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ALLOCATION, LEHRER MODELS, AND THE CONSENSUS OF PROBABILITIES

1. ALLOCATION PROBLEMS

Rational choice often involves the assignment of values to numerical decision variables. In some cases the most appropriate values of these variables are obvious and one can move with dispatch to apply an appropriate optimization algorithm and identify an optimal choice or set of choices. In general, however, the determination of such values is a difficult task. Typically, individuals are required to make subjective estimates of probabilities and utilities or other predictions of future outcomes. The experience and sophistication required to quantify the relevant aspects of a decision problem thus often call for the expertise of more than one individual. But the strategy of decision-making by groups raises a further problem: What is to be done if the experts disagree?

Suppose that a group of n experts, labeled $1, 2, \ldots, n$, is seeking numerical values of a sequence of k decision variables, $x_1, x_2, \ldots, x_k$. The outcome of the group's deliberation is an $n \times k$ matrix $A = (a_{ij})$, where a_{ij} is the value assigned by expert i to variable x_j. Intersubjective agreement on all of these values is reflected in a matrix with identical rows. Failing such agreement, and given the necessity of specifying a single value for each variable, how should the opinions registered in A be aggregated? Numerous possibilities from the realm of statistics (arithmetic and geometric means; medians; maxima, minima, and various combinations thereof) come to mind and have, indeed, frequently been employed in practice. Justifications for the choice of a particular method for aggregating group opinion have tended, however, to be piecemeal and anecdotal, relying heavily on tradition or simplicity of calculation.[1] Our aim in this paper is to begin, in a modest way, to rectify this situation by presenting an axiomatic characterization of weighted arithmetic averaging as a method of combining group opinion for a special class of decision problems called allocation problems. Our main result (Theorem 4) may be specialized to provide both a formal foundation for Keith Lehrer's

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iterated weighting model of rational group decision-making [7], [8], [13], as well as an axiomatic characterization of the method of producing a consensual probability distribution by arithmetic averaging.

Suppose that a group of experts is seeking numerical values of a sequence of k decision variables, $x_1, \ldots, x_k$. We shall call such a decision problem an *allocation problem* if the values of these variables are constrained by the requirements (1) $x_j \geqslant 0$, $j = 1, \ldots, k$ and (2) $x_1 + \ldots + x_k = s$, for some fixed $s > 0$. Examples of allocation problems abound, and include such familiar problems as (1) assigning probabilities to a sequence of k pairwise exclusive, exhaustive propositions ($s = 1$) and (2) allocating a fixed sum of money or other resource, s, among k projects.

We shall suppose that the constraints which define an allocation problem apply both to the individual opinions registered in the aforementioned matrix A (so that all entries of A are nonnegative, and each of its rows sums to s) and to the final set of values assigned to the x_j. Furthermore, we require that a method for aggregating group opinion be applicable to every possible configuration of opinions. Let $\mathscr{A}(n, k; s)$ denote the set of all $n \times k$ matrices A with nonnegative entries and all row sums equal to s, and let $\mathscr{A}(k; s)$ denote the set of all vectors $a = (a_1, \ldots, a_k)$ with nonnegative entries and $a_1 + \ldots + a_k = s$. Allowing for the widest possible initial range of aggregation methods, we make the following definition:

DEFINITION 1. An allocation aggregation method (AAM) is a function F: $\mathscr{A}(n, k; s) \to \mathscr{A}(k; s)$, for fixed positive integers n and k and fixed $s > 0$.

The familiar arithmetic mean yields an AAM by the rule: $F(A) = (a_1, \ldots, a_k)$, where a_j is the arithmetic mean, $(a_{1j} + \ldots + a_{nj})/n$, of the entries in the j-th column of A. This method gives, in some sense, equal weight to the opinions of all individuals in the decision-making group. More generally, any set of weights $w_1, \ldots, w_n$, nonnegative and summing to 1, yields an AAM by the rule: $F(A) = (a_1, \ldots, a_k)$, where a_j is the weighted arithmetic mean, $w_1 a_{1j} + \ldots + w_n a_{nj}$, of the entries in the j-th column of A. When individuals differ in expertise, a weighted arithmetic mean with unequal weights may be an appropriate way to reflect such differences. In the next section we present, among other results, a characterization of the set of AAMs based on weighted arithmetic averaging, thus providing criteria for employing such aggregation methods. Proofs of the theorems stated in the next sections appear in a technical appendix at the end of the paper.

