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A STOCHASTIC BEHAVIORAL MODEL AND A 'MICROSCOPIC' FOUNDATION OF EVOLUTIONARY GAME THEORY

ABSTRACT. A stochastic model is developed to describe behavioral changes by imitative pair interactions of individuals. 'Microscopic' assumptions on the specific form of the imitative processes lead to a stochastic version of the game dynamical equations, which means that the approximate mean value equations of these equations are the game dynamical equations of evolutionary game theory.

The stochastic version of the game dynamical equations allows the derivation of covariance equations. These should always be solved along with the ordinary game dynamical equations. On the one hand, the average behavior is affected by the covariances so that the game dynamical equations must be corrected for increasing covariances; otherwise they may become invalid in the course of time. On the other hand, the covariances are a measure of the reliability of game dynamical descriptions. An increase of the covariances beyond a critical value indicates a phase transition, i.e. a sudden change in the properties of the social system under consideration.

The applicability and use of the equations introduced are illustrated by computational results for the social self-organization of behavioral conventions.

KEY WORDS: Evolutionary game theory, behavioral model, imitative processes, self-organization of behavioral conventions, stochastic game theory, mean value equations, covariance equations, reliability of rate equations, expected strategy distribution, most probable strategy distribution.

1. INTRODUCTION

This paper treats a mathematical model of the temporal change of the proportions of individuals showing certain behavioral strategies. Models of this kind are of special interest for a *quantitative understanding* or *prognosis* of social developments. The description of the *competition* or *cooperation* in populations can be described by *game theoretical approaches* (cf. e.g. von Neumann and Morgenstern, 1994; Luce and Raiffa, 1957; Rapoport and Chammah, 1965; Axelrod, 1984). In order to cope with time-dependent problems the method of *iterated games* has been developed and has been in use for

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a long time. However, the *game dynamical equations* were discovered some years ago (Taylor and Jonker, 1978; Hofbauer *et al.*, 1979; Zeeman, 1980). These are ordinary differential equations which are related to the theory of evolution (Eigen, 1971; Fisher, 1930; Eigen and Schuster, 1979; Hofbauer and Sigmund, 1988; Feistel and Ebeling, 1989). Therefore, one can also speak of *evolutionary game theory*.

The game dynamical equations have the following advantages:

- They are continuous in time, which is more adequate for many problems.
- Ordinary differential equations are easier to handle than iterated formulations.
- Analytical results can be derived more easily (cf. e.g. Hofbauer and Sigmund, 1988; Helbing, 1992).

Up to now, there only exists a *'macroscopic'* foundation for the game dynamical equations, i.e. a derivation from a *collective* level of behavior (cf. Section 5.1). This paper presents a *'microscopic'* foundation, i.e. a derivation on the basis of the *individual* behavior. With this aim in view, we first develop a *stochastic* behavioral model for the following reasons:

- A stochastic model, i.e. a model that can describe random *fluctuations* of the quantities of interest and can cope with the fact that behavioral changes are not exactly predictable (which is a consequence of the 'freedom of decision-making').
- The phenomena appearing in the social system under consideration can be connected to the principles of *individual* behavior. As a consequence, processes on the 'macroscopic' (collective) level can be understood as effects of 'microscopic' (individual) interactions.
- The *probability of occurrence* can be calculated for each strategy. This is especially important for small social systems which are subject to large fluctuations (since they consist of only a few individuals).
- The stochastic model allows the derivation of *covariance equations* (cf. Section 4). Since the covariances influence the average temporal behavior, they are an essential criterion for the validity and reliability of behavioral descriptions using rate equations (which are actually approximate mean value equations). If the

covariances exceed a certain critical value, this indicates the occurrence of a *phase transition*, i.e. a sudden change of the properties of the social system under consideration.

Different stochastic methods have been developed for the description of systems that are subject to random fluctuations (cf. e.g. Gardiner, 1983; Weidlich and Haag, 1983; Helbing, 1992). One method is to delineate the temporal evolution of the *probability distribution* over the different possible states (which represent behavioral strategies here). This method is particularly suitable for an *'ensemble'* of similar systems or for frequently occurring processes. In the case of *discrete* states, the *master equation* has to be used, whereas in the case of *continuous* state variables the *Fokker–Planck equation* is normally preferred since it is easier to handle (Fokker, 1914; Planck, 1917). The Fokker–Planck equation can, to a good approximation, also be applied to systems with a large number of discrete states if state changes occur only between neighbouring states. It can be derived from the master equation by a *Kramers–Moyal expansion* (Kramers, 1940; Moyal, 1949), i.e. a second-order Taylor approximation.

