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THE VALUE OF PERFECT INFORMATION IN NONLINEAR UTILITY THEORY

ABSTRACT. Wakker (1988) has recently shown that, in contrast to an expected utility maximizer, the value of information will sometimes be negative for an agent who violates the independence axiom of expected utility theory. We demonstrate, however, that the value of *perfect* information will always be nonnegative if the agent satisfies a weak dominance axiom. This result thus mitigates to some degree the normative objection to nonlinear utility theory implicit in Wakker's finding.

Keywords: Value of perfect information, nonlinear utility.

1. INTRODUCTION

Researchers have devoted increasing attention in recent years to developing plausible alternatives to expected utility theory as a model of choice under uncertainty. An important part of this task is to identify which of the results known to be valid for expected utility theory do or do not hold in models not admitting of a representation of preferences that is "linear in the probabilities," and, for results in the former class, to identify precisely the set of preferences for which they are valid. An important property of the expected utility model is that the value of (costless) information is always nonnegative, i.e. information is never undesirable. This result holds whether information is perfect – any potential message completely resolves the underlying uncertainty – or imperfect. Wakker (1988) has demonstrated that this property fails if the independence axiom of expected utility theory is relaxed: if an agent violates the independence axiom, then there are circumstances under which he would strictly prefer not to receive costless information. Machina (1989) has argued that this result implicitly assumes that agents satisfy an assumption known as *consequentialism* (see, e.g., Hammond (1988)) on the relevant domain of decision trees and that, if one is willing to relax this assumption, then agents who violate the independence axiom need not ever refuse costless information. In this note we pursue this issue from a different viewpoint: we retain consequentialism on the relevant domain of decision trees and consider whether or not Wakker's result continues to hold if we restrict our attention to *perfect* information. Schlee (1990) has noted that for one nonlinear utility theory – Quiggin's (1982) anticipated utility theory – the answer is negative: an anticipated utility maximizer would never refuse perfect information. The purpose of this note is to generalize this result to a much wider class of choice models. In particular, we demonstrate that – under consequentialism – a necessary and sufficient condition for the value of perfect information to be nonnegative is that preferences satisfy a weak dominance axiom. This result implies that, in order to forge any qualitative differences between the valuation of information in linear and nonlinear utility theories, attention must almost certainly be restricted to the valuation of imperfect information. Moreover, although it would be desirable from a normative viewpoint to impose the condition that *all* information be of nonnegative value, that at least the value of perfect information is always nonnegative partly mitigates the normative objection to nonlinear utility implicit in Wakker's findings.

2. ANALYSIS

Let Z denote the set of outcomes of lotteries, here taken to be the nonnegative orthant of a finite-dimensional Euclidean space. For some fixed n $(n \geqslant 2)$, let S^n denote the set of nonnegative n-dimensional vectors whose coordinates sum to unity, and let $L = \{(z, p): z \in Z^n, p \in S^n\}$ denote the space of lotteries over elements of Z with at most n prizes (where Z^n denotes the n-fold Cartesian product of Z). An element $(z; p) = (z_1, \ldots, z_n; p_1, \ldots, p_n)$ of L yields a prize of $z_i \in Z$ with probability p_i for $i = 1, \ldots, n$. We shall maintain the following restrictions on preferences over L.

ASSUMPTION (A1). Preferences over L are representable by a continuous real-valued functional V.

ASSUMPTION (A2). Let $F_{(z;\, p)}$ denote the cumulative distribution function of the lottery $(z; p)$. For all $(z; p)$ and $(z'; p')$ in L, if $F_{(z;\, p)} = F_{(z';\, p')}$, then $V(z; p) = V(z'; p')$.

Assumptions (A1) and (A2) imply that preferences induced by V over sure prospects are representable by some continuous function $u: Z \to R$.

