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THE ORDINAL UTILITY UNDER UNCERTAINTY AND THE MEASURE OF RISK AVERSION IN TERMS OF PREFERENCES

1. INTRODUCTION

A choice is said to be rational if it follows a criterion, which is usually introduced as a preference model. The von Neumann–Morgenstern theory[1] not only assumes such a rationality but also other axioms with which the neo-Bernoullian utility is connected.

Since a preference model can be represented by ordinal utility also in the von Neumann–Morgenstern case,[2] a first analysis concerns the determination in this case of the relationships between the different utility indices of the actions on the one hand and the utilities of their consequences and probabilities on the other.

A second analysis concerns the measure of risk aversion, usually given by the Arrow–Pratt index, which is referred to the neo-Bernoullian utility. But a more general measure is necessary if we accept that a preference model can be considered without assuming, for instance, the independence axiom. A new index of risk aversion is proposed in this paper. It requires only the existence of a certainty equivalent for each action. This index turns out to be zero when the von Neumann–Morgenstern axioms hold and its derivative to be proportional to the Arrow–Pratt index.

It is also shown that the measure of risk aversion can be positive if the von Neumann–Morgenstern axioms are not all assumed.

2. THE ORDINAL UTILITY FUNCTION OF UNCERTAIN ACTIONS[3]

Every action is represented by a probability distribution of consequences. Then, by indicating the set of the consequences with C (where C is any set, not necessarily a Euclidean one), the actions are probability distributions on C, i.e., functions $a: C \to I$, where I is the real unitary interval, with $a(c) \geqslant 0$ and $\Sigma_{c \in C}\, a(c) = 1$. A set A of actions is considered as well as a preference

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system on the actions $a \in A$, which is indicated by R, so that $a_1 \, R \, a_2$ means that action a_1 is not worse than action a_2. Thus, the model (or relational system) $\langle A, R \rangle$ is considered, where R is a binary relation on the set A.

THEOREM 1. The model $\langle A, R \rangle$ admits a generalized (or non-standard) utility function if and only if R is a regular (i.e., total, reflexive, and transitive) preference on A.[4]

Proof. This theorem derives from a more general theorem.[5]

COROLLARY 2. The utility function is ordinal – or, better, order-homomorphic to the (non-standard) real numbers. In fact, since $a_1 \, R \, a_2$ if and only if $U(a_1) \geqslant U(a_2)$, it is again $a_1 \, R \, a_2$ if and only if $F(U(a_1)) \geqslant F(U(a_2))$, where F is any monotonically increasing function.

3. UTILITY INDICES WHEN THE VON NEUMANN–MORGENSTERN PREFERENCE AXIOMS HOLD

Now the case is considered where the model $\langle A, R \rangle$, with $C \subset A$,[6] admits a utility index of the von Neumann–Morgenstern type,[7] so that

$$(1) \qquad U_{\mathrm{NM}}(a) = \sum_{c \in C} a(c) \, U_{\mathrm{NM}}(c)$$

for any $a \in A$. The following simple theorem holds.

THEOREM 3. If a von Neumann–Morgenstern utility index exists for the model $\langle A, R \rangle$, then the following relationship holds for any utility index

$$(2) \qquad U(a) = F^{-1}\left(\sum_{c \in C} a(c) F(U(c)) \right),$$

where F is the monotonically increasing function by which $U_{\mathrm{NM}} = F(U)$, i.e., any utility index of uncertain actions is an associative mean of the utility indices of their consequences.[8]

Proof. Since utility is ordinal, any utility index is a monotonically increasing function of any other utility index. Then, if U_{NM} is such a utility index, any utility index U can be represented as $F^{-1}(U_{\mathrm{NM}})$ where F is a suitable monotonically increasing function. Thus, since $U_{\mathrm{NM}} = F(U)$ too, the relationship (1) implies the relationship (2).