2. AAMs BASED ON WEIGHTED MEANS

Consider an AAM F constructed as above from a sequence of weights $w_1, \ldots, w_n$. For each matrix $A \in \mathscr{A}(n, k; s)$ let A_j denote the j-th column of A, and let $F(A) = (a_1, \ldots, a_k)$. It is easy to check that F has the following properties:

(1) IA (*Irrelevance of Alternatives*): For all $A, B \in \mathscr{A}(n, k; s)$ $A_j = B_j \Rightarrow a_j = b_j$.

(2) Z (*Zero Unanimity*): For all $A \in \mathscr{A}(n, k; s)$, if A_j consists entirely of zeros, then $a_j = 0$.

(3) LN (*Label Neutrality*): If σ is a permutation on the set $\{1, \ldots, k\}$, for all $A, B \in \mathscr{A}(n, k; s)$, if $B_j = A_{\sigma(j)}$ for $j = 1, \ldots, k$, then $b_j = a_{\sigma(j)}$.

(4) SLN (*Strong Label Neutrality*): For all $A, B \in \mathscr{A}(n, k; s)$ $A_{j_1} = B_{j_2} \Rightarrow a_{j_1} = b_{j_2}$.

Property Z states a very weak unanimity condition: if all individuals assign x_j the value zero, F does the same. Properties IA, LN, and SLN are invariance conditions of varying strength. IA specifies that if values assigned by individuals to x_j are unchanged, changes in the values assigned by these individuals to variables other than x_j do not change the final value assigned to x_j by F. LN specifies that a variable be given no special consideration by F in virtue of its particular label-a relabeling of variables results, under F, in the same relabeling of the final values assigned to those variables. As noted below, SLN is equivalent to the conjunction of LN and IA.

THEOREM 1. *An AAM satisfies SLN if and only if it satisfies IA and LN.*

Note that when $k = 2$, IA holds trivially. When $k \geqslant 3$, however, IA is a property of some consequence. Indeed, when supplemented by the weak property Z, it implies SLN.

THEOREM 2. *If $k \geqslant 3$, an AAM F: $\mathscr{A}(n, k; s) \to \mathscr{A}(k; s)$ satisfies SLN if it satisfies IA and Z.*

It is of interest at this point to note an important consequence of the property SLN: If F satisfies SLN, then the value assigned by F to each variable x_j is a function exclusively of the values assigned by individuals to x_j, and independent of j. More formally, let $[0, s]$ denote the closed interval of real numbers between 0 and s, and let $[0, s]^n = \{(t_1, \ldots, t_n): t_i \in [0, s]\}$. (Note that the columns of a matrix $A \in \mathscr{A}(n, k; s)$ correspond in a natural way to members of $[0, s]^n$.)

THEOREM 3. *Let $F: \mathscr{A}(n, k; s) \to \mathscr{A}(k; s)$ satisfy SLN. Then there exists a function H: $[0, s]^n \to [0, s]$ such that for all $A = (a_{ij}) \in \mathscr{A}(n, k; s)$, $F(A) = (a_1, \ldots, a_k)$, where $a_j = H(a_{1j}, \ldots, a_{nj})$.*

We may now characterize, for the case $k \geqslant 3$, those AAMs based on weighted arithmetic averaging.

THEOREM 4. *If $k \geqslant 3$, an AAM F: $\mathscr{A}(n, k; s) \to \mathscr{A}(k; s)$ satisfies IA and Z if and only if there exists a sequence of weights $w_1, \ldots, w_n$, non-negative and summing to one, such that for all $A = (a_{ij}) \in \mathscr{A}(n, k; s)$, $F(A) = (a_1, \ldots, a_k)$, where $a_j = w_1 a_{1j} + \ldots + w_n a_{nj}$, $j = 1, \ldots, k$.*

We emphasize that the foregoing theorem holds only when there are at least three decision variables ($k \geqslant 3$). When $k = 2$, since IA holds trivially, it is easy to see that the conjunction of IA and Z need not imply SLN. The most we seem to be able to accomplish in this case is a characterization of AAMs satisfying SLN and Z. As noted below, such restrictions nevertheless admit a wide variety of nonlinear amalgamation methods.

