Efficacy, power and equity under approval voting

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Abstract

Approval voting allows each voter to vote for as many candidates as he wishes in a multicandidate election. Previous studies show that approval voting compares favorably with other practicable election systems. The present study examines the extent to which votes for different numbers of candidates can affect the outcome. It also considers generic powers of voters and the extent to which approval voting treats voters equitably.

If there are three candidates, votes for one or two candidates are equally efficacious in large electorates. For four or more candidates, votes for about half the candidates are most efficacious. Although inequities among voters can arise under approval voting, the common plurality voting system is considerably less equitable than approval voting.

An election by approval voting among three or more candidates allows each voter to vote for as many candidates as he wishes. The candidate with the most votes wins the election. Approval voting has recently been axiomatized [6, 7] and compared to other voting systems [2, 3, 5, 6, 8, 11, 13, 14, 19, 20, 21]. The impression given by these comparisons is that approval voting is competitive with and often superior to other systems. We believe it deserves serious consideration as an alternative to extant election systems.

The present study examines aspects of approval voting that have not been analyzed previously. It addresses the following questions:

1. To what extent does a voter's ability to affect the outcome of an approval voting election depend on how many candidates he votes for?
2. To what extent does a voter's 'power' depend on his specific preferences for the candidates?
3. To what extent does approval voting treat voters equitably?

The answers to these questions have an obvious bearing on the acceptability of approval voting. If it were true that votes for different numbers of

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Public Choice 37: 425-434 (1981) 0048-5829/81/0373-0425 $01.50.

candidates had significantly different abilities to affect the outcome, or if some voters were seriously disadvantaged because of their preference structures, then approval voting could seem less attractive than has been supposed by some advocates. Several people have also expressed concern about how approval voting would affect who enters an election and how it would influence candidates' strategies. Although we do not address this concern, it surely deserves examination.

We shall assume that the number $m \geq 3$ of candidates under consideration refers only to serious contenders; fringe candidates who have no chance of winning will be disregarded. In addition, we shall focus on large electorates and presume that voters' preferences are more or less evenly distributed over the different preference orders on the m candidates. If a final winner is assumed to be chosen randomly from a set of two or more candidates who are tied with the largest vote total, then our results lend themselves to a probabilistic interpretation. In effect, these results can be viewed as average-effects answers to the foregoing questions.

When there are $m = 3$ candidates, we shall show that (1) voting for either one or two candidates is equally efficacious, (2) all voters are equally powerful, whatever their preferences, and (3) voters are treated equitably under approval voting. However, when $m \geq 4$, (1) voting for about half the candidates is most efficacious, (2) voters' relative powers depend on their preferences, and (3) nontrivial inequities exist among voters. We shall also argue that, whereas the plurality (vote for one) system grants equal power to every voter, it is more inequitable than approval voting among voters with different types of preferences.

We should note that our analysis of inequity is confined to dichotomous voters, who divide the candidates into two indifference classes. This constraint is relaxed in [9], where it is shown that, even when more general preference orders are considered, plurality voting tends to be less equitable than approval voting.

1. Efficacy

A voter strategy S is a proper subset of candidates; a voter uses S when he votes for every candidate in S and does not vote for any other candidate. We exclude the set of all m candidates as a strategy since its effect on the outcome is the same as the effect of an abstention. The *outcome* of an election is the subset of candidates who have the largest vote total.

To assess the relative abilities of different strategies to affect the outcome in an approval voting election, we first define a measure $p(A, B)$ of the difference between two potential outcomes, or subsets of candidates, A and B:

$$p(A, B) = 1 - \frac{|A \bigcap B|}{|A \bigcup B|}.$$

Here $|X|$ is the number of candidates in X. This measure goes from 0, when the two outcomes are identical, to 1 when they are disjoint.