Another method – the *Langevin equation* (1908) (or *stochastic differential equation)* – is applied to the description of the temporal evolution of *single* systems affected by fluctuations. This consists of a deterministic dynamical part, which delineates systematic state changes, and a stochastic fluctuation term, which reflects random state variations. The Langevin equation can be reformulated in terms of a Fokker–Planck equation and vice versa (if the fluctuations are Gaussian and δ-correlated, which is normally the case; cf. Stratonovich, 1963, 1967; Weidlich and Haag, 1983).

Although these methods have their origin in statistical physics, the application to *interdisciplinary* topics has a long and successful tradition, beginning with the work of Weidlich (1971, 1972), Haken (1975), Prigogine (1976), and Nicolis and Prigogine (1977). For social and economic processes, too, Fokker–Planck equation models (cf. e.g. Weidlich and Haag, 1983; Topol, 1991), as well as the master equation models (cf. e.g. Weidlich and Haag, 1983; Weidlich, 1991; Haag *et al.*, 1993; Weidlich and Braun, 1992) have been proposed. In this paper we develop a behavioral model on the basis of the master equation (Section 2). For this purpose we have to specify the *transition rates*, i.e. the *probabilities* per time unit with which changes of behavioral strategies take place. The transition rates can be decomposed into

- rates describing *spontaneous* strategy changes, and
- rates describing strategy changes due to *pair interactions* of individuals.

In the following we will restrict our considerations to *imitative* pair interactions which seem to be the most important ones (Helbing, 1994). By distinguishing several *subpopulations a*, one can take account of different *types* of behavior or different *groups* of individuals. In order to connect the stochastic behavioral model with the game dynamical equations, the transition rates have to be chosen in such a way that they depend on the *expected successes* of the behavioral strategies (cf. Section 3.2). The ordinary game dynamical equations are the *approximate mean value* equations of the stochastic behavioral model (cf. Section 5.2).

For the approximate mean value equations correction terms can be calculated. These depend on the covariances (of the numbers of individuals pursuing a certain strategy) (cf. Section 4.1.4). If one neglects these corrections, the game dynamical equations may lose their validity after some time. Calculation of the covariances allows one to determine the time interval during which the game dynamical descriptions are reliable (cf. Section 4.1.5).

The equations introduced are illustrated by computational results for the self-organization of a behavioral convention by a competition between two alternative, but equivalent strategies (cf. Sections 3.3 and 4). These results are relevant for economics in regard to the rivalry between similar products (Arthur, 1988, 1989; Hauk, 1994).

2. THE STOCHASTIC BEHAVIORAL MODEL

Suppose we consider a social system with N individuals. These individuals can be divided into A *subpopulations* a consisting of N_a individuals, i.e.

$$\sum_{a=1}^{A} N_a = N.$$

Subpopulations allow one to distinguish different social groups (e.g. blue collar and white collar workers) or different characteristic *types of behavior.* In the following we will assume that individuals of the same subpopulation (group) behave *cooperatively* due to their common interests, whereas individuals of different subpopulations (groups) do not do so, having *conflicting* interests.

The N_a individuals of each subpopulation a are distributed over several *states*

$$i \in \{1, \dots, S\}$$

which represent the alternative *(behavioral) strategies* of an individual. For the time being we assume that every individual is able to choose each of the S strategies, i.e. the same strategy set is available for each subpopulation. If the *occupation number* $n_i^a(t)$ denotes the number of individuals of subpopulation a who use strategy i at the time t, we have the relation

$$(1) \qquad \sum_{i=1}^{S} n_i^a(t) = N_a.$$

Let

$$\mathbf{n} := (n_1^1, \dots, n_i^a, \dots, n_S^A)$$

be the vector consisting of all occupation numbers n_i^a. This vector is called the *socioconfiguration* since it contains all information about the distribution of the N individuals over the states i. $P(\mathbf{n}, t)$ denotes the *probability* of finding the socioconfiguration $\mathbf{n}$ at the time t. This implies

$$0 \leqslant P(\mathbf{n}, t) \leqslant 1 \quad \text{and} \quad \sum_{n} P(\mathbf{n}, t) = 1.$$

If transitions from socioconfiguration $\mathbf{n}$ to $\mathbf{n}'$ occur with a probability of $P(\mathbf{n}', t + \Delta t | \mathbf{n}, t)$ during a short time interval Δt, we have a *(relative) transition rate* of

$$w(\mathbf{n}'|\mathbf{n}; t) := \lim_{\Delta t \to 0} \frac{P(\mathbf{n}', t + \Delta t | \mathbf{n}, t)}{\Delta t}.$$

