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POSITIONALIST VOTING FUNCTIONS*

ABSTRACT. Positionalist voting functions are those social choice functions where the positions of the alternatives in the voter's preference orders crucially influence the social ordering of the alternatives. An important subclass consists of those voting functions where numbers are assigned to the alternatives in the preference orders and the social ordering is computed from these numbers. Such voting functions are called representable. Various well-known conditions for voting functions are introduced and it is investigated which representable voting functions satisfy these conditions. It is shown that no representable voting function satisfies the Condorcet criterion. This condition and Arrow's independence condition, which are typical non-positionalist conditions, are shown to be incompatible. The Borda function, which is a well-known positionalist voting function, is studied extensively, conditions uniquely characterizing it are given and some modifications of the function are investigated.

1. PROGRAM

In the theory of social choice as it has developed after Arrow, one can roughly distinguish two approaches to the problem of finding good social choice functions: firstly, the positionalist view, which supports the Borda function and similar voting functions, where the positions of an alternative in the preference orders play a crucial role for social choice; secondly, the non-positionalist view, where the social ordering is determined mainly from binary comparisons between the alternatives. Typical non-positionalist voting functions are those based on majority decisions.

Various conditions for voting functions have been proposed to insure that the admissible voting functions agree with the one or the other view. The best known non-positionalist conditions are Arrow's independence condition and the Condorcet condition.[1] Voting functions satisfying the Condorcet condition are favored by politicians and political scientists, since these functions seem to be the most natural extensions of the majority rule for two alternatives.

Bulky volumes have been written on Condorcet functions, where the so-called voting paradox (the term 'paradox' stresses the influence of the majority rule) is possible, and on how to avoid it. Devoted to these problems, very small efforts have been spared by theorists for positionalist vot-

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ing functions. At best, the Borda function figures as an example contrasted with the Condorcet functions. However, the fact that the majority rule is used in two-alternative voting does not imply that some majority method should be used when voting on several alternatives. In fact, all voting methods proposed as reasonable reduce to the majority rule when restricted to two alternatives.

In this paper we turn our attention to what we call positionalist voting functions. A very cheering fact is that we can prove that Arrow's independence condition and the Condorcet condition are incompatible. This can be interpreted as showing that an unrestricted non-positionalist view is inconsistent in itself.

Although the positionalist concept is somewhat vague, a typical positionalist voting function is a method where we assign numbers to the alternatives according to their positions in the preference orders and then determine the social ordering from the sum of these numbers for each alternative. Voting functions which can be defined in this manner we call representable. When placing certain conditions on the function, some of the number-assignments will be impossible. We investigate the effect of various well-known conditions. Remarkably, we find that there is no representable voting function which satisfies the Condorcet condition. We discuss to some extent the Borda function, which is the most famous positionalist voting function. We also state that a set of conditions uniquely determines the Borda function. The final section is devoted to an investigation of some modifications of the Borda function, one of which turns out to be a Condorcet function!

2. Preliminaries

The objects we study are voting functions. The arguments and values of voting functions are preference orders. In this section we will define these and other underlying concepts.

We use A to denote the set of *alternatives*. The set of voters we denote by V and its elements are called *persons* (or *individuals*). A and V are supposed to be non-empty and finite. We will use $x_1, x_2, \ldots x_n$ to denote the alternatives (for the reader's convenience sometimes also x, y, z and u) and p_1, $p_2, \ldots p_m$ to denote the persons. Here n means the number of alternatives and m the number of persons (sometimes also written $|A|$ and $|V|$ respectively).

Each member of V is required to arrange all alternatives according to his preferences. For each p_i, this yields an ordering R_i of A, which we call an *individual preference order*. From each preference order R (xRy intuitively meaning that x is at least as good as y) we define two other relations – strict preference and indifference – as follows:

DEFINITION 2.1. xPy iff xRy & $-(yRx)$

DEFINITION 2.2. xIy iff xRy & yRx

We assume the preference orders to be total weak orders, i.e. to fulfil the following axioms:

AXIOM R1. $xRy \vee yRx$

AXIOM R2. xRy & $yRz \rightarrow xRz$

It follows that R is reflexive, that P is irreflexive, asymmetric and transitive and that I is an equivalence relation. P's and I's derived from a certain R will always carry the same subscript as that R.

Two distinct alternatives x and y such that xIy are said to be in a *tie* in the preference order R. A preference order R without ties, i.e. an order where xPy or yPx holds for every pair of distinct alternatives, is called a *linear order*.

A set consisting of one preference order for each person will be called a *situation*. We use **a**, **b**, **c** etc. to range over situations. The set of all preference orders (for a given A) we denote D and so the set of all situations will be D^m. Thus we may regard a situation as an m-tuple of preference orders.

DEFINITION 2.3. A *voting function* is a function from D^m to D.

The individual preference order for p_i in situation **a** will be denoted $R_{i\mathbf{a}}$, and the value of a voting function F, which we call the *social preference order*, will be denoted $F(\mathbf{a})$ or $R_{\mathbf{a}}$ (when there is no risk of confusion).

Our voting functions take a situation as argument and have a preference order as their value. This is the kind of voting functions which Arrow [1] calls 'social welfare functions'. Other writers[2] use another kind of voting functions which have as their value a subset of A interpreted as the set of winning alternatives. These functions are sometimes called 'decision functions'. Here we shall not exploit the advantages of either treatment, but simply hold to the voting functions. It is, however, convenient to distinguish the set of top-ranked alternatives in any preference order.

DEFINITION 2.4. $T(R) = \{x: \forall y(xRy)\}$

We describe situations in the following manner:

$$
\begin{array}{ll}
\mathbf{a}: & 1.\ x_1, x_2, (x_3, x_4) \\
 & 2.\ x_2, (x_1, x_3, x_4) \\
 & 3.\ x_2, x_1, x_4, x_3 \\
\hline
 & R_{\mathbf{a}}.\ (x_1, x_2), x_4, x_3
\end{array}
$$

In this situation we have four alternatives and three persons, and the meaning of the scheme is that e.g. p_1 has the following preference order: $x_1 P_{1\mathbf{a}} x_2$, $x_2 P_{1\mathbf{a}} x_3$ and $x_3 I_{1\mathbf{a}} x_4$ (consequently $x_1 P_{1\mathbf{a}} x_3$, $x_1 P_{1\mathbf{a}} x_4$ and $x_2 P_{1\mathbf{a}} x_4$). We also have $T(R_{1\mathbf{a}}) = \{x_1\}$. $R_{\mathbf{a}}$ represents the social preference order (for an assumed voting function) which in this example is described by $x_1 I_{\mathbf{a}} x_2$, $x_1 P_{\mathbf{a}} x_4$ and $x_4 P_{\mathbf{a}} x_3$. Hence, $T(R_{\mathbf{a}}) = \{x_1, x_2\}$.