Remarks. Function F qualifies, in relation (2), the kind of average which is employed in order to obtain the utility indices of the actions. For instance, the arithmetic mean must be used with the von Neumann–Morgenstern utility index, as relationship (1) shows; the geometric mean must be used with the utility index $U_g = e^{U_{\mathrm{NM}}}$, since relationship (2) requires[9]

$$(3) \qquad U_g(a) = \prod_{c \in C} (U_g(c))^{a(c)};$$

the harmonic mean must be used with the utility index $U_h = -U_{\mathrm{NM}}^{-1}$, i.e.,

$$(4) \qquad U_h(a) = \left(\sum_{c \in C} a(c)(U_h(c))^{-1} \right)^{-1};$$

and so on. In other words, a different utility index is connected with each kind of associative mean (2).

Thus, we have ascertained that the ordinality of utility, which depends on the logical grounds of Corollary 2, requires the consideration of the generalized Bernoulli principle (as the relationship (2), or other equivalent ones, can be defined)[10] and that the cardinality of utility, which has been declared by von Neumann and Morgenstern and accepted by other scholars, derives from the consideration of the arithmetic Bernoulli principle which is expressed by relationship (1).

THEOREM 4. Let us assume that the preference model $\langle A, R \rangle$ admits both the von Neumann–Morgenstern utility index and a cardinal index of intensive utility (i.e., a measure of the intensity of preference). In this case the preceding analysis implies that a monotonically increasing function $U_{\mathrm{NM}} = F(U_A)$ exists, where U_A indicates the intensive utility index, and that the intensive utility of uncertain actions is determined by the relationship

$$U_A(a) = F^{-1}\left(\sum_{c \in C} a(c) F(U_A(c)) \right),$$

which shows that $U_A(a)$ depends generally (when F is not linear) not only on the expected utility $\Sigma_{c \in C}\, a(c) U_A(c)$, but also on the variance and the higher moments of the probability distribution of the utility. Moreover, function F is concave or convex if and only if the intensive utility $U_A(a)$ of any action is, respectively, not greater or not less than its expected utility $\Sigma_{c \in C}\, a(c) U_A(c)$.

Proof. For any pair $\bar{x}$, $\bar{y}$ with $\bar{y} = F(\bar{x})$, function $y = F(x)$ can be expressed as

$$x = \alpha + \beta y \pm f(y),$$

where $\bar{x} = \alpha + \beta\bar{y}$, $F'(\bar{x}) = 1/\beta$, f is a convex function, and we have a plus sign if F is concave and a minus sign if F is convex. Then since

$$U_{\mathrm{NM}}(a) = \sum_{c \in C} a(c) U_{\mathrm{NM}}(c)$$

and

$$U_{\mathrm{A}}(a) = F^{-1}(U_{\mathrm{NM}}(a)),$$

we have

$$U_{\mathrm{A}}(a) = \alpha + \beta \sum_{c \in C} a(c) U_{\mathrm{NM}}(c) \pm f\left(\sum_{c \in C} a(c) U_{\mathrm{NM}}(c)\right),$$

i.e.,

$$U_{\mathrm{A}}(a) = \alpha + \beta \sum_{c \in C} a(c) \left(-\frac{\alpha}{\beta} + \frac{1}{\beta} U_{\mathrm{A}}(c) \mp \frac{1}{\beta} f(U_{\mathrm{NM}}(c))\right) \pm$$
$$\pm f\left(\sum_{c \in C} a(c) U_{\mathrm{NM}}(c)\right)$$

so that

$$U_{\mathrm{A}}(a) - \sum_{c \in C} a(c) U_{\mathrm{A}}(c) = \pm f\left(\sum_{c \in C} a(c) U_{\mathrm{NM}}(c)\right) \mp$$
$$\mp \sum_{c \in C} a(c) f(U_{\mathrm{NM}}(c)).$$

Therefore, since f is convex, $U_A(a) - \Sigma_{c \in C}\, a(c) U_A(c)$ is non-positive if and only if F is concave and non-negative if and only if F is convex.

Remark. Theorem 4 implies that if two actions, which have an equal expected utility value in terms of the intensive utility but a different variance, are not indifferent – normally, the action with less variance is preferred to the one with greater variance – then the von Neumann–Morgenstern utility index (which is normally a concave transformation of the intensive index) requires that the two actions have different expected utility values. Conversely, if two actions have an equal expected utility value in terms of the von Neumann–Morgenstern index but a different variance, then the concave transformation of the intensive index in the von Neumann–Morgenstern one requires that the action with greater variance has a greater expected utility value in terms of the intensive index.