THEOREM 5. *An AAM F: $\mathscr{A}(n, 2; s) \to \mathscr{A}(2; s)$ satisfies SLN and Z if and only if there is a function h: $[-s/2, s/2]^n \to [-s/2, s/2]$, where*

(1) *h is odd* $(h(-\alpha_1, \ldots, -\alpha_n) = -h(\alpha_1, \ldots, \alpha_n))$,

and

(2) $h(s/2, \ldots, s/2) = s/2$,

such that for all $A = (a_{ij}) \in \mathscr{A}(n, 2; s)$, $F(A) = (a_1, a_2)$, *where* $a_j = h(a_{1j} - s/2, \ldots, a_{nj} - s/2) + s/2, j = 1, 2$.

In particular h (and hence, H) may be any weighted arithmetic mean. However, nonlinear functions such as $h(\alpha_1, \ldots, \alpha_n) = [(\alpha_1^3 + \ldots + \alpha_n^3)/n]^{1/3}$ also serve here to define AAMs satisfying SLN and Z^2. When $k = 2$, short of actually positing linearity, there seems to be no obvious way to isolate the AAMs based on arithmetic averaging.

Let us summarize our results thus far. In the case of an allocation problem involving at least three variables, Theorem 4 tells us that an AAM satisfying Z and IA (equivalently, Z and SLN) must be based on weighted arithmetic averaging. When there are only two variables, however, Z and SLN allow an AAM to be based on a class of odd multivariable functions which properly includes the class of weighted arithmetic means (Theorem 5). These results offer substantial guidance in the choice of an AAM, given a prior decision to employ an amalgamation method satisfying Z and IA (respectivley, Z and SLN). How reasonable is it to require that amalgamation satisfy these properties?

Consider first property Z, which, as noted earlier, is a very weak unanimity conditon. Unless one has reason to believe that the decision-makers are systematically biased in their evaluation of one or more variables, it is surely reasonable to respect their unanimity in assigning a variable the value zero. Moreover, if the output of an AAM is regarded simply as a summary of group opinion, unanimity will of necessity be respected. As for IA, it is easy to prove the following analogue of Theorem 3:

THEOREM 6. *Let* $F: \mathscr{A}(n, k; s) \to \mathscr{A}(k; s)$ *satisfy IA. Then for all* $j = 1, \ldots, k$, *there exist functions* H_j: $[0, s]^n \to [0, s]$ *such that for all* $A = (a_{ij}) \in \mathscr{A}(n, k; s)$, $F(A) = (a_1, \ldots, a_k)$, *where* $a_j = H_j(a_{1j}, \ldots, a_{nj})$

Thus if F satisfies IA, the value assigned by F to each variable x_j is a function (possibly dependent on j) exclusively of the values assigned by individuals to x_j. Given the values assigned by individuals to x_j, values assigned by individuals to alternative variables are, in short, irrelevant. Indeed, if $k \geqslant 3$ and IA is supplemented by Z, the above functions H_j are identical (Theorems 2 and 3) and equal to some weighted arithmetic mean (Theorem 4).

Despite its somewhat formidable name[3], IA is a rather weak condition. It postulates the irrelevance of the values of certain variables in a context where there are built-in constraints on certain sums of these variables. Because of these allocation constraints, issues have, in a sense, already been sorted out among variables evaluated in a matrix A. The value assigned by an individual to x_j is a function of s and the values he assigns to variables other than x_j. Thus, in assigning a final value to x_j an AAM satisfying IA 'ignores' the values assigned by individuals to variables other than x_j in a limited sense.

Indeed, we would argue that the burden of justification lies with those who propose not to adopt IA in choosing an AAM. It is easy, of course, to construct AAMs for which IA fails. For example the 'benefit-of-the-doubt AAM', which identifies the maximum values assigned by individuals to each variable and normalizes these so that the resulting sum is s, clearly violates IA[4]. This method can be attacked for giving too much weight to the opinion of a single individual, just as it can be defended for its (unidirectional) sensitivity to the perceptions of a single individual. As Steven Strasnick [11] has forcefully argued, however, such exchanges are not a fruitful way of evaluating a method of amalgamating opinion. No informed choice of an AAM can be made in the absence of an axiomatic characterization of that AAM, or of some natural class of methods of which it is a member.