3. Representable voting functions

If one attempts to let the positions of an alternative in the individual preference orders have a crucial influence on the social ordering, a very natural way is to provide some kind of utility measure of the positions. Our way to do this is to assign a number to each alternative in a given preference order and then for each alternative compute the sum of the numbers attached to it in a given situation. The social preference order is then determined according to the magnitude of these sums. A well-known voting function, which is representable in this manner, is the Borda function. Although the class of positionalist voting functions is not precisely defined, the class of representable voting functions certainly is an important subclass of it.

Definition 3.1. A *representation function* is a real-valued function having $D \times A$ as domain.

A representation function has two arguments; the first is a preference order and the second is an alternative. We will denote the value of a representation function f for arguments $R_{i\mathbf{a}}$ and x_j by $f(R_{i\mathbf{a}}, x_j)$ and this is according to the definition a real number. As shorthand for $\sum_i f(R_{i\mathbf{a}}, x_j)$ we will use $f_{\mathbf{a}}(x_j)$.

Definition 3.2. A voting function F is *representable* iff there exists a representation function f such that $x R_{\mathbf{a}} y$ iff $f_{\mathbf{a}}(x) \geqslant f_{\mathbf{a}}(y)$.

Our definition of a representable voting function is not the most general one that can be imagined, so we will discuss some possible extensions in the sequel.

A representable voting function to which we will devote some attention is the Borda function, characterized here by the number of alternatives that p_i thinks are worse than x, minus the number of alternatives that p_i prefers to x:

DEFINITION 3.3. A representation function f defines the *Borda function* iff $f(R_{i\mathbf{a}},x)=|\{y:xP_{i\mathbf{a}}y\}|-|\{y:yP_{i\mathbf{a}}x\}|$. For any given person p_i, situation **a** and position x, $f(R_{i\mathbf{a}},x)$ is called their *Borda number*. For further discussion, cf. sections 6 and 7.

Measuring the 'value' of an alternative by a representation function is, in one sense, *imposed* by the function and independent of the situation in which the preference order occurs[3] and, above all, also independent of the person for whom the order is given. Completely different problems arise if we allow the voters to *choose* the numbers to be assigned to the alternatives according to their preference intensities. This cannot be coped with within our theory, since to determine the result of such a voting method we must use more information than what we get solely from the preference orders.

Under the influence of majority decision methods, political scientists dissociate themselves from group decision methods where the individual preference intensities have any influence on the result. Often they fail to observe the distinction between the methods based on numbers chosen by the voters and those based on imposed numbers. Thus, the Borda function has been unjustly accused of being a (stupid) preference intensity amalgamating method.

Returning to the representable voting functions, we note that the different voters have the same influence on the social ordering. Voting functions with this property are called symmetric.

DEFINITION 3.4. Suppose situation **a** is like situation **b**, except that $R_{i\mathbf{a}}=R_{j\mathbf{b}}$ and $R_{i\mathbf{b}}=R_{j\mathbf{a}}$ for some p_i and p_j. A voting function satisfies the *symmetry* condition iff $R_{\mathbf{a}}=R_{\mathbf{b}}$.

The meaning of this condition (which we denote by S) is that all the opinions expressed by the individuals are of equal worth. It is sometimes referred to as the egalitarian principle. So, what is of interest is not who it is that votes for a certain alternative, but how many.

A more general approach in the definition of a representable voting function would have been to determine $f_{\mathbf{a}}(x)$ as a weighted sum, where the weight given to any individual is decided e.g. on basis of his position in the community or his number of stocks. Instead of symmetry, various questions concerning non-dictatorship arise. However, we will not pursue this theme.

For a given alternative, the numbers assigned to it in a situation can be regarded as the components of a vector. Then the social choice is determined as that alternative whose vector has the greatest summation norm. Another possible extension of our definition of a representable voting function could be that we not merely use the summation norm, but also exploit other vector norms. If we temporarily change the definition of $f_{\mathbf{a}}(x)$ to mean instead $\sum_i f(R_{i\mathbf{a}}, x)^k$, where k is some integer greater than 1, we will still get the same class of representable voting functions as before, since we can obtain all the old voting functions simply by a change of representation functions.

4. Non-positionalist conditions and representability

Typical non-positionalist conditions are formulated with the help of pairwise comparisons of the alternatives, disregarding their absolute positions. Arrow's independence condition and the Condorcet condition are the most familiar examples of non-positionalist conditions. We devote this section to a study of the compatibility of these conditions with representability.

Definition 4.1. A voting function satisfies the *weak Condorcet* condition (WC) iff the fact that an alternative x has a strict simple majority over all other alternatives in the situation $\mathbf{a}$ implies that $T(R_{\mathbf{a}}) = \{x\}$.

Definition 4.2. A voting function satisfies the *strong Condorcet* condition (SC) iff the fact that there is any alternative x such that x has a simple majority over or gets equally many votes against all other alternatives in the situation $\mathbf{a}$ implies that $T(R_{\mathbf{a}})$ is the set of all such alternatives.

Voting functions which satisfy any Condorcet condition are worshipped by politicians. Almost every decision procedure used in parliaments is based on majority decisions. Majority votings work well when we have only two alternatives and this is probably the reason why politicians cling to Condorcet functions for several alternatives. The Condorcet criterion is not without disadvantages and malign tongues have called it 'the principle of oppression by majority' since it allows an apathetic majority to defeat an engaged minority.

DEFINITION 4.3. Suppose situations **a** and **b** coincide as regards the orders between alternatives in a subset B of A. Then a voting function F satisfies *Arrow's independence* condition (AI) iff $F(\mathbf{a})$ and $F(\mathbf{b})$ coincide within B.

This is the famous 'independence of irrelevant alternatives' as proposed in Arrow's monograph [1]. A weaker independence condition has been proposed and studied by Hansson [7].