Example. Let $C = Re^+$, where Re^+ is the set of positive real numbers; $U_A(c) = \log(1 + c)$; and

$$U_A(a) = M \exp\left(-\sum_{n=2}^{\infty} (-1)^n \frac{M_n}{nM^n}\right),$$

where

$$M = \sum_{c \in C} a(c) U_A(c) = \sum_{c \in C} a(c) \log(1 + c),$$

$$M_n = \sum_{c \in C} a(c)(U_A(c) - M)^n.$$

This intensive utility index requires $\partial U_A(a)/(\partial M_2) < 0$. It is also

$$U_A(a) = \prod_{c \in C} U_A(c)^{a(c)}$$

since, considering Taylor's expansion of $\log U_A(c)$,

$$\log U_A(c) = \log M - \sum_{n=1}^{\infty} (-1)^n \frac{1}{nM^n} (U_A(c) - M)^n$$

so that

$$\prod_{c \in C} \exp(a(c) \log U_A(c)) = \exp\left(\sum_{c \in C} a(c) \log U_A(c)\right) =$$

$$= M \exp\left(-\sum_{n=2}^{\infty} (-1)^n \frac{M_n}{nM^n}\right).$$

Now, the von Neumann–Morgenstern index is the transformation $U_{NM} = F(U_A)$ of the intensive one which is expressed by the function

$$U_{NM} = \gamma + \delta \log U_A,$$

i.e.,

$$U_{NM}(c) = \gamma + \delta \log(1 + c),$$

where γ and $\delta > 0$ are arbitrary constants (since the von Neumann–Morgenstern index is defined up to a linear function). In fact, considering that

$$U_{\mathrm{NM}}(a) = F\left(\prod_{c\in C}(F^{-1}(U_{\mathrm{NM}}(c)))^{a(c)}\right)$$

we find

$$U_{\mathrm{NM}}(a) = \gamma + \delta \log \prod_{c\in C}\left[\exp\left(\frac{1}{\delta}(U_{\mathrm{NM}}(c)-\gamma)\right)\right]^{a(c)} =$$

$$= \gamma + \delta \sum_{c\in C} a(c)\frac{1}{\delta}(U_{\mathrm{NM}}(c)-\gamma) =$$

$$= \sum_{c\in C} a(c)U_{\mathrm{NM}}(c).$$

For instance, the two actions

$$a_1(c) = \begin{cases} 0.5 & \text{for } c_1 = e-1 \\ 0.5 & \text{for } c_2 = e^4-1 \\ 0 & \text{for } c \neq c_1, c_2 \end{cases} \quad a_2(c) = \begin{cases} 1 & \text{for } c_3 = e^2-1 \\ 0 & \text{for } c \neq c_3 \end{cases}$$

are indifferent, since we have, in terms of the intensive utility, $U_{\mathrm{A}}(a_1) = 2$ and $U_{\mathrm{A}}(a_2) = 2$ [but with $\Sigma_{c\in C}\, a_1(c)U_{\mathrm{A}}(c) = \frac{5}{2}$ and $\Sigma_{c\in C}\, a_2(c)U_{\mathrm{A}}(c) = 2$]. The indifference is obtained also considering the von Neumann–Morgenstern index: since

$$U_{\mathrm{NM}}(c_1) = \gamma + \delta \log U_{\mathrm{A}}(c_1) = \gamma,$$

$$U_{\mathrm{NM}}(c_2) = \gamma + \delta \log U_{\mathrm{A}}(c_2) = \gamma + \delta \log 4,$$

$$U_{\mathrm{NM}}(c_3) = \gamma + \delta \log U_{\mathrm{A}}(c_3) = \gamma + \delta \log 2,$$

we obtain

$$U_{\mathrm{NM}}(a_1) = \gamma + \tfrac{1}{2}\delta \log 4$$

and

$$U_{\mathrm{NM}}(a_2) = \gamma + \delta \log 2.$$

4. THE MEASURE OF RISK AVERSION IN TERMS OF PREFERENCES

The Arrow–Pratt measure of risk aversion

$$r(c) = -\frac{U''_{\mathrm{NM}}(c)}{U'_{\mathrm{NM}}(c)}$$

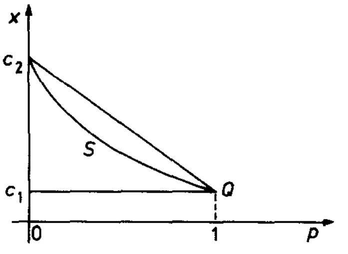

Fig. 1.

