)$ changed in the same direction. If this proportion is significantly different than 0.5, a general relationship can be assumed. The results of the second stage of this analysis, which are presented in Table 2,

Table 1. Correlation between measures of homogeneity and the probability of a Condorcet winner

n	$S^i(p)$							
	1	2	3	4	5	6	7	8
3	0.273	0.251	0.222	0.234	−0.204	0.257	−0.242	0.273
5	0.168	0.175	0.174	0.173	−0.106	0.171	−0.133	0.150
7	0.149	0.153	0.150	0.146	−0.047	0.129	−0.081	0.139
9	0.118	0.124	0.123	0.129	−0.035	0.110	−0.066	0.103
11	0.146	0.157	0.158	0.167	−0.038	0.141	−0.066	0.121
13	0.069	0.088	0.098	0.092	0.030	0.058	0.006	0.041
15	0.106	0.114	0.116	0.126	−0.038	0.110	−0.059	0.087
17	0.096	0.106	0.105	0.110	−0.007	0.085	−0.032	0.080
19	0.084	0.101	0.108	0.104	0.013	0.073	−0.003	0.044
21	0.052	0.073	0.086	0.086	0.015	0.058	0.001	0.023

Table 2. Proportions of times that measures of homogeneity and probability of a Condorcet winner change in the same direction

n	$S^i(p)$							
	1	2	3	4	5	6	7	8
3	0.673	0.666	0.661	0.629	0.371	0.648	0.331	0.669
5	0.641	0.631	0.626	0.611	0.409	0.614	0.371	0.638
7	0.629	0.619	0.619	0.603	0.410	0.617	0.378	0.631
9	0.619	0.613	0.615	0.612	0.431	0.604	0.403	0.611
11	0.618	0.618	0.612	0.605	0.428	0.617	0.400	0.625
13	0.580	0.583	0.585	0.581	0.464	0.577	0.438	0.587
15	0.598	0.596	0.596	0.591	0.429	0.596	0.408	0.604
17	0.613	0.609	0.605	0.597	0.437	0.593	0.415	0.610
19	0.610	0.611	0.611	0.603	0.422	0.600	0.403	0.605
21	0.611	0.605	0.606	0.597	0.431	0.602	0.415	0.597

give little support to the statement that a strong positive relationship exists between social homogeniety and $P_n(p)$. None of the non-profile specific measures emerges as being consistently best over the range of n and we can generally expect $S^i(p)$ and $P_n(p)$ to change in the same direction in only about 61 percent or less of all cases. As with the correlation results, trends in the data suggest that the weak relationship that does exist generally tends to decrease for all $S^i(p)$ as n increases. Due to the large sample size, a hypothesis test thay any of these proportions differ significantly from 0.5 would be accepted, even for minute probabilities of a type one error. However, the evidence suggesting a strong positive relationship is very weak.

5. Profile specific measures of social homogeneity

The simulation results of the previous section suggest that we must consider more sophisticated measures of social homogeneity if we wish to find a strong relationship between homogeneity and the probability that a Condorcet winner exists. Thirteen different profile specific measures are considered in this part of the study. Each uses information beyond that given in the p vector. Additional information is given in terms of characteristics of the specific linear order that each p_i is associated with.

One concern to a profile specific measure of homogeneity is the probability that a candidate will be ranked first or last by a randomly selected voter. Let FA, FB and FC define the probability that an individual's first ranked candidate is A, B or C respectively. Then:

$$FA = p_1 + p_2 \quad FB = p_3 + p_4 \quad FC = p_5 + p_6.$$

Similarly, let LA, LB and LC define the probability that an individual's last ranked candidate is A, B or C respectively. Then:

$$LA = p_4 + p_6 \quad LB = p_2 + p_5 \quad LC = p_1 + p_3.$$

Another concern of profile specific measures relates to the likelihood that a randomly selected voter would tend to rank one candidate over another. One way to indirectly measure this in a simple fashion for two candidates, say A and B, is to take the sum of probabilities for rankings with A ranked over B and subtract the probabilities for rankings with B ranked over A. Define this measure as AOB and

$$AOB = p_1 + p_2 + p_4 - p_3 - p_5 - p_6.$$

For the other pairs of candidates we find.