DEFINITION 4.4. Suppose situations **a** and **b** coincide as regards the preference relations between x and the other alternatives. Then a voting function F satisfies the *weak independence* condition (WI) iff $F(\mathbf{a})$ and $F(\mathbf{b})$ coincide as regards x's relations to the other alternatives.

We now proceed to examine the relations between the representable voting functions and the Condorcet conditions.

THEOREM 4.1.[4] There is no representable voting function which satisfies the weak Condorcet function, if $|A| \geqslant 3$ and $|V| \geqslant 5$.

Proof. We prove the theorem in the case where $|A|=3$ and $|V|=5$. Obvious additions of alternatives and individuals will prove the theorem in an analogous manner for the rest of the cases. Consider the six possible linear orders with three alternatives, $R_1: x, y, z$; $R_2: x, z, y$; $R_3: y, x, z$; $R_4: y, z, x$; $R_5: z, x, y$ and $R_6: z, y, x$. For an arbitrary representation function f, suppose $f(R_1, x) - f(R_1, y) = a_1$, $f(R_1, y) - f(R_1, z) = b_1$, $f(R_2, x) - f(R_2, z) = a_2$, $f(R_2, z) - f(R_2, y) = b_2$ etc. Consider the following situation:

a:
1. x, y, z
2. x, y, z
3. $x. y, z$
4. y, z, x
5. y, z, x

If we assume WC, we have $T(R_{\mathbf{a}}) = \{x\}$. Turning to the representation function, we conclude that $3 \cdot a_1 > 2 \cdot a_4 + 2 \cdot b_4$. If we now permute x and y we get the following situation:

b:
1. y, x, z
2. y, x, z
3. y, x, z
4. x, z, y
5. x, z, y

A similar argument gives $3 \cdot a_3 > 2 \cdot a_2 + 2 \cdot b_2$. Treating the remaining four permutations of the alternatives in a similar manner yields the following equations:

$$3 \cdot a_2 > 2 \cdot a_6 + 2 \cdot b_6$$
$$3 \cdot a_4 > 2 \cdot a_5 + 2 \cdot b_5$$
$$3 \cdot a_5 > 2 \cdot a_1 + 2 \cdot b_5$$
$$3 \cdot a_6 > 2 \cdot a_3 + 2 \cdot b_3$$

Adding these inequalities together gives $3 \cdot \Sigma a_i > 2 \cdot \Sigma a_i + 2 \cdot \Sigma b_i$, which reduces to $\Sigma a_i > 2 \cdot \Sigma b_i$. Now consider the following situation:

c:
1. x, y, z
2. x, y, z
3. z, y, x
4. z, y, x
5. y, x, z

According to WC we have $T(R_{\mathbf{c}}) = \{y\}$, i.e. $a_3 + 2 \cdot b_6 > 2 \cdot a_1$. Permuting and calculating as before, we derive $\Sigma a_i + 2 \cdot \Sigma b_i > 2 \cdot \Sigma a_i$, i.e. $2 \cdot \Sigma b_i > \Sigma a_i$. This contradicts the previous inequality, so there is no representable voting function which also satisfies WC.

THEOREM 4.2. There is no representable voting function which satisfies the weak Condorcet condition, if $|A| \geqslant 4$ and $|V| \geqslant 3$.

Proof-sketch. Consider the following situation:

a:
1. x, y, z, u
2. z, x, y, u
3. y, u, x, z

According to WC, $T(R_{\mathbf{a}}) = \{x\}$. Proceed as in the proof of Theorem 4.1, diligently permuting and calculating.

For the cases where $|A| = 3$ and $|V| = 3$ or $|V| = 4$ we cannot derive a contradiction since e.g. assignments of 4, 2 and 1 respectively to linear orders with three alternatives are perfectly compatible with the weak Condorcet condition.

We omit the proof of the following theorem, since it is tedious and similar to that of Theorem 4.1.

THEOREM 4.3. There is no representable voting function which satisfies the strong Condorcet condition if $|A| \geqslant 3$ and $|V| \geqslant 3$.

As regards Arrow's independence condition versus the representable voting functions, we first note that all representation functions which assign a fixed number to an alternative independent of the preference order yield voting functions which satisfy AI. A less trivial example arises from the following representation function: For two special alternatives x and y, we have $f(R_i, x)=2$ iff xP_iy, $f(R_i, y)=2$ iff yP_ix, $f(R_i, x)=$ $=f(R_i, y)=1$ iff xI_iy and otherwise $f(R_i, x)=f(R_i, y)=0$. For any alternative z, distinct from x and y, we always have $f(R_i, z)=0$.

If we assume neutrality, these examples do not work, but then we can apply a theorem of Hansson's[5] (where we don't even need representability, but only symmetry) to show that even the function which for every situation always yields ties satisfies AI and is representable.

We have found that in all the interesting cases there is no representable voting function satisfying AI or WC.

These conditions are the clearest expressions of the non-positionalist view. Arrow's independence condition is explicitly against the view that the positions of an alternative will have a direct influence on the social ordering. Applying the Condorcet conditions, one completely ignores the positions of an alternative and finds the optimal alternatives by pairwise majority votings. We will now prove the somewhat striking result that the non-positionalist view is inconsistent in itself, or more exactly stated: no voting function (operating on at least three alternatives) satisfies both AI and WC. It seems as if Arrow was not aware of this fact, but rather thought that the conditions were compatible. He mentions[6] the connection between the conditions as follows:

> ... the chief contribution has been what might be termed the Condorcet criterion, that a candidate who receives a majority against each other candidate should be elected. This implicitly accepts the view of what I have termed the independence of irrelevant alternatives.

In fact, we can prove an even stronger result than that AI and WC are incompatible.

THEOREM 4.4.[7] There is no voting function which satisfies the weak independence condition and the weak Condorcet condition, if $|A|\geqslant 3$ and $|V|\geqslant 3$.