While Theorem 4 specifies necessary and sufficient conditions for the use of weighted arithmetic averaging to construct AAMs, it furnishes no guidance in the choice of weights. Weights should be expected to reflect the relative expertise of the individuals to which they are assigned, but such an observation barely escapes being a tautology. It is doubtful that appropriate methods of determining weights will be discovered by a priori analysis. Decisionmaking groups will have to experiment with different sorts of scoring rules and compare the outcomes of employing these rules. Computer simulation may prove to be a useful tool in this enterprise, and statistical decision theory may provide theoretical guidance, but the problem of determining weights is, at the core, an empirical one.

3. APPLICATIONS

3.1. *Lehrer Models*

Keith Lehrer [7], [8] has proposed the following normative model of rational group decisionmaking: A group of n individuals is attempting to determine consensual values of k numerical decision variables. Their initial opinions are registered in an $n \times k$ matrix $A = (a_{ij})$, where a_{ij} is the value assigned by individual i to variable j. Failing consensus in A, the individuals attempt to find a consensual set of weights, nonnegative and summing to one, with which to average the entries in the columns of A. The result is an $n \times n$ matrix $W_1 = (w_{ij})$, where w_{ij} is the weight which individual i deems it most appropriate to assign to individual j. Failing consensus in W_1, the group seeks a consensual set of second order weights with which to average the entries of the columns of W_1. Opinions as to the most appropriate values of these second order weights are registered in a matrix W_2. In theory, this process might be iterated indefinitely. Significantly, the prospect of an infinite regress need not doom this model to failure, for under some weak conditions of respect among individuals, the group will converge to consensus regarding the first order weights [12], [13].

Using Theorem 4, we may identify a set of decisionmaking conditions from which Lehrer's model may be derived. They are:

(I) The original decision problem is an allocation problem involving at least three variables.

(II) There are at least three individuals.

(III) Failing consensus, the values assigned by these decision-makers to the initial variables are to be amalgamated by a method satisfying Z and IA.

(IV) These same decision-makers are responsible for determining the values of any auxiliary decision variables required as a consequence of satisfying condition (III), and, failing consensus, their opinions regarding the most appropriate values of such auxiliary variables are to be amalgamated by a method satisfying Z and IA.

The derivation of Lehrer's model from these four principles is straightforward. By (I), (III), and Theorem 4, opinions registered in A, failing consensus, must be amalgamated by weighted averaging with weights nonnegative and summing to one. Determination of these weights is therefore an allocation problem which, by (II), involves at least three first order auxiliary weight variables. By (IV), the evaluation of these variables is to be carried out, failing consensus, by amalgamating individual opinions on the matter by a method satisfying Z and IA. Hence, by Theorem 4, the group must seek a set of consensual second order weights, nonnegative and summing to one, with which to average opinions regarding the first order weights. Determination of these weights poses a further allocation problem involving at least three second order auxiliary decision variables. Thus by (IV) and Theorem 4, failing consensus on this issue, the group must seek consensual third order weights with which to average their opinions regarding values of the second order weights. Each time consensus fails the above conditions dictate the introduction of higher order auxiliary variables. Condition (IV) is essentially a 'looping' instruction. As stated, it specifies no limit on the number of higher order weight matrices constructed by the group, but as a practical matter, it might be supplemented by an instruction to stop the process when an acceptable approximation of consensus emerges, or when no such approximation has emerged at some predetermined level.

3.2. *The Consensus of Probabilities*

As noted in §1, the assignment of probabilities to a sequence of pairwise exclusive, exhaustive propositions is an allocation problem, with $s = 1$. Specializations of the theorems of §2 thus yield interesting results on the amalgamation of probability distributions of a set of individuals. In particular, if there are at least three propositions, and if an AAM assigns a probability to each proposition purely as a function of the probabilities assigned to that proposition by individuals, and respects their agreement in assigning a proposition the probability zero, then the AAM is based on weighted arithmetic averaging[5]. Stone [10] has called such probability amalgamation methods *opinion pools*. We conclude this section with a theorem which removes the finiteness restriction on the set of alternatives. We thus switch to a measure-theoretic point of view and consider the amalgamation of a sequence of probability measures on a fixed, possibly infinite, σ-algebra of events.