We shall also assume that agents satisfy consequentialism on the domain of lotteries we consider. To formalize this condition, we follow the notation of Karni and Schmeidler (1990). For any $(z; p) \in L$ and $y \in \{z_1, \ldots, z_n\}$, let $(y \mid (z; p))$ denote the event that a prize y is received from the lottery $(z; p)$. Let $\Psi(L) = \{(y \mid (z; p)) : (z; p) \in L$, $y \in \{z_1, \ldots, z_n\}$ and $p_i > 0$ for some i with $y = z_i\}$ denote the set of prizes conditioned on the lotteries in which they are obtained. The version of consequentialism relevant to our analysis is a restriction on preferences over elements of $\Psi(L)$.

ASSUMPTION (A3). Preferences over $\Psi(L)$ are representable by a function v satisfying $v(y \mid (z; p)) = u(y)$ for all $(y \mid (z; p)) \in \Psi(L)$.

(A3) asserts that preferences over prizes are independent of the lotteries in which they are obtained. The only other restriction that we will consider is the following dominance axiom for risky prospects:

ASSUMPTION (A4). For any z and z' in Z^n, if $u(z_i') \geqslant u(z_i)$ for $i = 1, \ldots, n$, then $V(z'; p) \geqslant V(z; p)$ for all $p \in S^n$.

Consider now the following choice problem. An agent's final outcome depends upon a choice variable, $\alpha \in R_+$, and an exogenous univariate random variable, X, with finite range. Let $L^* = \{(x; p) : x \in R^n, p \in S^n\}$ denote the space of such random variables whose range contains at most n elements, where the ith coordinate of p denotes the probability that X equals x_i, the ith coordinate of x. Let F denote the set of continuous mappings from $R_+ \times R$ into Z; an element f of F determines the final outcome as a function of α and realizations of X. Observe that any lottery in L can be generated by a lottery in L^* together with a suitable mapping from F. For convenience, we assume that α is chosen from a nonempty compact set $d \subset R_+$; let D denote the set of nonempty compact subsets of R_+.

Suppose now that the agent is given the option of delaying the choice of α until after X is realized, rather than choosing α before X is realized. The agent must come to a decision on this option before a value of X is realized. We shall say that an agent prefers perfect to partial information if he prefers to delay his choice of α until X is realized.

PROPOSITION. *Under assumptions (A1)–(A3), an agent will weakly prefer perfect to partial information for all* $((x; p), f, d) \in L^* \times F \times D$ *if and only if assumption (A4) holds.*

Proof. If α is to be chosen before X is realized, then the maximum *ex ante* utility is

$$U((x; p), f, d) \equiv \max_{\alpha \in d} V(f(\alpha, x_1), \ldots, f(\alpha, x_n); p_1, \ldots, p_n),$$

which is well defined by Assumption (A1), the continuity of f, and the compactness of d. If, on the other hand, the choice of α is delayed until X is realized, then, by (A2) and (A3), the *ex ante* utility is

$$U^*((x; p), f, d) \equiv V(f(\alpha_1^*, x_1), \ldots, f(\alpha_n^*, x_n); p_1, \ldots, p_n),$$

where $\alpha_i^* = \mathrm{argmax}_{\alpha_i \in d}\{u(f(\alpha_i, x_i))\}$. If (A4) holds, then clearly $U((x; p), f, d) \leqslant U^*((x; p), f, d)$ for all $((x; p), f, d) \in L^* \times F \times D$.

To prove the necessity of (A4), suppose there is an z' and z in Z^n and a $p \in S^n$ such that $u(z_i') \geqslant u(z_i)$ for $i = 1, \ldots, n$, but $V(z'; p) < V(z; p)$. Let $d = \{0, 1\}$ and let f be any element of F satisfying $f(0, x_i) = z_i$ and $f(1, x_i) = z_i'$ for $i = 1, \ldots, n$. Then $U((x; p), f, d) = V(z; p) > V(z'; p) = U^*((x; p), f, d)$, so that partial information is strictly preferred to perfect information. □

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