The *absolute* transition rate of changes from $\mathbf{n}$ to $\mathbf{n}'$ is the product $w(\mathbf{n}'|\mathbf{n}; t)P(\mathbf{n}, t)$ of the probability $P(\mathbf{n}, t)$ that we have configuration $\mathbf{n}$ and the *relative* transition rate $w(\mathbf{n}'|\mathbf{n}; t)$ to $\mathbf{n}'$ given the configuration $\mathbf{n}$. Whereas the *inflow* into $\mathbf{n}$ is defined by the sum over all absolute transition rates of changes from an *arbitrary* configuration $\mathbf{n}'$ to $\mathbf{n}$, the *outflow* from $\mathbf{n}$ is determined by the sum over all absolute transition rates of changes from $\mathbf{n}$ to *another* configuration $\mathbf{n}'$. Since the temporal change of the probability $P(\mathbf{n}, t)$ is determined by the inflow into $\mathbf{n}$ reduced by the outflow from $\mathbf{n}$, we find the socalled *master equation*

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}P(\mathbf{n},t) &= \text{inflow into } \mathbf{n} - \text{outflow from } \mathbf{n} \\
&= \sum_{n'} w(\mathbf{n}|\mathbf{n}';t)P(\mathbf{n}',t) \\
&\quad - \sum_{n'} w(\mathbf{n}'|\mathbf{n};t)P(\mathbf{n},t) \qquad (2)
\end{aligned}
$$

(Pauli, 1928; Haken, 1979; Weidlich and Haag, 1983; Weidlich, 1991).

It will be assumed that two processes contribute to a change of the socioconfiguration $\mathbf{n}$:

- Individuals may change their strategy i spontaneously and independently of each other to another strategy i' with an *individual* transition rate $\hat{w}_a(i'|i;t)$. These changes correspond to transitions of the socioconfiguration from $\mathbf{n}$ to
$$\mathbf{n}^a_{i'i} := (n^1_1, \dots, (n^a_{i'} + 1), \dots, (n^a_i - 1), \dots, n^A_S)$$
with a *configurational* transition rate $w(\mathbf{n}^a_{i'i}|\mathbf{n};t) = n^a_i \hat{w}_a(i'|i;t)$ which is proportional to the number n^a_i of individuals who can change strategy i.
- An individual of subpopulation a may change the strategy from i to i' during a pair interaction with an individual of some subpopulation b who changes the strategy from j to j'. Let transitions of this kind occur with a probability $\hat{w}_{ab}(i',j'|i,j;t)$ per time unit. The corresponding change of the socioconfiguration from $\mathbf{n}$ to
$$
\begin{aligned}
\mathbf{n}^{ab}_{i'j'ij} := (&n^1_1, \dots, (n^a_{i'} + 1), \dots, (n^a_i - 1), \dots, \\
&(n^b_{j'} + 1), \dots, (n^b_j - 1), \dots, n^A_S)
\end{aligned}
$$
leads to a configurational transition rate $w(\mathbf{n}^{ab}_{i'j'ij}|\mathbf{n};t) = n^a_i n^b_j \hat{w}_{ab}(i',j'|i,j;t)$ which is proportional to the number $n^a_i n^b_j$

of possible pair interactions between individuals of subpopulations a and b who pursue strategy i and j, respectively. (Exactly speaking – in order to exclude self-interactions – $n_i^a n_i^a \hat{w}_{aa}(i', j'|i, i; t)$ has to be replaced by $n_i^a(n_i^a - 1)\hat{w}_{aa}(i', j'|i, i; t)$ if $\sum_{j'} \hat{w}_{aa}(i', j'|i, i; t) \ll \hat{w}_a(i'|i; t)$ is invalid and $P(\mathbf{n}, t)$ is not negligible where $n_i^a \gg 1$ is not fulfilled.)

The resulting configurational transition rate $w(\mathbf{n}'|\mathbf{n}; t)$ is given by

$$w(\mathbf{n}'|\mathbf{n};t) := \begin{cases} n_i^a \hat{w}_a(i'|i;t) & \text{if } \mathbf{n} = \mathbf{n}^a_{i'i} \\ n_i^a n_j^b \hat{w}_{ab}(i',j'|i,j;t) & \text{if } \mathbf{n}' = \mathbf{n}^{ab}_{i'j'ij} \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

As a consequence, the explicit form of Master Equation (2) is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} P(\mathbf{n}, t) = & \sum_{a,i,i'} [(n_{i'}^a + 1)\hat{w}_a(i|i';t)P(\mathbf{n}^a_{i'i}, t) - \\ & -n_i^a \hat{w}_a(i'|i;t)P(\mathbf{n},t)] + \\ & + \frac{1}{2} \sum_{a,i,i'} \sum_{b,j,j'} [(n_{i'}^a + 1)(n_{j'}^b + 1) \times \\ & \times \hat{w}_{ab}(i,j|i',j';t)P(\mathbf{n}^{ab}_{i'j'ij}, t) - \\ & -n_i^a n_j^b \hat{w}_{ab}(i',j'|i,j;t)P(\mathbf{n},t)] \end{aligned} \tag{4}$$

(cf. Helbing, 1992a).