requires that $C = Re^+$ (c is normally defined as agent's wealth) and that the von Neumann–Morgenstern utility index exists. Now, the measure of risk aversion can be defined without assuming that all the preference axioms of the von Neumann–Morgenstern theory hold, but only assuming that the preference model $\langle A, R\rangle$ admits, for any action with two possible consequences, a certainty equivalent, i.e., a consequence $x \in C$ which is indifferent to the action. In this case, for any action with the possible consequences c_1 and $c_2 \in C$,

$$a(c) = \begin{cases} p & \text{for } c = c_1 \\ 1-p & \text{for } c = c_2 \end{cases}$$

we can draw (see Fig. 1) the curve c_2SQ, which represents function $x(p)$ and we can define the ratio of area c_2SQ to area c_2c_1Q, as the measure of risk aversion between c_1 and c_2.

That is, consequently,

$$\mu(c_1, c_2) = \frac{c_2 + c_1}{c_2 - c_1} - \frac{2}{c_2 - c_1}\int_0^1 x(p)\,\mathrm{d}p.$$

A positive value of $\mu(c_1, c_2)$ means risk aversion while a negative value means risk attraction. An equivalent expression of $\mu(c_1, c_2)$, when $x(p)$ is a decreasing function, is

$$\mu(c_1, c_2) = 1 - \frac{2}{c_2 - c_1}\int_{c_1}^{c_2} p(x)\,\mathrm{d}x,$$

where $p(x)$ is the inverse function of $x(p)$.

The measure of the local risk aversion in the point c_1 of C can be defined considering the limit

$$\mu(c_1) = \lim_{c_2 \to c_1} \mu(c_1, c_2).$$

If this limit is zero, then we can consider the limit

$$\rho(c_1) = \lim_{c_2 \to c_1} \frac{1}{c_2 - c_1} \mu(c_1, c_2),$$

which represents the velocity by which $\mu(c_1, c_2)$ tends to zero when c_2 approaches to c_1, i.e.,

$$\rho(c_1) = \lim_{dc \to 0} \frac{\mu(c_1, c_1 + dc)}{dc}.$$

5. THE RISK AVERSION MEASURE, WHEN THE VON NEUMANN–MORGENSTERN AXIOMS HOLD, AND ITS CONNECTION WITH THE ARROW–PRATT INDEX

THEOREM 5. If the von Neumann–Morgenstern utility index exists, then $\mu(c_1) = 0$ and

$$\rho(c_1) = -\frac{1}{6} \frac{U''_{\mathrm{NM}}(c_1)}{U'_{\mathrm{NM}}(c_1)},$$

i.e., $\rho(c_1)$ is proportional (by factor $\frac{1}{6}$) to the Arrow–Pratt measure of local risk aversion.

Proof. Since

$$U_{\mathrm{NM}}(x) = p(x) U_{\mathrm{NM}}(c_1) + (1 - p(x)) U_{\mathrm{NM}}(c_2),$$

i.e.,

$$p(x) = \frac{U_{\mathrm{NM}}(c_2) - U_{\mathrm{NM}}(x)}{U_{\mathrm{NM}}(c_2) - U_{\mathrm{NM}}(c_1)},$$

which is a continuously decreasing function of x, we have

$$\mu(c_1, c_2) = 1 - \frac{2}{c_2 - c_1} \frac{1}{U_{\mathrm{NM}}(c_2) - U_{\mathrm{NM}}(c_1)} \times$$
$$\times \left((c_2 - c_1) U_{\mathrm{NM}}(c_2) - \int_{c_1}^{c_2} U_{\mathrm{NM}}(x)\, \mathrm{d}x \right).$$