Consider a focal voter, with the votes of all other voters fixed. Let A be the outcome when the focal voter abstains, and let B be the outcome when he uses strategy S. By using S instead of abstaining, the focal voter might either create new tied winners (making B the larger set: $A \subset B$) or break old ties (making B the smaller set: $B \subset A$), but he can affect the outcome in no other way. If we assume that each candidate in an outcome has an equal chance of winning (random tie-breaking), then it is not difficult to prove that, given the fixed votes of voters other than the focal voter,

$$\begin{aligned} p(A, B) = \quad & Pr \text{ (an } x \in S \text{ wins when the focal voter uses } S) \\ & - Pr \text{ (an } x \in S \text{ wins when the focal voter abstains).} \end{aligned}$$

Thus, under random tie-breaking, $p(A, B)$ is the amount by which the focal voter's vote for S *increases* the probability that some candidate in S will win.

The efficacy of strategy S for our focal voter will be defined on the basis of equal weights for all combinations of other voters' votes. We shall let m and n be, respectively, the number of candidates ($m \geq 3$) and the number of other voters who do not abstain. Since there are $2^m - 2$ nonempty proper subsets of candidates, each other voter has $2^m - 2$ nonabstention strategies to choose from. Thus there are

$$H(m, n) = (2^m - 2)^n$$

possible ways that the others might vote. We index these by $h = = 1, 2, \ldots, H(m, n)$, and for each h let $A_h(S)$ and $B_h(S)$ be, respectively, the outcomes that obtain when the focal voter abstains and when he uses S. The *efficacy of strategy* S is then defined as the average value of $p(A_h(S), B_h(S))$ over all h.

Because the ways that others might vote are weighted equally, every S that contains the same number of candidates has the same efficacy. Therefore, when S contains k candidates, the efficacy of S is given by

$$E_k(m, n) = \sum_{h=1}^{H(m,n)} p(A_h(S), B_h(S))/H(m, n).$$

It is easily seen that $E_k(m, n) = 0$ if and only if $k = 0$; only the abstention strategy has zero efficacy. Moreover, since the focal voter can never produce a totally different outcome by not abstaining ($A_h(S) \bigcap B_h(S)$ is never empty),

$p(A_h(S), B_h(S)) < 1$ for all h, and therefore the efficacy of a nonabstention strategy is always strictly between 0 and 1.

As defined here, $E_k(m, n)$ is simply the average change that the focal voter can effect by voting for k candidates rather than abstaining. However, if we assume both that ties are broken randomly and that each of the n other voters votes independently and has probability $1/(2^m - 2)$ of using each of the $2^m - 2$ nonabstention strategies, then

$$E_k(m, n) = Pr \text{ (an } x \in S \text{ wins when the focal voter uses } S) - Pr \text{ (an } x \in S \text{ wins when the focal voter abstains).}$$

However one interprets $E_k(m, n)$, it must approach zero for all fixed k and m as n gets large since the proportion of h values for which $p(A_h(S), B_h(S)) > 0$ approaches zero as n gets large. In other words, one voter has almost no ability to change the outcome by his vote in a large electorate. This does not, however, say anything about the *relative* abilities of votes for different numbers of candidates to affect the outcome. Hence, to assess relative efficacies and provide an answer to our first question, we consider ratios of efficacies, defined by

$$r_k^j(m, n) = \frac{E_j(m, n)}{E_k(m, n)}$$

for $j, k \in \{1, 2, \ldots, m - 1\}$. The limit of this ratio as the number of voters gets large is

$$r_k^j(m) = \lim_{n \to \infty} r_k^j(m, n).$$

Although we shall not prove it here, it is not hard to show that $r_2^1(3, n) > 1$ for all n. Hence, in a three-candidate election, a vote for one candidate is more efficacious than a vote for two candidates, regardless of the number of other voters. (With three other voters, for example, $E_1(3, 3) = 177/648$, $E_2(3, 3) = 135/648$, and $r_2^1(3, 3) = 1.31$.) However, as a consequence of the ensuing theorem, $r_2^1(3) = 1$, so that votes for one candidate and for two candidates are approximately equally efficacious in a large electorate. Thus, in elections involving three candidates and more than several voters, an individual who votes for two candidates has practically the same ability to affect the outcome (small as it may be) as an individual who votes for one candidate.