We have here restricted our considerations to pair interactions, since they normally play the most significant role. Even in groups the most frequent interactions are alternating pair interactions – not always, but in many cases. In situations where simultaneous interactions between more than two individuals are essential (one example for this is *group pressure*), the above master equation must be extended with higher-order interaction terms. The corresponding procedure is discussed by Helbing (1992, 1992a).

3. STOCHASTIC VERSION OF THE GAME DYNAMICAL EQUATIONS

3.1. *Specification of the Transition Rates*

The pair interactions

$$i', j' \leftarrow i, j \tag{5}$$

of two individuals of subpopulations a and b who change their strategy from i and j to i' and j', respectively, can be classified into three different *kinds* of processes: imitative processes, avoidance processes, and compromising processes. These are discussed in detail and simulated in several publications (Helbing, 1992, 1992b, 1994). In the following we will focus on *imitative processes* (processes of persuasion) which describe the tendency to take over the strategy of another individual. These are of the special form

(6a) $\quad i, i \leftarrow i, j \quad (i \neq j),$

(6b) $\quad j, j \leftarrow i, j \quad (i \neq j).$

The corresponding pair interaction rates read

$$\begin{aligned}
&\text{(7a)} \quad \hat{w}_{ab}(i', j'|i, j; t) = \hat{\nu}_{ab} p_{ba}(i|j; t)\delta_{ii'}\delta_{ij'}(1 - \delta_{ij}) + \\
&\text{(7b)} \quad\quad + \hat{\nu}_{ab} p_{ab}(j|i; t)\delta_{jj'}\delta_{ji'}(1 - \delta_{ij}),
\end{aligned}$$

where the Kronecker symbol δ_{ij} is defined by

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

The factors $(1 - \delta_{ij})$ result from the constraint $i \neq j$, whereas factors of the form δ_{ij} correspond to conditions of the kind $i = j$ which follow by comparison of (6a) and (6b), respectively, with (5). The parameter

(8) $\quad \nu_{ab} := N_b \hat{\nu}_{ab}$

represents the *contact rate* between an individual of subpopulation a with individuals of subpopulation b. $p_{ab}(j|i; t)$ denotes the probability that an individual of subpopulation a will change the strategy from i to j during an imitative pair interaction with an individual of subpopulation b, i.e.

$$\sum_j p_{ab}(j|i; t) = 1.$$

For $j \neq i$ we will assume

(9) $\quad p_{ab}(j|i; t) := f_{ab} \hat{R}_a(j|i; t)$

where the parameter f_{ab} is a measure of the *frequency* of imitative pair interactions among individuals of subpopulation a when confronted with an individual of subpopulation b. $\hat{R}_a(j|i;t)$ is a measure of the *readiness* of individuals belonging to subpopulation a to change the strategy from i to j during a pair interaction.

3.2. *'Microcsopic' Foundation of Evolutionary Game Theory*

The problem of this section is to specify the frequency f_{ab} and the readiness $\hat{R}_a(j|i;t) \equiv \hat{R}_a(j|i;\mathbf{n};t)$ in an adequate way. For this we make the following assumptions:

- By experience each individual knows – at least approximately – the *expected success* of the strategy used. We will define the expected success of a strategy i for an individual of subpopulation a in interactions with other individuals by

$$(10) \quad \hat{E}_a(i,t) \equiv \hat{E}_a(i,\mathbf{n};t) := \sum_b \sum_j r_{ab} E_{ab}(i,j) \frac{n_j^b(t)}{N_b}.$$

  Here, the parameter

$$r_{ab} = \frac{\nu_{ab}}{\sum_c \nu_{ac}}$$

  represents the *relative contact rate* of an individual of subpopulation a with individuals of subpopulation b. $n_j^b(t)/N_b$ is the probability that an interaction partner of subpopulation b uses strategy j. $E_{ab}(i,j)$ is an exogenously given quantity that denotes the *success* of strategy i for an individual of subpopulation a during an interaction with an individual of subpopulation b who uses strategy j. Since all these quantities can be determined by each individual, the evaluation of the expected success $\hat{E}_a(i,t)$ is obviously possible.
- In interactions with individuals of the *same* subpopulations an individual tends to take over the strategy of another individual if the expected success would increase: When an individual who uses strategy i meets another individual of the same subpopulation who uses strategy j, they compare their expected successes $\hat{E}_a(i,t)$ and $\hat{E}_a(j,t)$, respectively, by exchange of their experiences. (Remember that individuals of the same subpopulation were assumed to cooperate.) The individual with strategy i will *imitate* the other's strategy j with a probability $p_{ab}(j|;t)$ that grows with the expected increase

$$\Delta_{ji}\hat{E}_a := \hat{E}_a(j,t) - \hat{E}_a(i,t)$$

of success. If a change of strategy would imply a *decrease* of success ($\Delta_{ji}\hat{E}_a < 0$), the individual will not change the strategy i. Therefore, the readiness to replace the strategy i by j during an interaction within the same subpopulation can be assumed to be

$$(11) \quad \hat{R}_a(j|t;t) := \max(\hat{E}_a(j,t) - \hat{E}_a(i,t), 0)$$

where $\max(x, y)$ is the maximum of the two numbers x and y. This describes an individual *optimization* or *learning process.*