Then, using l'Hospital's rule

$$\mu(c_1) = \lim_{c_2 \to c_1} \frac{-(c_2 - c_1)(U_{\mathrm{NM}}(c_2) + U_{\mathrm{NM}}(c_1)) + 2\int_{c_1}^{c_2} U_{\mathrm{NM}}(x)\,\mathrm{d}x}{(c_2 - c_1)(U_{\mathrm{NM}}(c_2) - U_{\mathrm{NM}}(c_1))} =$$

$$= \lim_{c_2 \to c_1} \frac{U_{\mathrm{NM}}(c_2) - U_{\mathrm{NM}}(c_1) - (c_2 - c_1)U'_{\mathrm{NM}}(c_2)}{U_{\mathrm{NM}}(c_2) - U_{\mathrm{NM}}(c_1) + (c_2 - c_1)U'_{\mathrm{NM}}(c_2)} =$$

$$= \lim_{c_2 \to c_1} \frac{-(c_2 - c_1)U''_{\mathrm{NM}}(c_2)}{2U'_{\mathrm{NM}}(c_2) + (c_2 - c_1)U''_{\mathrm{NM}}(c_2)} = 0,$$

and

$$\rho(c_1) = \lim_{c_2 \to c_1} \frac{-(c_2 - c_1)(U_{\mathrm{NM}}(c_2) + U_{\mathrm{NM}}(c_1)) + 2\int_{c_1}^{c_2} U_{\mathrm{NM}}(x)\,\mathrm{d}x}{(c_2 - c_1)^2 (U_{\mathrm{NM}}(c_2) - U_{\mathrm{NM}}(c_1))} =$$

$$= \lim_{c_2 \to c_1} \frac{U_{\mathrm{NM}}(c_2) - U_{\mathrm{NM}}(c_1) - (c_2 - c_1)U'_{\mathrm{NM}}(c_2)}{2(c_2 - c_1)(U_{\mathrm{NM}}(c_2) - U_{\mathrm{NM}}(c_1)) + (c_2 - c_1)^2 U'_{\mathrm{NM}}(c_2)} =$$

$$= \lim_{c_2 \to c_1} \frac{-(c_2 - c_1)U''_{\mathrm{NM}}(c_2)}{2U_{\mathrm{NM}}(c_2) - 2U_{\mathrm{NM}}(c_1) + 4(c_2 - c_1)U'_{\mathrm{NM}}(c_2) + (c_2 - c_1)^2 U''_{\mathrm{NM}}(c_2)} =$$

$$= \lim_{c_2 \to c_1} \frac{-U''_{\mathrm{NM}}(c_2) - (c_2 - c_1)U'''_{\mathrm{NM}}(c_2)}{6U'_{\mathrm{NM}}(c_2) + 6(c_2 - c_1)U''_{\mathrm{NM}}(c_2) + (c_2 - c_1)^2 U'''_{\mathrm{NM}}(c_2)} =$$

$$= -\frac{1}{6}\frac{U''_{\mathrm{NM}}(c_1)}{U'_{\mathrm{NM}}(c_1)}.$$

COROLLARY 6. If, instead of the von Neumann–Morgenstern index, another index of utility is used, the measure of local risk aversion is

$$\rho(c_1) = -\frac{1}{6}\frac{U''_{\mathrm{NM}}(c_1)}{U'_{\mathrm{NM}}(c_1)} = -\frac{1}{6}\left(\frac{U''(c_1)}{U'(c_1)} + U'(c_1)\frac{F''(U)}{F'(U)}\right),$$

where the utility index U is such that $U_{\mathrm{NM}} = F(U)$. Consequently, if $F'' < 0$ (and $F' > 0$), we find $-U''(c_1)/U'(c_1) < -U''_{\mathrm{NM}}(c_1)/U'_{\mathrm{NM}}(c_1)$, without meaning any difference in risk aversion.

6. THE RISK AVERSION MEASURE WHEN THE INDEPENDENCE AXIOM DOES NOT HOLD

COROLLARY 7. If the preference model $\langle A, R\rangle$ does not admit the von Neumann–Morgenstern index (for instance, since the independence axiom does not hold), then the measure of risk aversion $\mu(c_1, c_2)$ does not tend necessarily to zero when c_2 tends to c_1. (In this way Corollary 7 justifies Allais's opinion[11] that the von Neumann–Morgenstern theory excludes the risk aversion.)