$$AOC = p_1 + p_2 + p_3 - p_4 - p_5 - p_6$$

$$BOC = p_1 + p_3 + p_4 - p_2 - p_5 - p_6$$

$$BOA = -AOB$$

$$COA = -AOC$$

$$COB = -BOC.$$

Using these definitions, the thirteen profile specific measures of social homogeneity are defined by:

$$T^1(p) = \text{Max}\ \{FA, FB, FC\}$$

$$T^2(p) = \text{Max}\ \{LA, LB, LC\}$$

$$T^3(p) = \text{Min}\ \{FA, FB, FC\}$$

$$T^4(p) = \text{Min}\ \{LA, LB, LC\}$$

$$T^5(p) = T^1(p) - T^3(p)$$

$$T^6(p) = T^2(p) - T^4(p)$$

$$T^7(p) = FA^2 + FB^2 + FC^2$$

$$T^8(p) = LA^2 + LB^2 + LC^2$$

$$T^9(p) = \text{Abs}\,\{T^8(p) - T^7(p)\}$$

$$T^{10}(p) = \text{Max}\,\{\text{Abs}(AOB*AOC), \text{Abs}(BOA*BOC), \text{Abs}(COA*COB)\}$$

$$T^{11}(p) = \text{Min}\,\{\text{Abs}(AOB*AOC), \text{Abs}(BOA*BOC), \text{Abs}(COA*COB)\}$$

$$T^{12}(p) = T^{10}(p) - T^{11}(p)$$

$$T^{13}(p) = \{(SA - X)^2 + (SB - X)^2 + (SC - X)^2\}/2n^2$$

where

$$SA = FA + 3LA + 2(1 - FA - LA) = 2 - FA + LA$$

$$SB = FB + 3LB + 2(1 - FB - LB) = 2 - FB + LB$$

$$SC = FC + 3LC + 2(1 - FC - LC) = 2 - FC + LC$$

$$X = (SA + SB + SC)/3.$$

In the $T^i(p)$ definitions, Max $\{a, b, c\}$ is the maximum of a, b and c; Min $\{a, b, c\}$ is the minimum of a, b and c; and Abs(a) is the absolute value of a. These thirteen measures of homogeneity cover a wide range of possibilities. The measure $T^{13}(p)$ is Kendall's Coefficient of Concordance.

In order to determine the strength of the relationship between these measures and the probability that a Condorcet winner exists, simulation analysis was used again. By the nature of their definitions, we expect to observe a negative relationship when comparing $T^3(p)$ and $T^4(p)$ with the probability that a Condorcet winner exists.

6. Simulation results on profile specific measures

A simulation was run on the thirteen profile specific measures of social homogeneity in the same fashion as the analysis of non-profile specific measures. Table 3 presents coefficients of correlation between the measures of homogeneity and the probability that a Condorcet winner exists. Table 4 indicates the percentage of times that the homogeneity measures changed

Table 3. Correlation between measures of homogeneity and the probability of a Condorcet winner

n	$T^i(p)$												
	1	2	3	4	5	6	7	8	9	10	11	12	13
3	0.404	0.427	−0.461	−0.479	0.459	0.482	0.462	0.483	0.543	0.410	0.323	0.374	0.470
5	0.423	0.414	−0.479	−0.451	0.481	0.460	0.474	0.459	0.459	0.440	0.333	0.399	0.520
7	0.396	0.386	−0.418	−0.424	0.434	0.431	0.426	0.435	0.441	0.414	0.296	0.375	0.500
9	0.385	0.411	−0.438	−0.447	0.438	0.456	0.427	0.441	0.442	0.427	0.326	0.391	0.509
11	0.385	0.405	−0.413	−0.438	0.425	0.447	0.416	0.433	0.406	0.408	0.329	0.358	0.502
13	0.395	0.389	−0.441	−0.428	0.444	0.437	0.427	0.420	0.405	0.418	0.304	0.382	0.508
15	0.372	0.408	−0.407	−0.447	0.414	0.453	0.392	0.425	0.404	0.387	0.285	0.353	0.489
17	0.372	0.388	−0.408	−0.417	0.416	0.429	0.393	0.407	0.387	0.379	0.290	0.346	0.488
19	0.385	0.375	−0.425	−0.417	0.431	0.424	0.402	0.401	0.380	0.388	0.273	0.367	0.491
21	0.388	0.380	−0.420	−0.401	0.431	0.417	0.395	0.397	0.391	0.362	0.271	0.326	0.482