- In interactions with individuals of *other* subpopulations (who behave in a non-cooperative way), no imitative processes will normally take place. During these interactions the expected success $\hat{E}_b(j, t)$ of the interaction partner can at best be *estimated* by *observation* since he will not tell of his experiences. Moreover, due to different criteria for the grade of success, the expected success of a strategy j will normally be varying with the subpopulation (i.e. $\hat{E}_a(i,t) \neq \hat{E}_b(i,t)$ for $a \neq b$). As a consequence, an imitation of the strategy of individuals belonging to *another* subpopulation would be very risky since it would probably be connected with a *decrease* of expected success. Hence the assumption

$$(12) \quad f_{ab} := \delta_{ab} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b. \end{cases}$$

will normally be justified.
Relation (12) also results in cases where the strategies of the respective other sub-populations *cannot* be imitated due to different (disjunct) strategy sets. Then, we need not to assume that individuals of the same subpopulations cooperate, whereas individuals of different subpopulations do not.

In Section 5 it will turn out that the game dynamical equations are the approximate mean value equations of the stochastic behavioral model defined by Expressions (7)–(12). In this sense, the model of this section can be regarded as stochastic version of the game dynamical equations. Moreover, the assumptions made above are a 'microscopic' foundation of evolutionary game theory since they allow a derivation of the game dynamical equations on the basis of individual behavior patterns.

3.3. *Self-Organization of Behavioral Conventions by Competition Between Strategies*

As an example of the stochastic game dynamical equations we will consider a case with one subpopulation only ($A = 1$). In this case we can omit the indices a, b, and the summation over b. Let us assume that the individuals choose between two *equivalent* strategies $i \in \{1, 2\}$, i.e. the *success matrix* $\mathbf{E} \equiv (E(i, j))$ is symmetric:

$$\mathbf{E} := \begin{pmatrix} B+C & B \\ B & B+C \end{pmatrix}. \tag{13}$$

According to the relation

$$n_1(t) + n_2(t) = N$$

(cf. (1)), $n_2(t) = N - n_1(t)$ is already determined by $n_1(t)$. For spontaneous strategy changes due to *trial and error* we will take the simplest form of transition rates:

$$w(j|i;t) := W. \tag{14}$$

A situation of the above kind is the avoidance behavior of pedestrians (cf. Helbing, 1991, 1992). In pedestrian crowds with two opposite directions of motion, pedestrians sometimes have to avoid each other in order to prevent a collision. For an avoidance maneuver to be successful, both pedestrians concerned have to pass the other pedestrian either on the right hand side (strategy $i = 1$) or on the left hand side (strategy $i = 2$). Otherwise, both pedestrians have to stop (cf. Figure 1a). Here, both strategies are equivalent, but the success of a strategy increases with the number n_i of individuals who use the *same* strategy. In Success Matrix (13) we then have

$$C > 0.$$

Empirically one finds that the probability P_1 of choosing the right hand side is usually different from the probability $P_2 = 1 - P_1$ of choosing the left hand side. Consequently, opposite directions of motion normally use separate lanes (cf. Figure 1b).

We will now examine whether our behavioral model can explain this *symmetry breaking* (the fact that $P_1 \neq P_2$). Figure 2 shows some computational results for $C = 1$ and different values of W/ν. If