Example. Let us assume $C = Re^+$ and a preference model $\langle A, R\rangle$ represented by the utility index

$$U(a) = \max \{\tfrac{1}{2}(M + U(c_m)), 2M - U(c_M)\},$$

where $M = \Sigma_{c\in C}\, a(c)U(c)$, $c_m = \min \{c \in C\colon a(c) > 0\}$ and $c_M = \max \{c\in C\colon a(c) > 0\}$. This preference model implies that the agent is influenced more by the less good consequences than by the better ones.

The preference model represented by this utility index does not obey the preference axioms of the von Neumann–Morgenstern theory. In particular, von Neumann–Morgenstern's axiom (3 : B : a) is not satisfied, exactly in the same manner as considered by Allais (1979): for instance, actions

$$a_1 = \begin{cases} 1/3 & \text{for } c = 1 \\ 2/3 & \text{for } c = 3 \end{cases} \qquad a_2 = \{1 \quad \text{for } c = 2$$

$$a_3 = \begin{cases} 1/2 & \text{for } a_1 \\ 1/2 & \text{for } a_2 \end{cases}, \qquad \text{i.e.,} \qquad a_3 = \begin{cases} 1/6 & \text{for } c = 1 \\ 1/2 & \text{for } c = 2 \\ 1/3 & \text{for } c = 3 \end{cases}$$

have utilities $U(a_1) = \tfrac{2}{3}U(1) + \tfrac{1}{3}U(3)$, $U(a_2) = U(2)$ and $U(a_3) = \max \{\tfrac{1}{12}(7U(1) + 3U(2) + 2U(3), \tfrac{1}{3}(U(1) + 3U(2) - U(3))\}$, so that, if $2U(1) + U(3) < 3U(2)$ and $U(1) + 2U(3) > 3U(2)$, we find $U(a_2) > U(a_1) > U(a_3)$ while axiom (3 : B : a) would require $U(a_1) < U(a_3)$. (The preceding condition is satisfied, for instance, by functions $U(c) = k + c$ and $U(c) = \log (k + c)$ for any $k \geqslant 0$.)

Considering the measure of risk aversion for this preference model, the utility of the certainty equivalent for action

$$a(c) = \begin{cases} p & \text{for } c = c_1 \\ 1 - p & \text{for } c = c_2, \end{cases}$$

where $c_2 > c_1$, is

$$U(x) = \max\left\{\frac{1+p}{2}U(c_1)+\frac{1-p}{2}U(c_2),\right.$$
$$\left. 2pU(c_1)-(1-2p)U(c_2)\right\},$$

i.e.,

$$U(x) = \begin{cases} 2pU(c_1)+(1-2p)U(c_2) & \text{for } 0 \leqslant p \leqslant \frac{1}{3} \\ \dfrac{1+p}{2}U(c_1)+\dfrac{1-p}{2}U(c_2) & \text{for } \frac{1}{3} \leqslant p \leqslant 1, \end{cases}$$

or

$$p(x) = \begin{cases} \dfrac{U(c_1)+U(c_2)-2U(x)}{U(c_2)-U(c_1)} & \text{for } U(c_1) \leqslant U(x) \leqslant \frac{2}{3}U(c_1)+\frac{1}{3}U(c_2) \\ \dfrac{1}{2}\dfrac{U(c_2)-U(x)}{U(c_2)-U(c_1)} & \text{for } \frac{2}{3}U(c_1)+\frac{1}{3}U(c_2) \leqslant U(x) \leqslant U(c_2). \end{cases}$$

Since

$$\frac{1-\mu(c_1)}{2} = \lim_{c_2\to c_1}\frac{1}{c_2-c_1}\int_{c_1}^{c_2} p(x)\,\mathrm{d}x,$$

we find

$$\frac{1-\mu(c_1)}{2} = \lim_{c_2\to c_1} \times$$
$$\times\ \frac{(U(c_1)+U(c_2))(\hat{c}-c_1)-2\int_{c_1}^{\hat{c}}U(x)\,\mathrm{d}x+\frac{1}{2}U(c_2)(c_2-\hat{c})-\frac{1}{2}\int_{\hat{c}}^{c_2}U(x)\,\mathrm{d}x}{(c_2-c_1)(U(c_2)-U(c_1))},$$