Proof. Consider the following situation:

a: 1. $x_1, x_2, (x_3, x_4 \ldots x_n)$
2. $x_2, (x_3, x_4 \ldots x_n), x_1$

3. $(x_3, x_4 \ldots x_n), x_1, x_2$
4.-m. $(x_1, x_2 \ldots x_n)$

$T(R_{\mathbf{a}})$ must be a non-empty set (it is of course possible that it contains more than one element). We examine three cases:

(i) $x_1 \in T(R_{\mathbf{a}})$. Then compare situation **a** with the following:

b:
1. as in **a**
2. $x_3, x_2, (x_4, x_5 \ldots x_n), x_1$
3. as in **a**
4.-m. as in **a**

This situation satisfies what is needed for the application of WI as regards x_1. So by this condition, $x_1 \in T(R_{\mathbf{a}})$ iff $x_1 \in T(R_{\mathbf{b}})$. On the other hand, x_3 now has a strict majority over the rest of the alternatives, so WC demands that $T(R_{\mathbf{b}}) = \{x_3\}$. This contradiction shows that $x_1 \notin T(R_{\mathbf{a}})$.

(ii) $x_2 \in T(R_{\mathbf{a}})$. Compare situation (a) with the following:

c:
1. as in **a**
2. as in **a**
3. $x_1, (x_3, x_4 \ldots x_n), x_2$
4.-m. as in **a**

For the same reasons as before, WI demands $x_2 \in T(R_{\mathbf{a}})$ iff $x_2 \in T(R_{\mathbf{c}})$, while WC demands $T(R_{\mathbf{c}}) = \{x_1\}$. Hence, $x_2 \notin T(R_{\mathbf{a}})$

(iii) Some of the alternatives x_3, $x_4 \ldots x_n$, say x_i, belong to $T(R_{\mathbf{a}})$. Then compare **a** with the following situation:

d:
1. $x_2, x_1, (x_3, x_4 \ldots x_n)$
2. as in **a**
3. as in **a**
4.-m. as in **a**

As before, by WI, $x_i \in T(R_{\mathbf{a}})$ iff $x_i \in T(R_{\mathbf{d}})$ and by WC, $T(R_{\mathbf{d}}) = \{x_2\}$. Therefore, $x_i \notin T(R_{\mathbf{a}})$. We have now considered all posibilities for the top-ranked alternatives and in every case we reached a contradiction. This proves the theorem.

COROLLARY. There is no voting function satisfying both Arrow's independence condition and the weak Condorcet condition, if $|A| \geqslant 3$ and $|V| \geqslant 3$.

Notice that the situation **a** from which the proof starts is a variant of the voting paradox.

This theorem shows that an unrestrained non-positionalistic view is untenable. We take this as a good reason for turning to positionalist voting functions.

5. Various Conditions and Representability

In this section we introduce some conditions for voting functions, familiar from other publications, and examine the properties of the representation functions for representable voting functions satisfying these conditions.

5.1. *Neutrality*

Definition 5.1. Suppose situation **a** is like situation **b** except that the alternatives x and y have changed places in every individual preference order. A voting function F satisfies the *neutrality* condition (N) iff $F(\mathbf{a})$ is like $F(\mathbf{b})$ except that x and y have changed places.

In short, this means that no alternative is favored for other reasons than the preferences expressed by the individuals. Neutrality plays the same role for the alternatives that symmetry plays for the individuals.

Lemma 5.1. For any voting function satisfying neutrality and symmetry, the social relation is $xI_{\mathbf{a}}y$ in any situation of the following type:

a: $1-k.\ \ldots x, \ldots y, \ldots$
$(k+1)-2k.\ \ldots y, \ldots x, \ldots$
$(2k+1)-m.$ (any preference order with x and y in a tie)

(The dots indicate that the individual preference orders are similar as regards all other alternatives).

Proof. If we interchange x and y, we get a situation which has the same individual preference orders, but at different places in the situation. By S and N we can conclude $xR_{\mathbf{a}}y$ iff $yR_{\mathbf{a}}x$. Hence, $xI_{\mathbf{a}}y$.

Lemma 5.2. Suppose the preference orders $R_{1\mathbf{a}}$, $R_{2\mathbf{a}}$ and $R_{3\mathbf{a}}$ are identical, except that x, y and z are located in the orders as indicated in the following situation. For any voting function satisfying neutrality and symmetry we must have $xI_{\mathbf{a}}y$ and $yI_{\mathbf{a}}z$.[8]

a:
1. $\dots x, \dots y, \dots z, \dots$
2. $\dots y, \dots z, \dots x, \dots$
3. $\dots z, \dots x, \dots y, \dots$

4.-m. (any preference order with x, y and z in the same tie)

Proof. Compare **a** with the following situations:

b:
1. $\dots y, \dots x, \dots z, \dots$
2. $\dots x, \dots z, \dots y, \dots$
3. $\dots z, \dots y, \dots x, \dots$

4.-m. as in **a**

c:
1. $\dots y, \dots z, \dots x, \dots$
2. $\dots z, \dots x, \dots y, \dots$
3. $\dots x, \dots y, \dots z, \dots$

4.-m. as in **a**

Situation **b** has arisen from **a** by way of interchanging x and y. Likewise **c** is constructed from **b** by interchanging x and z. Suppose we have $xR_{\mathbf{a}}y$ and $yR_{\mathbf{a}}z$. Using N, we conclude $xR_{\mathbf{b}}z$. By N again, $zR_{\mathbf{c}}x$. Since **c** is like **a** except for a different numbering of the preference orders, we have by S that $zR_{\mathbf{a}}x$. From $xR_{\mathbf{a}}y$, $yR_{\mathbf{a}}z$ and $zR_{\mathbf{a}}x$, we derive $xI_{\mathbf{a}}y$ and $yI_{\mathbf{a}}z$. If we start with another supposition of the order $R_{\mathbf{a}}$, we will find exactly the same relations by an analogous proof.

A representable voting function need not be neutral. It may happen that a certain alternative, e.g. standing for the status quo, is assigned a greater number than is assigned to any other alternative (at the corresponding places in the preference orders). We now proceed to find out which representable voting functions are neutral.

DEFINITION 5.2. Two preference orders R_1 and R_2 *have the same frame* iff there exists a mapping φ from A onto A, such that xR_1y iff $\varphi(x)R_2\varphi(y)$ for all alternatives x and y.

LEMMA 5.3. (The Dalby lemma) Let F be a neutral representable voting function and f an arbitrary representation function for it. Suppose that for the preference order R_1 we have $f(R_1, x)-f(R_1, y)=a$ and $f(R_1, y)-f(R_1, z)=b$. Then for every preference order R_k, which has the same frame as R_1 (under the mapping φ) and where every alternative except x, y and z has the same place in the frame, it holds that $f(R_k, \varphi(x))-f(R_k, \varphi(y))=a$ and $f(R_k, \varphi(y))-f(R_k, \varphi(z))=b$.