$$\kappa := 1 - \frac{4W}{\nu C} < 0, \tag{15}$$

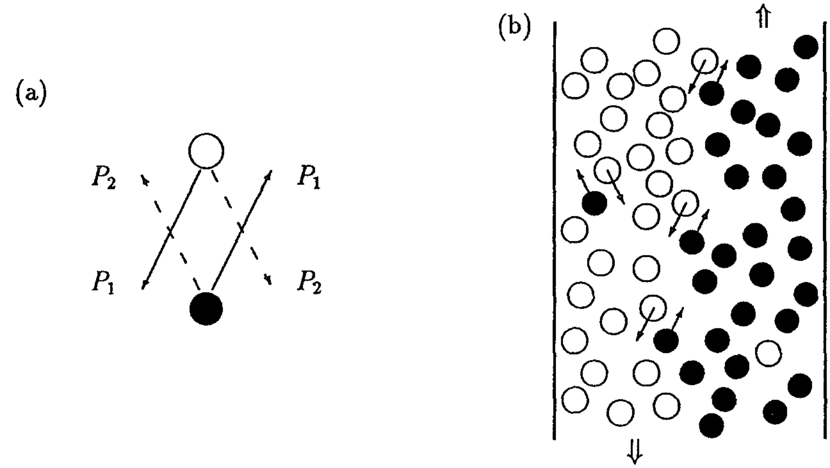

Fig. 1. (a) For pedestrians with an opposite direction of motion it is advantageous if both prefer either the right-hand side or the left-hand side when trying to pass each other. Otherwise, they would have to stop in order to avoid a collision. The probability P_1 of choosing the right-hand side is usually different from the probability $P_2 = 1 - P_1$ of choosing the left-hand side. (b) Opposite directions of motion normally use separate lanes. Avoidance maneuvers are indicated by arrows.

the configurational distribution is unimodal and symmetrical with respect to $n_1 = N/2 = n_2$, i.e. both strategies will be chosen by about one half of the individuals. A *phase transition (bifurcation)* appears at the *critical point* $\kappa = 0$. This is indicated by the broadness of the probability distribution $P(\mathbf{n}, t) \equiv P(n_1, n_2; t) = P(n_1, N - n_1; t)$ which comes from so-called *critical fluctuations* (cf. Haken, 1983). The term 'critical fluctuations' denotes the fact that the fluctuations become particularly large at a critical point since the system's behavior is then unstable. Whereas the individuals behave more or less independently *before* the phase transition ($\kappa < 0$), around the critical point the individuals begin to act in correlation due to their (imitative) interactions. However, the spontaneous strategy changes (represented by W) still prevent the formation of a behavioral preference. Above the critical point (i.e. for $\kappa > 0$) the correlation of individual behaviors is strong enough for the *self-organization (emergence) of a behavioral convention*: the configurational distribution becomes multimodal in the course of time with maxima at $n_1 \neq N/2$ so that one of the two equivalent strategies will very

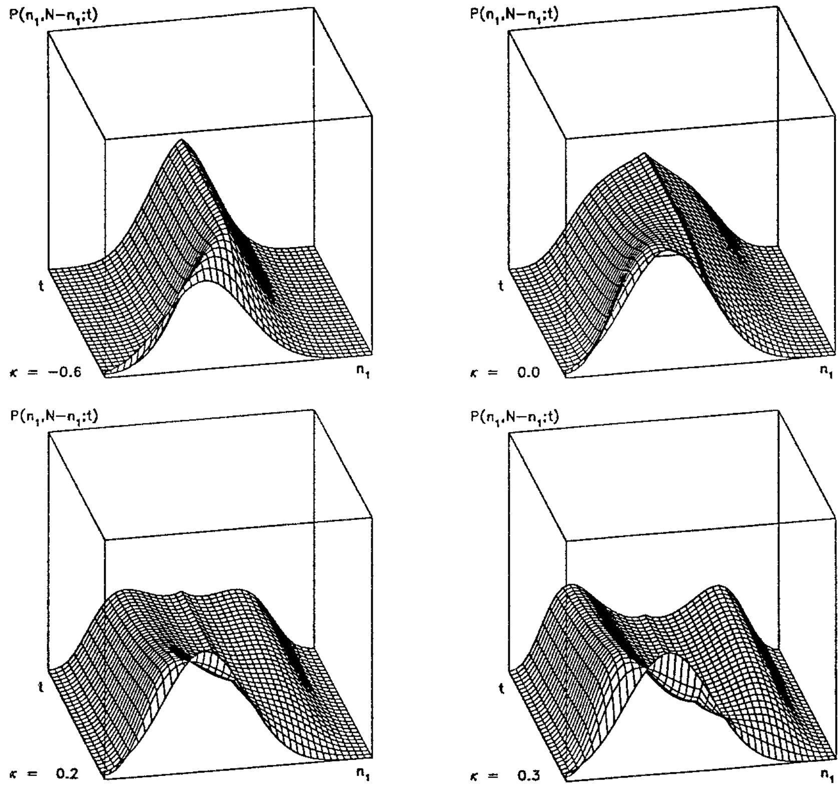

Fig. 2. Probability distribution $P(\mathbf{n}, t) \equiv P(n_1, N-n_1; t)$ of the socioconfiguration $\mathbf{n}$ for varying values of the control parameter κ. For $\kappa = 0$ a phase transition occurs: Whilst for $\kappa < 0$ both strategies are used by about one half of the individuals, for $\kappa > 0$ very probably one of the strategies will be preferred after some time. That means that a behavioral convention develops by social self-organization.

probably be chosen by a *majority* of individuals. In this connection one also speaks of *symmetry breaking* (Haken, 1979, 1983).

Behavioral conventions often obtain a law-like character after some time. Which one of two equivalent strategies will win the majority is completely random. It is possible that conventions differ from one region to another. This is, for example, the case for the prescription of which side of the road cars are to be driven.