where

$$U(\hat{c}) = \tfrac{2}{3}U(c_1)+\tfrac{1}{3}U(c_2),$$

thus with

$$U'(\hat{c})\frac{\mathrm{d}\hat{c}}{\mathrm{d}c_2} = \tfrac{1}{3}U'(c_2).$$

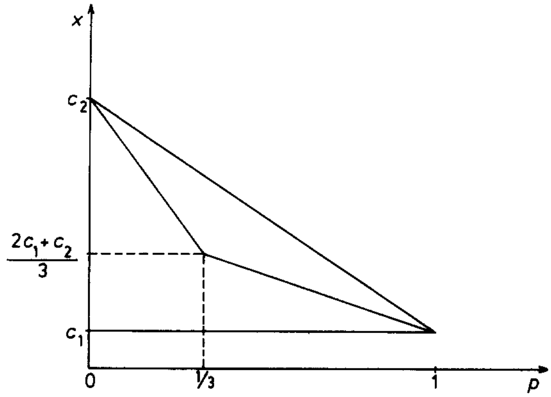

Fig. 2.

Then we have

$$\frac{1-\mu(c_1)}{2} = \lim_{c_2 \to c_1} \frac{(\frac{1}{2}c_2 + \frac{1}{2}\hat{c} - c_1)U'(c_2)}{U(c_2) + U(c_1) + (c_2 - c_1)U'(c_2)} =$$

$$= \lim_{c_2 \to c_1} \frac{\left(\frac{1}{2} + \frac{1}{2}\frac{\mathrm{d}\hat{c}}{\mathrm{d}c_2}\right) U'(c_2) + (\frac{1}{2}c_2 + \frac{1}{2}\hat{c} - c_1)U''(c_2)}{2U'(c_2) + (c_2 - c_1)U''(c_2)}$$

and consequently

$$\frac{1-\mu(c_1)}{2} = \lim_{c_2 \to c_1} \frac{1}{4} + \frac{1}{4}\frac{\mathrm{d}\hat{c}}{\mathrm{d}c_2} = \frac{1}{3},$$

i.e., $\mu(c_1) = \frac{1}{3}$ for any smooth function $U(c)$. For instance, if $U(c) = c$, we have the function $x(p)$ of Figure 2, for which

$$\mu(c_1, c_2) = \frac{1}{c_2 - c_1} \begin{vmatrix} 1 & c_1 & 1 \\ 0 & c_2 & 1 \\ \frac{1}{3} & \frac{2c_1 + c_2}{3} & 1 \end{vmatrix} = \frac{1}{3}.$$

NOTES

[1] Von Neumann and Morgenstern (1953, pp. 15–31 and 617–632).

[2] For instance, Baumol (1958) and Green (1978, pp. 220–226).

[3] This chapter and the first part of the following one synthesize a previous paper (Montesano, 1982).
[4] A preference is total if $a_1 R a_2$ and/or $a_2 R a_1$ for any pair a_1, $a_2 \in A$; reflexive if $a R a$ for any $a \in A$; transitive if $a_1 R a_3$ for any triplet $a_1, a_2, a_3 \in A$ for which $a_1 R a_2$ and $a_2 R a_3$. For instance, von Neumann and Morgenstern (1953, p. 26), assume a regular preference by means of axiom (3: A). A (non-standard) utility function is a function $U: A \to Re$ such that $a_1 R a_2$ for any pair $a_1, a_2 \in A$ if and only if $U(a_1) \geqslant U(a_2)$, where Re represents the set of (non-standard) real numbers and $\geqslant$ the relation of "greater or equal": for the non-standard utility (Richter, 1971).
[5] Richter (1971, p. 43, Theorem 9).
[6] The requirements $C \subset A$ means that the actions with a sure consequence (i.e., the actions with a positive probability only for one element $c \in C$) are included in A. These actions can be indicated by c and C is their set.
[7] Such a utility index is admitted by the usual theories of choice under uncertainty: von Neumann and Morgenstern (1953), Marschak (1950), Savage (1954), etc. Nevertheless, there are theories which do not admit this utility index: Allais (1979).
[8] The notion of associative mean has been considered by De Finetti: see Daboni (1982).
[9] Green (1982, p. 225) considers an example of this type.
[10] By extending a remark of Chipman (1960, p. 219).
[11] Allais (1979, pp. 597–598).

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