Proof. There are three places in the frame at our disposal. The six possible arrangements of x, y and z in these places are R_1: x, y, z; R_2: x, z, y; R_3: y, x, z; R_4: y, z, x; R_5: z, x, y and R_6: z, y, x. Using the fact that a representable voting function is symmetric and Lemma 5.1, we derive that $f(R_1, y)+f(R_2, y)=f(R_1, z)+f(R_2, z)$. Hence $f(R_2, z)-$ $-f(R_2, y)=f(R_1, y)-f(R_1, z)=b$. Similarly, $f(R_3, y)-f(R_3, x)=a$ and $f(R_6, z)-f(R_6, x)=a+b$. If we now assume $f(R_2, x)-f(R_2, z)=c$ and $f(R_3, x)-f(R_3, z)=d$, we can by diligent use of Lemma 5.1 compute the remaining differences between the values of the representation function. The result is shown in Table I.

TABLE I

R_1.	x	(a)	y	(b)	z
R_2.	x	(c)	z	(b)	y
R_3.	y	(a)	x	(d)	z
R_4.	y	$(b+c-d)$	z	(d)	x
R_5.	z	(c)	x	$(a+d-c)$	y
R_6.	z	$(b+c-d)$	y	$(a+d-c)$	x

If we apply Lemma 5.2 on R_1, R_4 and R_5, using the situation described there, we get the following equations:

(1) $a+(a+d-c)=(c+b-d)+d$ (from the fact that xI_ay)

(2) $a+b=c+d$ (from the fact that xI_az)

Adding (1) and (2), we find that $a=c$ and hence also $b=d$. If we substitute these values in the table above, we see that the difference between the numbers assigned to the first and second alternatives is always a and similarly the difference for the second and third alternatives is b. The lemma is proved.

THEOREM 5.1. Let F be a representable voting function and f an arbitrary representation function for it. Then the following statements are equivalent:

(1) If R_1 is like R_2 except that x and y have changed places, then $f(R_1, x)-f(R_1, z)=f(R_2, y)-f(R_2, z)$, for any z distinct from x and y.

(2) If R_1 and R_2 have the same frame (under the mapping φ), then $f(R_1, x)-f(R_1, y)=f(R_2, \varphi(y)) -f R_2, \varphi(y))$ for any alternatives x and y.

(3) F is neutral.

Proof. (1) says that if R_2 arises from R_1 through a transposition of two alternatives, then the assignments to the positions in the preference orders shall be essentially the same. (2) says that if R_2 arises from R_1 through a permutation of the alternatives, then the assignments shall be essentially the same. Since any permutation is a composition of transpositions, (2) follows from (1). That (2) implies (3) is a direct consequence of the definitions. Finally, that (3) implies (1) follows from Lemma 5.3.

This theorem shows that the neutral representable voting functions have remarkably simple representation functions. One need only know the numbers assigned to one preference order of each frame to determine the voting function.

5.2. *Pareto-Optimality*

DEFINITION 5.3. Suppose $xP_{i\mathrm{a}}y$ for every p_i. A voting function satisfies *weak Pareto-optimality* (WP) iff $xP_{\mathrm{a}}y$.

DEFINITION 5.4. Suppose $xR_{i\mathrm{a}}y$ for every p_i and $xP_{j\mathrm{a}}y$ for some p_j. A voting function satisfies *strong Pareto-optimality* (SP) iff $xP_{\mathrm{a}}y$.

The meaning of this condition is that if everybody thinks x is better than y (or at least as good as), then this shall be the case in the social ordering too. The conditions are also referred to as unanimity conditions. Obviously, any voting function which satisfies SP also satisfies WP.

The following theorem shows the connection between Pareto-optimality and the representation functions.

THEOREM 5.2. Let F be a representable voting function and f an arbitrary representation function for it. Then the following statements are equivalent:

(1) xRy iff $f(R, x) \geqslant f(R, y)$ for any preference order R and any alternatives x and y.

(2) $F(\langle R, R, \ldots R\rangle) = R$ for any preference order R (the m-tuple is the situation).

They imply the following:

(3) F is strongly Pareto-optimal.

Furthermore, if F is neutral and weakly Pareto-optimal, then it satisfies (1) and (2).

Proof. That (1) and (2) are equivalent follows directly from the definitions. That (1) implies (3) is trivial. If F is neutral and xIy in the preference order R, we conclude that xIy in $F(\langle R, R, \ldots R\rangle)$. If F is weakly

Pareto-optimal and xPy in R, then xPy in $F(<R, R, \ldots R>)$. Thus (2) is satisfied and the theorem is proved.

So, if F is a representable and neutral voting function, it is weakly Pareto-optimal iff it is strongly Pareto-optimal.

5.3. *Monotonicity*

DEFINITION 5.5. Suppose situation **a** is like situation **b** except that, for some p_i and some x, there is a z for which either $xI_{i\mathbf{a}}z$ and $xP_{i\mathbf{b}}z$ or $zP_{i\mathbf{a}}x$ and $xR_{i\mathbf{b}}z$, and furthermore $xP_{i\mathbf{a}}y$ iff $xP_{i\mathbf{b}}y$ and $yP_{i\mathbf{a}}x$ iff $yP_{i\mathbf{b}}x$ for every $y \neq z$. A voting function satisfies the *monotonicity* condition (M) iff $xI_{\mathbf{a}}y$ implies $xR_{\mathbf{b}}y$ and $xP_{\mathbf{a}}y$ implies $xP_{\mathbf{b}}y$ for any alternative y. A voting function satisfies the *strong monotonicity* condition (SM) iff $xR_{\mathbf{a}}y$ implies $xP_{\mathbf{b}}y$ for any alternative y such that $yR_{i\mathbf{a}}z$ iff $yR_{i\mathbf{b}}$ and $zR_{i\mathbf{a}}y$ iff $zR_{i\mathbf{b}}y$.

In outline, this means that if p_i changes his mind in favour of x, then, according to monotonicity, x is not moved backwards in the social ordering, and furthermore, according to strong monotonicity, if x is tied with some alternatives in $R_{\mathbf{a}}$, x is ranked before these in $R_{\mathbf{b}}$. Thus, a voting function which is strongly monotonic avoids a large number of ties in the social preference order and generally such a function is very decisive. Decisiveness seems to be a desirable property of a voting function, but still some people regard the strong monotonicity condition as too strict.