The model of this section can also be applied to the competition between the two video systems VHS and Betamax, which were initially equivalent with respect to technology and price (Hauk, 1994). In the course of time VHS won this rivalry since (for reasons of *compatibility* concerning copying, selling or hiring of video tapes) it was advantageous for new purchasers to decide for that video system which gained a small majority at some moment. Other examples of the emergence of a behavioral convention are the direction in which clock hands revolve, the direction of writing, etc. It is easy to generalize the above model to the case of more than two alternative strategies. Of course, the model can also be adapted to situations where one behavioral alternative is superior to the others. However, the formation of a behavioral convention is then trivial.

Finally, we should mention some related models that were proposed during recent years for the description of symmetry breaking phenomena in economics: Orléan (1992, 1993) and Orléan and Robin (1992) presented a phase transition model using polynomial transition rates which base on a Bayesian rational. Durlauf (1989, 1991) used Markovian fields to explain the non-ergodic (i.e. path-dependent) behavior of some economic systems. Föllmer (1974) applied the Ising model paradigm (1925) to the modelling of an economy of many interacting agents and discussed the conditions under which a symmetry breakdown occurs. A similar model of polarization effects in opinion formation had already been suggested by Weidlich (1972). Last but not least, Topol (1991) presented a Fokker–Planck equation model for the explanation of bubbles in stock markets by mimetic contagion (i.e. some kind of imitative interactions) between agents.

4. MOST PROBABLE AND EXPECTED STRATEGY DISTRIBUTION

Because of the huge number of possible socioconfigurations $\mathbf{n}$, in more complex cases than in Section 3.3 the master equation for the determination of the configurational distribution $P(\mathbf{n}, t)$ is usually difficult to solve (even with a computer). However,

- in cases of the description of single or rare social processes the *most probable* strategy distribution

$$P_i^a(t) := \frac{\hat{n}_i^a(t)}{N_a} \qquad (16)$$

  is the quantity of interest, whereas

- in cases of frequently occurring social processes the interesting quantity is the *expected* strategy distribution

$$(17) \quad P_i^a(t) := \frac{\langle n_i^a \rangle_t}{N_a}.$$

$P_i^a(t)$ is the *proportion* of individuals within subpopulation a using strategy i so that

$$P_i^a(t) \geqslant 0 \quad \text{and} \quad \sum_i P_i^a(t) = 1.$$

Equations for the *most probable occupation numbers* $\hat{n}_i^a(t)$ can be deduced from a Langevin equation (1908) for the temporal development of the socioconfiguration $\mathbf{n}(t)$. For the *mean values* $\langle n_i^a \rangle_t$ of the *occupation numbers* n_i^a only *approximate* closed equations can normally be derived. Measures of the reliability of $\hat{n}_i^a(t)$ and $\langle n_i^a \rangle_t$ with respect to the possible temporal developments of $n_i^a(t)$ are the variances $\sigma_{ii}^{aa}(t)$ of $n_i^a(t)$. If the standard deviation $\sqrt{\sigma_{ii}^{aa}(t)}$ becomes comparable to $0.12\hat{n}_i^a(t)$ or $0.12\langle n_i^a \rangle_t$, the values of $\hat{n}_i^a(t)$ and $\langle n_i^a \rangle_t$, respectively, are no longer representative of $n_i^a(t)$ (cf. Section 4.1.5). In the case that $P(\mathbf{n}, t)$ is normally distributed this would imply a probability of 34% (5%) that the value of $n_i^a(t)$ deviates more than 12% (24%) from $\hat{n}_i^a(t)$ and $\langle n_i^a \rangle_t$, respectively. Moreover, if the variances $\sigma_{ii}^{aa}(t)$ become large, this may indicate a phase transition, i.e. a non-ergodic (path-dependent) temporal evolution of the system (see Figure 2).

4.1. *Mean Value and Covariance Equations*

The *mean value* of a function $f(\mathbf{n}, t)$ is defined by

$$\langle f(\mathbf{n}, t) \rangle_t \equiv \langle f(\mathbf{n}, t) \rangle := \sum_n f(\mathbf{n}, t) P(\mathbf{n}, t).$$

Master Equation (4) can be used to derive the fact that the mean values of the occupation numbers $f(\mathbf{n}, t) = n_i^a$ are determined by the equations

$$(18) \quad \frac{\mathrm{d}\langle n_i^a \rangle}{\mathrm{d}t} = \langle m_i^a(\mathbf{n}, t) \rangle$$

with *drift coefficients*

$$m_i^a(\mathbf{n}, t) := \sum_{n'} (n_i'^a - n_i^a) w(\mathbf{n}' | \mathbf{n}; t)$$

$$= \sum_{i'} [\bar{w}^a(i|i';t)n_{i'}^a - \bar{w}^a(i'|i;t)n_i^a] \tag{19}$$

and *effective transition rates*

$$\bar{w}^a(i'|;t) := \hat{w}_a(i'|i;t) + \sum_b \sum_{j'} \sum_j \hat{w}_{ab}(i',j'|i,j;t)n_j^b \tag{20}$$

(cf. Helbing, 1992, 1992a). Obviously, the contributions $\hat{w}_{ab}(i',j'|i,j;t)n_j^b$ due to pair interactions are proportional to the number n_j^b of possible interaction partners.