Let X be a nonempty set and $\mathscr{S}$ a σ-algebra on X. Let $\mathscr{P}(\mathscr{S})$ denote the set of probability measures on $\mathscr{S}$ and let $\mathscr{P}(\mathscr{S})^n = \{(p_1, \ldots, p_n)$ where $p_i \in \mathscr{P}(\mathscr{S})$, $i = 1, \ldots, n\}$. A *probability amalgamation method* (PAM) is a function $F: \mathscr{P}(\mathscr{S})^n \to \mathscr{P}(\mathscr{S})$. Clearly each sequence of weights, $w_1, \ldots, w_n$, nonnegative and summing to one, gives rise to a PAM by the rule: $F(p_1, \ldots, p_n) = p$, where, for all $A \in \mathscr{S}, p(A) = w_1 p_1(A) + \ldots + w_n p_n(A)$. Such PAMs have the property of determining the measure of each event purely as a function of the individual measures ascribed to that event, and independently of the label of that event. Indeed, with a single exception, if a PAM satisfies this property, then it must be based on weighted arithmetic averaging. In view of Theorem 5 we would of course not expect such a result to hold for 'small' σ-algebras such as $\mathscr{S}_A = \{\phi, A, X\text{-}A, X\}$, where A is a proper nonempty subset of X. Let us call a nontrivial σ-algebra on X *tertiary* if it is not equal to any $\mathscr{S}_A$. It is easy to see that $\mathscr{S}$ is tertiary if and only if it contains at least three nonempty pairwise disjoint events.[6] A modification of the proof of Theorem 4 yields a proof of the following theorem:

THEOREM 7. *Let $\mathscr{S}$ be a tertiary σ-algebra on X and let $F: \mathscr{P}(\mathscr{S})^n \to \mathscr{P}(\mathscr{S})$ be a PAM for which there exists a function H: $[0, 1]^n \to [0, 1]$ such that $F(p_1, \ldots, p_n) = p$, where for all $A \in \mathscr{S}, p(A) = H(p_1(A), \ldots, p_n(A))$. Then H is weighted arithmetic mean.*[7]

We remark in conclusion that for the trivial σ-algebra $\mathscr{S} = \{\phi, X\}$ every PAM is based on weighted arithmetic averaging, since $\mathscr{S}$ admits only one probability measure. As for the σ-algebras $\mathscr{S}_A = \{\phi, A, X\text{-}A, X\}$, it is easy to prove a result complementary to that of Theorem 7, where the function H must belong to the class described in Theorem 5.

4. TECHNICAL APPENDIX

THEOREM 1. *An AAM satisfies SLN if and only if it satisfies IA and LN.*

Proof. It is clear that SLN implies IA and LN. Suppose that F satisfies IA and LN and let $A, B \in \mathscr{A}(n, k; s)$ be such that $A_{j_1} = B_{j_2}$. Denote by B' the matrix resulting from the interchange of columns j_1 and j_2 of B. Then $A_{j_1} = B'_{j_1}$, and so $a_{j_1} = b'_{j_1}$ by IA. But by LN, $b'_{j_1} = b_{j_2}$. Hence $a_{j_1} = b_{j_2}$, as required.

THEOREM 2. *If $k \geqslant 3$, an AAM $F: \mathscr{A}(n, k; s) \to \mathscr{A}(k; s)$ satisfies SLN if it satisfies IA and Z.*

Proof. Let $A, B \in \mathscr{A}(n, k; s)$ be such that $A_{j_1} = B_{j_2}$, and let $[\alpha_1, \ldots, \alpha_n]^T$ denote the elements in column j_1 of A (identically, in column j_2 of B). Now choose an index j_3 different from both j_1 and j_2 and define matrices A' and B' as follows: Column j_1 of A' is $[\alpha_1, \ldots, \alpha_n]^T$; column j_3 of A' is $[s-\alpha_1, \ldots, s-\alpha_n]^T$; and all other columns of A' consist entirely of zeros. Column j_2 of B' is $[\alpha_1, \ldots, \alpha_n]^T$; column j_3 of B' is $[s-\alpha_1, \ldots, s-\alpha_n]^T$; and all other columns of B' consist entirely of zeros. By Z, F assigns the value zero to all variables corresponding to zero columns. Furthermore, the values assigned by F to all variables must sum to s for every matrix. Hence $a'_{j_1} = s - a'_{j_3}$ and $b'_{j_2} = s - b'_{j_3}$. But by IA, $a'_{j_3} = b'_{j_3}$, $a_{j_1} = a'_{j_1}$, and $b_{j_2} = b'_{j_2}$. Hence $a_{j_1} = b_{j_2}$, as required.