The general limit result for $m \geq 3$ is given by the following theorem. Its proof is outlined in the Appendix.

Theorem:

$$r_k^j(m) = j(m-j)/[k(m-k)] \text{ for all } m \geq 3$$
$$\text{and all } j, k \in \{1, \ldots, m-1\}.$$

Thus, for large electorates and $m \geq 4$, efficacy is essentially single-peaked and symmetric about $m/2$, *where* $j(m-j)$ is maximized. Values of $|S|$ that are equidistant from $m/2$ have approximately equal efficacies, and values of $|S|$ closest to $m/2$ have the largest efficacies.

To view this in a slightly different way, suppose m and n are fixed with n large. Let $c(m, n)$ equal $E_1(m, n)/(m-1)$. Then, since $E_k(m, n)/E_1(m, n)$ is approximately equal to $k(m-k)/(m-1)$, the Theorem says that

$$E_k(m, n) = k(m-k)c(m, n) \quad \text{for} \quad k = 1, \ldots, m-1$$

with negligible error when n is large.

To illustrate this result, when $m = 4$, one-candidate and three-candidate strategies are about equally efficacious, whereas two-candidate strategies are about 30 percent more efficacious than the others. For $m = 5$, the two-candidate and three-candidate strategies are about 50 percent more efficacious than the one-candidate and four-candidate strategies.

2. Power and equity

Although a voter's power has been conceived of in various ways, we shall focus here on the usual conception of power as a measure of the effect of an individual's vote on the outcome of an election [1, 10, 16, 17]. Our notion of power in approval voting will therefore tie into efficacy, but it will also account for the individual's preferences, as recommended in [15].

When a voter votes for k of m candidates and there are n other voters, we define his *power* as simply the efficacy $E_k(m, n)$ of his voting strategy. To refine this definition to take account of preferences, we consider the effect of preferences on strategy selection. This refinement is developed more fully in [9] and will only be outlined here.

We shall assume that ties in outcome are broken randomly, and that each of the n other voters votes independently with probability $1/(2^m - 2)$ for each of the nonabstention strategies. We suppose further that the focal voter's preferences are characterized by a von Neumann-Morgenstern [4, 16] utility function u on the m candidates, and that he votes to maximize his expected utility. Let $\bar{u}$ denote his average utility – the sum of the $u(x)$ divided by m – and assume that n is large. Then, as shown in [9], the voter will vote for every candidate whose utility exceeds $\bar{u}$, so that strategy $S = \{x: u(x) > \bar{u}\}$ maximizes his expected utility. This conclusion about an optimal strategy has been obtained also by Hoffman [11], Merrill [13], and Weber [19]. Hoffman's derivation is similar to ours, while Merrill and Weber use different approaches.

Under the assumptions of the preceding paragraph, the power of a voter for whom exactly k candidates have utilities for him that exceed his $\bar{u}$ is $E_k(m, n)$. Because $E_1(m, n) \approx E_2(m, n)$ when $m = 3$ and n is large, voters are equally powerful in three-candidate elections. Significant power differentials occur, however, for different k when $m \geq 4$.

The notions of efficacy and power that we have developed provide a basis for answering our question concerning equity among voters. As noted earlier, we shall consider only dichotomous voters in addressing the equity issue. A more general analysis appears in [9].

A dichotomous voter partitions the candidates into nonempty subsets M and L such that he is indifferent among all candidates in M, indifferent among all candidates in L, and prefers every candidate in M to every candidate in L. The preceding analysis with $\bar{u}$ points to M as a dichotomous voter's optimal strategy; an even stronger case for M as his uniquely best strategy has been made in [2]. Hence, a dichotomous voter's power will be $E_{|M|}(m, n)$.