LEMMA 5.4.[9] Any voting function which satisfies strong monotonicity and neutrality also satisfies strong Pareto-optimality.

Proof. In any situation **a** where $xI_{i\mathbf{a}}y$ for every p_i, we conclude by N that $xI_{\mathbf{a}}y$. So in any situation **b** where $xR_{i\mathbf{b}}y$ for every p_i and $xP_{j\mathbf{b}}y$ for some p_j, we can by iterated use of SM conclude that $xP_{\mathbf{b}}y$. Hence the voting function is strongly Pareto-optimal.

The effects of the monotonicity conditions as regards the representation functions are not easily grasped, but the next theorem shows some connections.

DEFINITION 5.6. A preference order R_2 is a *straightening* of R_1 iff xP_1y implies xP_2y, for every x and y, and there is some z such that zI_1x but zP_2x, for some x.

THEOREM 5.3. Let F be a representable voting function and f an arbitrary representation function for it. Then the following statements are equivalent:

(1) F is [strongly] monotonic.

(2) If situations **a** and **b** and alternatives x and y are as in Definition 5.5, $f(R_{i\mathbf{a}}, x) - f(R_{i\mathbf{a}}, y) \leqslant [<] f(R_{i\mathbf{b}}, x) - f(R_{i\mathbf{b}}, y)$.

If F is neutral, these statements imply:

(3) If R_2 is a straightening of R_1 such that xI_1z but xP_2z, then $f(R_2, x) - f(R_2, y) \geqslant [>] f(R_1, x) - f(R_1, y)$ for any $y \neq x$, and $f(R_2, z) - f(R_2, y) \leqslant [<] f(R_1, z) - f(R_1, y)$ for any $y \neq z$.

Proof. That (1) is equivalent to (2) is just a consequence of the definitions. Suppose R_2 is a straightening of R_1. According to Lemma 5.2 it is always possible to find a situation **a** involving R_1 such that $xI_{\mathbf{a}}y$. In a situation **b** where R_1 is replaced by R_2, we have $xR_{\mathbf{b}}y$ [$xP_{\mathbf{b}}y$] since we assume [strong] monotonicity. Hence $f(R_2, x) - f(R_2, y) \geqslant [>] f(R_1, x) - f(R_1, y)$. Similarly, using Lemma 5.2. again, it is always possible to find a situation **c** involving R_2 such that $zI_{\mathbf{c}}y$. In a situation **d** where R_2 is replaced by R_1, we have, by [strong] monotonicity, $yR_{\mathbf{d}}z$ [$yP_{\mathbf{d}}z$]. Hence $f(R_2, z) - f(R_2, y) \leqslant [<] f(R_1, z) - f(R_1, y)$.

6. Conditions Characterizing the Borda Function

The most familiar representable voting function is the Borda function, which will be the focus of our study in this section. The function was defined in Section 3, and before turning to the uniqueness theorems, we want to make some remarks on the properties of the function.

Borda's original definition[10] of the function was intended for linear preference orders only. Essentially, he proposed that the least preferred alternative (the last one in the preference order) should be assigned a rank of 0, the next to the last 1, the next again 2, etc. Our definition in Section 3 is a generalization that can deal with weak orders, too. As is easily proved, the procedures are equivalent when restricted to linear orders. On the other hand, Borda's original definition can be extended to weak orders if we add the provision that to each alternative in a tie is assigned the mean value of what they would have been assigned if they were ranked in a linear order. For example, the preference order $(x_1, x_2), x_3, (x_4, x_5, x_6)$ can be straightened to a linear order as e.g. $x_2, x_1, x_3, x_4, x_6, x_5$ with the Borda assignments 5, 4, 3, 2, 1 and 0. Hence x_1 and x_2 shall be assigned $1/2 \cdot (5+4)$, i.e. 4.5 points each and x_4, x_5 and x_6 shall be assigned

$1/3 \cdot (2+1+0)$, i.e. 1 point each (x_3 is still assigned 3 points). This procedure yields the same social ordering as the one defined in Section 3.

Summing up, every representation function for the Borda function is characterized by the following properties:

(i) The difference between the numbers assigned to any two consecutive alternatives in a linear order is a positive constant.

(ii) The number assigned to any alternative in a tie is determined as the mean value of the numbers assigned to the alternatives in a corresponding linear straightening.

Furthermore we note that, for the Borda numbers as defined in section 3, the sum $\sum_{x \in A} f_{\mathbf{a}}(x)$ is always zero, a fact which we will use in the sequel.

Before stating the uniqueness theorems, we informally describe two independence conditions as they ought to be according to the positionalistic view.

In outline, an independence condition tells us how much we can change an individual preference order without changing the social ordering of a certain subset B of the alternatives. AI, which is the strongest among the independence conditions introduced in this paper, allows us to move around any alternative, except those in B, as much as we want, still having the same social ordering as regards the alternatives in B. WI entails that we regard an individual preference order as consisting of three subgroups; those preferred to x, those considered equally good as x and those considered worse than x. In these subgroups, one may move around the alternatives as one likes (but not from one subgroup to another), without altering the social relationships involving x.

The intuitive idea behind a positionalist independence condition can be described in a similar manner. We regard an individual preference order as divided into five subgroups; those ranked before both x and y, those on a par with x, those between x and y, those on a par with y and those ranked after both x and y. We are now allowed to move around the alternatives within each subgroup freely, without changing the social relationship between x and y. The definition of this condition, hereafter called positionalist independence (PI), can be found in Hansson [7], p. 46 this issue.

A stronger condition comes up if one also demands that if an alternative is not tied to x or y or ranked between them, then it does not matter for the social relation between x and y whether it is ranked before or after both x

and y. The effects of this condition, which we call strong positionalist independence (SPI), can be described as for PI, but where the first and fifth subgroups coincide. A precise definition is given in Hansson [7], where also the relative strengths of the different independence conditions are investigated (p. 47 this issue).

THEOREM 6.1. Suppose F is a representable voting function which satisfies neutrality, weak Pareto-optimality and strong positionalist independence. Then for any situation which only contains linear orders, the social ordering is the same as for the Borda function.