THEOREM 3. *Let $F: \mathscr{A}(n, k; s) \to \mathscr{A}(k; s)$ satisfy SLN. Then there exists a function $H: [0, s]^n \to [0, s]$ such that for all $A = (a_{ij}) \in \mathscr{A}(n, k; s)$, $F(A) = (a_1, \ldots, a_k)$, where $a_j = H(a_{1j}, \ldots, a_{nj})$, $j = 1, \ldots, k$.*

Proof. Let $(\alpha_1, \ldots, \alpha_n) \in [0, s]^n$. Let A be a matrix whose first column is $[\alpha_1, \ldots, \alpha_n]^T$, whose second column is $[s-\alpha_1, \ldots, s-\alpha_n]^T$, and whose remaining columns, if any, consist entirely of zeros. Set $H(\alpha_1, \ldots, \alpha_n) = a_1$, the value assigned by F to x_1. It follows immediately by the invariance property SLN that H has the desired property.

THEOREM 4. *If $k \geqslant 3$, an AAM $F: \mathscr{A}(n, k; s) \to \mathscr{A}(k; s)$ satisfies IA and Z if and only if there exists a sequence of weights $w_1, \ldots, w_n$, nonnegative and summing to 1, such for all $A = (a_{ij}) \in \mathscr{A}(n, k; s)$, $F(A) = (a_1, \ldots, a_k)$, where $a_j = w_1 a_{1j} + \ldots + w_n a_{nj}$.*

Proof. It is easy to check that an AAM based on weighted arithmetic averaging satisfies IA and Z. Suppose, conversely, that F satisfies IA and Z. By Theorem 2, F satisfies SLN, and hence by Theorem 3 there exists a function $H: [0, s]^n \to [0, s]$ such that for all $A = (a_{ij}) \in \mathscr{A}(n, k; s)$, $F(A) = (a_1, \ldots, a_k)$, where $a_j = H(a_{1j}, \ldots, a_{nj})$, $j = 1, \ldots, k$. Thus we need only show that H is a weighted arithmetic mean. Let $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$ be members of $[0, s]^n$ such that $(\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$ is also a member of $[0, s]^n$, and define matrices A and $B \in \mathscr{A}(n, k; s)$ as follows: The first column of A is $[\alpha_1, \ldots, \alpha_n]^T$; the second column of A is $[\beta_1, \ldots, \beta_n]^T$; the third column of A is $[s - \alpha_1 - \beta_1, \ldots, s - \alpha_n - \beta_n]^T$; and all other columns of A, if any, consist entirely of zeros. The first column of B is $[\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n]^T$; the third column of B is $[s - \alpha_1 - \beta_1, \ldots, s - \alpha_n - \beta_n]^T$; and all other columns of B consist entirely of zeros.

By property $Z, H(0, \ldots, 0) = 0$. Thus, considering the action of H on the columns of A and B, we see that $H(\alpha_1, \ldots, \alpha_n) + H(\beta_1, \ldots, \beta_n) + H(s - \alpha_1 - \beta_1, \ldots, s - \alpha_n - \beta_n) = s$, and $H(\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n) + H(s - \alpha_1 - \beta_1, \ldots, s - \alpha_n - \beta_n) = s$. Hence H satisfies the multivariable Cauchy equation $H(\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n) = H(\alpha_1, \ldots, \alpha_n) + H(\beta_1, \ldots, \beta_n)$, where $\alpha_i, \beta_i, \alpha_i + \beta_i \in [0, s]$, $i = 1, \ldots, n$. It follows as a corollary to a well known theorem on functional equations[8] that there exists a set of real weights $w_1, \ldots, w_n$ such that $H(\alpha_1, \ldots, \alpha_n) = w_1\alpha_1 + \ldots + w_n\alpha_n$ for all $(\alpha_1, \ldots, \alpha_n) \in [0, s]^n$. It is trivial to show that these weights are nonnegative and sum to one.