We shall say that two voters are *treated equally* if and only if their expected gains from voting optimally are equal. To operationalize this definition for dichotomous voters, we shall again adopt the independent-voters model with equiprobable strategy choices along with random tie-breaking. We suppose further that, as a normalizing convention, a dichotomous voter receives a utility of 1 if one of his more-preferred candidates is elected, and a utility of 0 if one of his less-preferred candidates is elected. Then the increase or gain in expected utility that accrues from the vote of a dichotomous voter is simply $E_{|M|}(m, n)$.

It follows that, when n is large, all dichotomous voters are treated approximately equally by approval voting when there are three candidates. However, when $m \geq 4$, nontrivial inequities can arise since voters whose sets of more-preferred candidates contain about $m/2$ candidates stand to gain more than dichotomous voters whose values of $|M|$ are nearer to 1 or to $m - 1$.

3. Plurality comparison

The foregoing analysis shows that approval voting is inherently inequitable among certain types of voters when there are four or more serious contenders. This inequity can also arise for other simple voting systems. In particular, we shall now argue that probably the most commonly used multicandidate voting procedure, namely plurality voting, is more inequitable than approval voting.

Since each voter can vote for only $k = 1$ candidate under plurality voting, there is only one efficacy number for each (m, n) under this system. We denote this number as $E(m, n)$. It equals the average value of $p(A_h, B_h)$ over all ways that others can vote under the plurality system, where A_h is the outcome when the focal voter abstains and B_h is the outcome when he votes for a specific candidate. If we assume that ties are broken randomly and each other voter votes independently with probability $1/m$ for each nonabstention strategy, then

$$E(m, n) = \quad Pr\,(x \text{ wins when the focal voter votes for } x) - Pr\,(x \text{ wins when the focal voter abstains}).$$

The latter probability equals $1/m$ according to the symmetry assumptions.

By analogy with our approval-voting discussion, it is evident that every voter in a plurality-voting election has the same power, namely $E(m, n)$. To consider the equity issue for dichotomous voters, we suppose as before that a dichotomous voter gets a utility of 1 if one of his more-preferred candidates is elected and a utility of 0 otherwise. Such a voter maximizes his expected utility by voting for a more-preferred candidate, say x. Then, under the symmetry assumptions of the preceding paragraph, his vote will increase x's chance of election by $E(m, n)$ and decrease the chance of election for each of the other $m - 1$ candidates by $E(m, n)/(m - 1)$. Since there are $|M| - 1$ candidates besides x in the voter's preferred subset M, his gain in expected utility by voting instead of abstaining is

$$E(m, n) - (|M| - 1)E(m, n)/(m - 1) = \frac{(m - |M|)E(m, n)}{m - 1}.$$

Clearly, a dichotomous voter's expected-utility gain under plurality voting is very sensitive to $|M|$, and inequities arise in this case even when $m = 3$. For $m = 5$, a dichotomous voter with a single more-preferred candidate ($|M| = 1$) stands to gain four times as much as a dichotomous voter with four more-preferred candidates ($|M| = 4$).

To compare approval voting and plurality voting directly, suppose n is large. Then, under approval voting, the ratio of the largest expected-utility gain that (at $|M| = (m - 1)/2$ for odd m, and $|M| = m/2$ for even m) to the smallest expected-utility gain (at $|M| = 1$ or $m - 1$) for dichotomous voters is

$$\frac{m + 1}{4} \quad \text{for odd } m,$$

$$\frac{m^2}{4(m - 1)} \quad \text{for even } m.$$

Under plurality voting, the ratio of the largest (at $|M| = 1$) to smallest (at $|M| = m - 1$) expected-utility gain for dichotomous voters is $m - 1$. These ratios for m from 3 to 10 are:

m:	3	4	5	6	7	8	9	10
Approval voting:	1	1.33	1.5	1.8	2	2.29	2.5	2.78
Plurality voting:	2	3	4	5	6	7	8	9

According to the ratios, inequities under plurality voting are much greater than those under approval voting for dichotomous voters. Although there are other ways to compare inequities [12], it seems reasonable to presume that other measures will yield a similar conclusion.