Proof. Compare the following situations:

a:
1. $x_1, x_2, x_3 \dots x_n$
2. $x_2, x_1, x_3 \dots x_n$

3.-m. $(x_1, x_2 \dots x_n)$

b:
1. $x_3, x_4 \dots x_k, x_1, x_2, x_{k+1} \dots x_n$
2. as in **a**

3.-m. as in **a**

According to Lemma 5.1., it holds that $x_1 I_{\mathbf{a}} x_2$. Hence by SPI, $x_1 I_{\mathbf{b}} x_2$. For any representation function f for F it holds that $f(R_{2\mathbf{b}}, x_2) - f(R_{2\mathbf{b}}, x_1) = f(R_{1\mathbf{b}}, x_1) - f(R_{1\mathbf{b}}, x_2)$. Since $R_{1\mathbf{a}}$ has the same frame as $R_{1\mathbf{b}}$ and $R_{2\mathbf{b}}$, we can apply Theorem 5.1. We conclude that $f(R_{1\mathbf{a}}, x_1) - f(R_{1\mathbf{a}}, x_2) = f(R_{1\mathbf{a}}, x_k) - f(R_{1\mathbf{a}}, x_{k+1})$ for any $k < n$. So, the difference between the numbers assigned to any two consecutive alternatives is a constant, say c. From Theorem 5.2. and the fact that we assume WP, it follows that $c > 0$. Therefore, for linear orders the representation function is essentially the same as for the Borda function and the theorem is proved.

Suppose a voter changes his mind in favor of an alternative, but only slightly, e.g. straightens a tie to a strict preference. Then it seems reasonable to demand that there is no revolutionary change in the social preference order. This is the rationale behind the next condition, which demands that the 'effect' in the social order is not greater than the 'cause'.

DEFINITION 6.3. The situations **a** and **b** are *almost equal with respect to x* iff there is at most one p_i and at most one z such that $R_{i\mathbf{a}}$ is the same ordering as $R_{i\mathbf{b}}$ on the set $A - \{z\}$ and $zP_{i\mathbf{a}}x$ and $xI_{i\mathbf{b}}z$ or $xI_{i\mathbf{a}}z$ and $xP_{i\mathbf{b}}z$.

The intended meaning of this somewhat messy definition is that x is considered better in **b** than in **a** by p_i, but only by a minimal amount.

DEFINITION 6.4. A voting function satisfies the *stability* condition (ST) iff not both $yP_{\mathbf{a}}x$ and $xP_{\mathbf{b}}y$ for any $y \neq z$ (where z is as in Definition 6.3) whenever **a** and **b** are almost equal with respect to x.

In one sense, the stability condition is against decisiveness since it prevents radical changes in the social order. So it might be thought that stability and strong monotonicity are incompatible. This is not the case; but for all we know, there is only one reasonable voting function satisfying both these conditions, namely the Borda function. We take this as a good argument for the Borda function.

THEOREM 6.2. The only representable voting function satisfying neutrality, strong monotonicity and stability is the Borda function, if we assume that $|A| \geqslant 3$ and $|V| \geqslant 3$.

Proof omitted. It is by induction on the number of alternatives involved in ties: first it can be proved that any representation applied to a linear preference order yields the same result as the Borda function. Then the induction base is established for preference orders with only two alternatives and a single tie, first for $|A| \geqslant 3$ and $|V| \geqslant 4$, then for $|A| \geqslant 4$ and $|V| \geqslant 3$. The induction step is established by constructing combinations of situations like those used for proving the induction base and by applying SM, ST and Theorem 5.1 as done there. Finally, the case for $|A|=3$ and $|V|=3$ is taken and it is shown how the corresponding voting function must be the Borda function.

7. SOME MODIFICATIONS OF THE BORDA FUNCTION

We first define three variants of the Borda function which are representable. Borda's original definition was intended for linear preference orders only, and the voting function we call the Borda function is one possible extension of the definition to weak orders in general. There are, however, other possibilities.

DEFINITION 7.1. The *restricted Borda function* (RB) is the voting function which has $f(P_{i\mathbf{a}}, x) = |\{y : xP_{i\mathbf{a}}y\}|$ as one of its representation functions.

DEFINITION 7.2. Two alternatives x and y are at the same *ranking level* in the preference order R iff xIy. This is an equivalence relation and the equivalence classes are called *ranking levels.*

DEFINITION 7.3. The *ranking-level function* (RL) is the voting function which is determined by the representation function f, where $f(R_{i\mathbf{a}}, x)$ is the number of ranking levels ranked after x in $R_{i\mathbf{a}}$.

Both these modifications yield the same social ordering as the Borda function, when applied to situations containing only linear orders. In the Borda function, the alternatives in a tie were assigned the mean value of what they would have become in any straightening to a linear order, while in the restricted Borda function they are assigned the minimum. In the ranking level function the difference between the numbers assigned to any two consecutive levels is constant. In connection with Theorem 5.3 this shows that neither RB nor RL satisfy strong monotonicity.

DEFINITION 7.4. Let $f(R_{i\mathbf{a}}, x)$ be as for the Borda function in Section 3. The *squared Borda function* (SB) is the voting function which is determined by the representation function defined as $(f(R_{i\mathbf{a}}, x))^2$ if $f(R_{i\mathbf{a}}, x)$ is non-negative and $-(f(R_{i\mathbf{a}}, x))^2$ if $f(R_{i\mathbf{a}}, x)$ is negative.

This voting function is interesting because it satisfies positionalist independence and strong monotonicity, but not strong positionalist independence.

The next two modifications are not representable, as we will prove in the sequel, so we describe how to construct the social preference order.

DEFINITION 7.5. The social ordering of the *iterated Borda function* (IB) is determined as follows: First use the Borda function to determine a preliminary social preference order. Then, if there are any ties in this order, regard the individual preference orders between the alternatives in any tie as determining a new situation. Use the Borda method on this restricted situation to determine the ordering of the alternatives previously involved in a tie. Repeat this procedure until there is no change in the social ordering.

EXAMPLE 7.1. In the following situation, the procedure for the iterated Borda function works as follows:

a:
1. x, y, z
2. x, y, z
3. y, z, x

The Borda function yields the social ordering $(x, y), z$. Since there is a tie between x and y, we study the restricted situation

a′:
1. x, y
2. x, y
3. y, x

Applied to this situation, the Borda function yields the ordering xPy. There remain no ties, so the social ordering for the iterated Borda function in situation **a** is $R_{\mathbf{a}}$. x, y, z.