THEOREM 5. *An AAM* $F: \mathscr{A}(n, 2; s) \to \mathscr{A}(2; s)$ *satisfies SLN and Z if and only if there is a function* $h: [-s/2, s/2]^n \to [-s/2, s/2]$, *where*

(1) *h is odd* $(h(-\alpha_1, \ldots, -\alpha_n) = -h(\alpha_1, \ldots, \alpha_n))$, *and*

(2) $h(s/2, \ldots, s/2) = s/2$,

such that for all $A = (a_{ij}) \in \mathscr{A}(n, 2; s)$ $F(A) = (a_1, a_2)$, where $a_j = h(a_1 - s/2, \ldots, a_{nj} - s/2) + s/2, j = 1, 2$.

Proof. It is easy to see that functions h satisfying (1) and (2) yield AAMs satisfying SLN and Z. Conversely, by Theorem 3 it follows from SLN that there exists a function H: $[0, s]^n \to [0, s]$ such that for all $A = (a_{ij}) \in \mathscr{A}(n, 2; s)$, $F(A) = (a_1, a_2)$, where $a_j = H(a_{1j}, \ldots, a_{nj}), j = 1, 2$. By Z,

(i) $H(0, \ldots, 0) = 0$.

By considering the action of F on a matrix A whose first column is $[\alpha_1, \ldots, \alpha_n]^T$ and whose second column is $[s - \alpha_1, \ldots, s - \alpha_n]^T$, we see that

(ii) $H(\alpha_1, \ldots, \alpha_n) + H(s - \alpha_1, \ldots, s - \alpha_n) = s$.

It is easy to check (See Aczél, Kannappen and Ng [2]) that a function H satisfies (i) and (ii) if and only if $H(\alpha_1, \ldots, \alpha_n) = h(\alpha_1 - s/2, \ldots, \alpha_n - s/2) + s/2$, where h satisfies conditions (1) and (2) specified in the present theorem.

THEOREM 7. *Let $\mathscr{S}$ be a tertiary σ-algebra on X and let F: $\mathscr{P}(\mathscr{S})^n \rightarrow \mathscr{P}(\mathscr{S})$ be a PAM for which there exists a function H: $[0, 1]^n \rightarrow [0, 1]$ such that $F(p_1, \ldots, p_n) = p$, where for all $A \in \mathscr{S}, p(A) = H(p_1(A), \ldots, p_n(A))$. Then H is a weighted arithmetic mean.*

Proof. We show that H satisfies the multivariable Cauchy equation $H(\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n) = H(\alpha_1, \ldots, \alpha_n) + H(\beta_1, \ldots, \beta_n)$, where α_i, β_i, $\alpha_i + \beta_i \in [0, 1]$, $i = 1, \ldots, n$, from which it follows, as in the proof of Theorem 4, that H is a weighted arithmetic mean. Let A_1, A_2, and A_3 be a sequence of nonempty pairwise disjoint events in $\mathscr{S}$, and choose a sequence x_1, x_2, x_3 of elements of X with $x_k \in A_k$, $k = 1, 2, 3$. Given numbers $\alpha_i, \beta_i \in [0, 1]$ with $\alpha_i + \beta_i \in [0, 1]$, $i = 1, \ldots, n$, define a sequence of probability measures (p_i), $i = 1, \ldots, n$, on $\mathscr{S}$ by $p_i(A) = \alpha_i I_1(A) + \beta_i I_2(A) + (1 - \alpha_i - \beta_i) I_3(A)$, where, for $k = 1, 2, 3$, $I_k(A) = 1$ if $x_k \in A$ and $I_k(A) = 0$ if $x_k \notin A$. Then $p_i(A_1) = \alpha_i$, $p_i(A_2) = \beta_i$ and $p_i(A_1 \cup A_2) = \alpha_i + \beta_i$, $i = 1, \ldots, n$.

Suppose that $F(p_1, \ldots, p_n) = p$, where for all $A \in \mathscr{S}$, $p(A) = H(p_1(A), \ldots, p_n(A))$. Since p is a probability measure $p(A_1 \cup A_2) = p(A_1) + p(A_2)$, and so $H(\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n) = H(\alpha_1, \ldots, \alpha_n) + H(\beta_1, \ldots, \beta_n)$, as required.

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NOTES

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[1] By way of contrast, the aggregation of individual orderings or utilities into consensual orderings, has been the subject of extensive axiomatic analysis by social choice theorists. See [1], [4], [6], and [11].