4. Conclusions

The efficacy of a voting strategy is a measure of that strategy's ability, on average, to change the outcome of an election from what it would be if the voter in question abstained. For large electorates and approval voting, the efficacies of votes for one and for two candidates in a three-candidate election are approximately equal. However, for $m \geq 4$, larger efficacies result from votes for about half the candidates, with votes for 1 and $m - 1$ candidates having the least ability to affect the outcome.

When a voter's power is conceived of as the efficacy of his utility-maximizing strategy, it follows for approval voting that voters with different preferences or utilities can have different generic powers, provided $m \geq 4$. When $m = 3$ with a large electorate, all voters are equally powerful.

Given that other voters are equally likely to choose any nonabstention strategy, that ties in outcomes are broken randomly, that the focal voter's utilities are scaled from 0 to 1, and that he is dichotomous with k candidates in his more-preferred subset, his gain in expected utility from voting optimally rather than abstaining equals his power. We envision the relative equity between two dichotomous voters as the ratio of their expected-utility gains when they vote optimally. Approval voting treats such voters equitably when $m = 3$ and the electorate is large, but not when $m \geq 4$.

When the latter result is compared to a similarly defined equity notion for plurality voting, it is seen that plurality voting is much more inequitable across dichotomous voters than is approval voting. Elsewhere [9], a similar but less pronounced trend is noted for general preference orders.

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Appendix

This appendix outlines the proof that $r_k^j(m) = j(m-j)/[k(m-k)]$. We let S be a fixed set of k candidates with efficacy $E_k(m, n)$ and consider the nonzero terms in the sum that defines this efficacy value.

The only nonzero terms involve cases in which the n other voters have two or more candidates (the 'contenders') within one vote of each other, with all other candidates two or more votes behind the leader. Let E denote a generic event that includes all such cases in which a specified set of contenders have fixed vote differences (0, +1 or −1) with respect to each other, and let $P(E)$ be the probability of E under the symmetry assumption for other voters' choices of strategies. For two such events, say E and F, $P(E)/P(F) \to 0$ as $n \to \infty$ if E has more candidates in contention than does F. This follows from the central limit theorem, which says that the limiting distribution of $(y_1, \ldots, y_m)$ – where $y_i = (\bar{x}_1 - 1/2)\sqrt{n}$, and $\bar{x}_i$ is the number of votes for candidate i, divided by n – is m-variate normal with mean $(0, 0, \ldots, 0)$ and correlation matrix ρ for which $\rho_{ii} = 1$ and $\rho_{ij} = -2(2^m - 2)$ for $i \neq j$. For large n, the probability of having $k + 1$ 'nearly tied winners' is very small compared to the probability of having k 'nearly tied winners', and the ratio of these probabilities vanishes as $n \to \infty$.

Call E a *prime* event if it has exactly two candidates in contention. Then the ratio of $E_k(m, n)$ to the sum of its terms that are involved in prime events approaches 1 as n gets large. Moreover, p is either 0 or $\frac{1}{2}$ in the prime event cases. Exactly two types of prime events have $p = \frac{1}{2}$. The first has contenders x and y with $x \in S$, $y \notin S$ and x and y tied. We denote such an event as E_0. Since x can be any one of k candidates, and y can be any one of $m - k$ candidates, there are $k(m - k)E_0$-type events, all of which have the same probability $P(E_0)$ by the symmetry assumption. The second type of prime event with $p = \frac{1}{2}$ has contenders x and y with $y \in S$, $y \notin S$, and y one vote ahead of x. We denote such an event as E_1. There are $k(m - k)E_1$-type events, all of which have the same probability $P(E_1)$. It follows that

$$E_k(m, n) \sim \tfrac{1}{2}k(m - k)[P(E_0) + P(E_1)],$$

where $\sim$ means that the ratio of the two sides approaches 1 as $n \to \infty$. Consequently, $E_j(m, n)/E_k(m, n) \to j(m - j)/[k(m - k)]$ as $n \to \infty$.