DEFINITION 7.6. The social ordering for the *elimination Borda function* (EB) is constructed in the following manner. First use the Borda function to determine a preliminary social ordering. The alternatives in the last ranking level will remain there in the final social ordering. The preference orders restricted to alternatives not in the last ranking level yield a new situation to which the Borda function is applied again. The last alternative in this new order will become the next to last in the final order. Repeat this procedure until there is not more than one ranking level in the social order.

EXAMPLE 7.2. We illustrate the procedure for the following situation:

a:
1. x, y, z, u
2. z, x, y, u
3. $y, u, (x, z)$

The social ordering according to the Borda function is y, x, z, u. Hence u is last in the final order for the elimination Borda function. When u is eliminated, the restricted situation is:

a′:
1. x, y, z
2. z, x, y
3. $y, (x, z)$

The result by the Borda function is x, y, z, so z is next to last in the final order. When z is also eliminated, the restricted situation becomes:

a″:
1. x, y
2. x, y
3. y, x

The result according to the Borda function is x, y. Hence the social ordering for the elimination Borda function in situation **a** is x, y, z, u.

We next introduce a condition, hitherto not discussed, which is of interest when comparing the different modifications of the Borda function here presented. In some sense, the condition expresses the principle that 'the worse something is, the better is its absence'.

DEFINITION 7.7. Suppose we get situation **b** from **a** by reversing all the individual preference orders. Then **b** is called the *dual* of **a**.

DEFINITION 7.8. A voting function satisfies the *duality* condition (DU) iff for any alternatives x and y and any situation **a**, $xR_{\mathbf{a}}y$ iff $yR_{\mathbf{b}}x$, where **b** is the dual of **a**.

We have defined five modifications of the Borda function. All of them fulfil symmetry, neutrality, strong Pareto-optimality and monotonicity. In Table II we show how they satisfy some other conditions.

TABLE II

	representable	SM	WC	PI	SPI	ST	DU
Borda	yes	yes	no	yes	yes	yes	yes
RB	yes	no	no	yes	yes	yes	no
RL	yes	no	no	no	no	yes	yes
SB	yes	yes	no	yes	no	no	yes
IB	no	yes	no	no	no	no	yes
EB	no	no	yes	no	no	no	no

Most of the facts stated in this table are easily verified but some need proof.

LEMMA 7.1. The iterated Borda function is not representable, if $|A|\geqslant 3$ and $|V|\geqslant 3$.

Proof. We prove the lemma for the case $|A|=3$ and $|V|=3$. This proof can easily be extended to a general proof. Suppose that IB can be represented by the representation function f. In the following situation we have three preference orders of the same frame and since IB is neutral we may apply Theorem 5.1.

a:
1. x, y, z
2. x, y, z
3. y, z, x

$R_{\mathbf{a}}$. x, y, z

Using the representation function, we conclude $f(R_{1\mathbf{a}}, x)+f(R_{2\mathbf{a}}, x)+$ $+f(R_{3\mathbf{a}}, x)>f(R_{1\mathbf{a}}, y)+f(R_{2\mathbf{a}}, y)+f(R_{3\mathbf{a}}, y)$. According to Theorem 5.1., this implies that $f(R_{1\mathbf{a}}, x)+f(R_{1\mathbf{a}}, z)>2\cdot f(R_{1\mathbf{a}}, y)$. Now compare with this situation:

b:
1. x, y, z
2. z, y, x
3. (x, y, z)

$R_{\mathbf{b}}$. (x, y, z)

Hence, $f(R_{1b}, x)+f(R_{2b}, x)=f(R_{1b}, y)+f(R_{2b}, y)$. According to Theorem 5.1, this implies $f(R_{1a}, x)+f(R_{1a}, z)=2\cdot f(R_{1a}, y)$ since $R_{1a}=R_{1b}$. This contradicts the previous inequality, so the assumption that IB is representable must be false.

Perhaps surprisingly, the elimination Borda function is a Condorcet function, as is proved in the following lemma.

LEMMA 7.2. The elimination Borda function satisfies the weak Condorcet condition.

Proof. Suppose x has a simple majority over all other alternatives in situation **a**. Taking the sum over all individual preference orders in **a**, we conclude that the number of alternatives ranked after x must be greater than the number of alternatives ranked before x, i.e. the Borda number $f_{\mathbf{a}}(x)$ is positive. Since the sum of the Borda numbers for all the alternatives is zero, there must be at least one alternative with a negative Borda number. So x will not be in the last ranking level. This holds also when some of the alternatives are eliminated from the preference orders since x will still be a Condorcet alternative. Hence, when the elimination procedure is completed only x will be top-ranked.

In fact, every alternative which should be top-ranked, according to the strong Condorcet condition, will be top-ranked, but there may be some other alternatives, too.

The elimination Borda function is not representable in general, as we know from Theorem 4.1. That EB does not satisfy PI can be seen from the following two situations:

a:	1. x, y, u, z	**b**:	1. as in **a**
	2. y, z, u, x		2. y, u, z, x
	3. z, x, u, y		3. as in **a**
	$R_{\mathbf{a}}$. $(x, y, z), u$		$R_{\mathbf{b}}$. $x, y, (z, u)$

Since $xI_{\mathbf{a}}y$, PI demands $xI_{\mathbf{b}}y$, which is false for EB.

We let this voting function, which is somewhat of a hybrid between positionalist and non-positionalist methods, be a final example which shows the difficulty of drawing a sharp line between the two views.

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NOTES

* My thanks are due to professor Bengt Hansson for encouragement and several helpful suggestions.

[1] For a definition of these conditions, see Section 4.

[2] See e.g. [5].

[3] In [5] ch. 17 a more general approach is discussed, where the numbers assigned are dependent on the situation.

[4] This theorem is due to Bengt Hansson.

[5] See [6]. For a definition of the neutrality condition, see Definition 5.1.

[6] In [1], pp. 94–95.

[7] The proof of this theorem was found in collaboration with Bengt Hansson.

[8] The situation we start from is again a variant of the voting paradox.

[9] This is also proved by Sen in [8].

[10] Originally in [3]. For a discussion of this paper see [2].

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