[2] This example is due to Aczél, Kannappan, and Ng [2].

[3] Our condition IA plays a role similar to that of Arrow's classical 'independence of irrelevant alternatives', but there are substantial differences in the structure and strength of the two conditions. Arrow's condition is binary, requiring the consensual ordering of any two alternatives to depend only on the individual orderings of these alternatives or, in recent formulations, on the utilities assigned to just these alternatives.

[4] Other averaging functions, such as geometric and harmonic means, and medians, also fail in general, without subsequent normalization, to yield consensual values of the variables which sum to s. And normalization violates IA. See Aczél and Wagner [3].

[5] An example of an aggregation method violating these conditons is the 'parimutuel method' of Eisenberg and Gale [5]. It should be noted that these authors have characterized parimutuel consensus as 'somewhat 'pathological.''

[6] Equivalently, $\mathscr{S}$ is tertiary if and only if it contains at least two non-empty disjoint events whose union is a proper subset of X.

[7] For economy of exposition we have postulated the existence of the function H among the hypotheses of Theorem 7. Alternatively, we might have postulated an invariance condition analogous to SLN.

The present theorem is related to some interesting recent results of McConway [9], which deal with the class of amalgamation methods for the class of *all* σ-algebras on a set X. McConway exhibits necessary and sufficient conditions for the amalgamation method of each class to be based on the same weighted arithmetic mean. As McConway points out, to the extent that different σ-algebras represent different assessment situations, employing the same weights in all of these situations may be undesirable. Our Theorem 7, which deals with a fixed σ-algebra, is a localized version of McConway's theorem which escapes the aforementioned difficulty.

It should be noted that when the probability measures in question come from a class of natural conjugate distributions, there may be good reasons for employing an amalgamation method not based on arithmetic averaging. See Winkler [14].

[8] See Aczél, Kannappan, and Ng [2] or McConway [9] for a proof of this corollary, based on the classical single variable theorem.

REFERENCES

[1] Arrow, K. J.: 1963, 'Social Choice and Individual Values', (John Wiley and Sons, New York).

[2] Aczél, J., Kannappan, P., and Ng, C. T.: 'Rational Group Decision-Making Revisited: A More Natural Characterization of Arithmetic Means', to appear.

[3] Aczél, J. and Wagner, C.: 1980, 'A Characterization of Weighted Arithmetic Means', SIAM *Journal of Algebraic and Discrete Methods* 1, 259–260.

[4] d'Aspremont, Claude and Gevers, Louis: 1977, 'Equity and the Informational Basis of Collective Choice', *Review of Economic Studies* 46, 199–210.

[5] Eisenberg, E. and Gale, D.: 1959, 'Consensus of Subjective Probabilities: The Pari-Mutuel Method', *Annals of Mathematical Statistics* 30, 165–168.

[6] Gevers, Louis: 1979, 'On Interpersonal Comparability and Social Welfare Orderings', *Econometrica* 47, 75–89.

[7] Lehrer, Keith: 1976, 'When Rational Disagreement is Impossible', *Noûs* 10, 327–332.

[8] Lehrer, Keith: 1978, 'Consensus and Comparison: A Theory of Social Rationality' in *Foundations and Applications of Decision Theory*, Hooker, C.A. *et al.* (eds.), (Reidel, Dordrecht, Holland), 283–309.
[9] McConway, K. J.: 'Marginalization and Linear Opinion Pools', to appear.
[10] Stone, M.: 1961, 'The Opinion Pool', *Annals of Mathematical Statistics* **32**, 1339–1342.
[11] Strasnick, Steven: 1979, 'Moral Structures and Axiomatic Theory', *Theory and Decision* **11**, 195–206.
[12] Wagner, Carl: 1978, 'Consensus Through Respect: A Model of Rational Group Decision-Making', *Philosophical Studies* **34**, 335–349.
[13] Wagner, Carl: 1980, 'The Formal Foundations of Lehrer's Theory of Consensus', in *Profile*: Keith Lehrer, Radu J. Bogdan, (ed.), (Reidel, Dordrecht, Holland).
[14] Winkler, Robert L.: 1968, 'The Consensus of Subjective Probability Distributions', *Management Science* **15**, B-61-B-75.