Table 4. Proportions of times that measures of homogeneity and probability of a Condorcet winner change in the same direction

n	$T^i(p)$												
	1	2	3	4	5	6	7	8	9	10	11	12	13
3	0.647	0.655	0.334	0.326	0.660	0.679	0.663	0.679	0.713	0.637	0.630	0.611	0.668
5	0.666	0.660	0.314	0.327	0.686	0.685	0.683	0.686	0.689	0.673	0.655	0.644	0.704
7	0.661	0.650	0.323	0.331	0.681	0.667	0.680	0.664	0.681	0.695	0.654	0.675	0.727
9	0.649	0.664	0.338	0.300	0.674	0.699	0.676	0.694	0.675	0.705	0.660	0.681	0.742
11	0.662	0.661	0.323	0.331	0.685	0.684	0.681	0.682	0.672	0.700	0.659	0.672	0.739
13	0.662	0.661	0.315	0.306	0.684	0.691	0.685	0.691	0.665	0.716	0.689	0.691	0.743
15	0.655	0.697	0.323	0.301	0.680	0.713	0.676	0.713	0.682	0.728	0.677	0.711	0.778
17	0.663	0.674	0.320	0.315	0.687	0.699	0.688	0.701	0.674	0.744	0.671	0.716	0.776
19	0.696	0.671	0.304	0.299	0.707	0.692	0.708	0.696	0.674	0.747	0.672	0.733	0.774
21	0.685	0.664	0.305	0.311	0.699	0.698	0.703	0.700	0.694	0.752	0.680	0.722	0.780

in the same direction as the probability that a Condorcet winner exists over consecutive p vectors in the sample.

There are some distinct differences between the simulation results on the two types of homogeneity measures. The profile specific measures generally show a stronger relationship to the probability that a Condorcet winner exists. This stronger relationship is present in both the correlation results and the percentage change in the same direction results. Some minor exceptions to these results do occur for the three voter case.

The simulation results on the profile specific measures show that Kendall's Coefficient of Concordance demonstrates the strongest relationship with the probability that a Condorcet winner exists. Kendall's Coefficient is generally superior to the other twelve measure based on both methods of comparison, with some minor exceptions for the case of three voters. The measure $T^{10}(p)$ exhibits unusual behavior since it shows a very weak relationship with the probability that a Condorcet winner exists when using correlation for comparison, but it is superior to all methods, excluding Kendall's Coefficient of Concordance, for n greater than five under the change in direction criterion. The results also indicate some interesting differences between the two methods of comparison. That is, both $T^{10}(p)$ and Kendall's Coefficient consistently increase for percentage change in the same direction as n increases while both show a general decrease in the coefficient of correlation as n increases.

6. Conclusion

Several results clearly emerge from the comparative analysis of twenty one different measures of social homogeneity. First, if we wish to have a measure of homogeneity that tends to have a strong relationship with the probability that a Condorcet winner exists, then attention should be restricted to profile specific measures. Very different results can be obtained according to the method of comparison used to measure the strength of the relationship between the measures of social homogeneity and the probability that a Condorcet winner exists. The only result that remained consistent over both methods of comparison was the general superiority of Kendall's Coefficient of Concordance. Finally, under the comparison by percentage change in the same direction, $T^{10}(p)$ was only dominated by Kendall's Coefficient for 7 or more voters.

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