4.1.1. *Approximate Mean Value Equations*

Equations (18) are not closed equations, since they depend on the mean values $\langle n_i^a n_j^b \rangle$, which are not determined by (18). We therefore have to find a suitable approximation. Using a *first order* Taylor approximation we obtain the *approximate mean value equations*

$$\begin{aligned} \frac{\partial \langle n_i^a \rangle}{\partial t} &\approx \left\langle m_i^a(\langle \mathbf{n} \rangle, t) + \sum_{b,j} (n_j^b - \langle n_j^b \rangle) \frac{\partial m_i^a(\langle \mathbf{n} \rangle, t)}{\partial \langle n_j^b \rangle} \right\rangle \\ &= m_i^a(\langle \mathbf{n} \rangle, t). \end{aligned} \tag{21}$$

These are applicable if the configurational distribution $P(\mathbf{n}, t)$ has only small covariances

$$\begin{aligned} \sigma_{ij}^{ab} &:= \langle (n_i^a - \langle n_i^a \rangle)(n_j^b - \langle n_j^b \rangle) \rangle \\ &= \langle n_i^a n_j^b \rangle - \langle n_i^a \rangle \langle n_j^b \rangle \approx 0. \end{aligned} \tag{22}$$

Condition (22) corresponds to the limit of *statistical independence* $\langle n_i^a n_j^b \rangle = \langle n_i^a \rangle \langle n_j^b \rangle$ of the occupation numbers (and, therefore, of the individual behaviors).

4.1.2. *Boltzmann-like Equations*

Inserting (17), (19), and (20) into (21) the resulting approximate equations for the expected strategy distribution $P_i^a(t)$ are

$$\frac{\mathrm{d}}{\mathrm{d}t} P_i^a(t) = \sum_{i'} [w^a(i|i';t)P_{i'}^a(t) - w^a(i'|i;t)P_i^a(t)] \tag{23}$$

with the *mean transition rates*

$$w^a(i'|i;t) = \hat{w}_a(i'|i;t) + \sum_b \sum_{j'} \sum_j N_b \hat{w}_{ab}(i',j'|i,j;t)P_j^b(t). \tag{24}$$

Equations (23), (24) are called *Boltzmann-like equations* (Boltzmann, 1964; Helbing, 1992, 1992a) since the mean transition rates (24) depend on the strategy distributions $P_j^b(t)$ due to pair interactions. Assuming (7), (8), and (9) we obtain the formula

$$w^a(i|i';t) = \hat{w}_a(i|i';t) + R_a(i|i';t)\sum_b \nu_{ab} f_{ab} P_i^b(t) \tag{25}$$

with $R_a(i|i';t) := \hat{R}_a(i|i';\langle\mathbf{n}\rangle;t)$ for the mean transition rates. Equations (23) and (25) are a special case of more general equations introduced by Helbing (1992, 1992b, 1994) to describe the temporal development of the expected strategy distribution in a social system consisting of a huge number $N \gg 1$ of individuals.

4.1.3. *Approximate Covariance Equations*

In many cases, the configuration $\mathbf{n}_0$ at an initial time t_0 is known by empirical evaluation, i.e. the initial distribution is

$$P(\mathbf{n}, t_0) = \delta_{nn_0}.$$

As a consequence, the covariances σ_{ij}^{ab} vanish at time t_0 and remain small during a certain time interval. For the temporal development of σ_{ij}^{ab}, the equations

$$\begin{aligned} \frac{d\sigma_{ij}^{ab}}{dt} &= \langle m_{ij}^{ab}(\mathbf{n},t)\rangle + \langle (n_i^a - \langle n_i^a\rangle) m_j^b(\mathbf{n},t)\rangle \\ &\quad + \langle (n_j^b - \langle n_j^b\rangle) m_i^a(\mathbf{n},t)\rangle \end{aligned} \tag{26}$$

can be derived from Master Equation (4) (cf. Helbing, 1992, 1992a). Here,

$$\begin{aligned} m_{ij}^{ab}(\mathbf{n},t) &:= \sum_{n'} (n_i'^a - n_i^a)(n_j'^b - n_j^b) w(\mathbf{n}'|\mathbf{n};t) \\ &= \delta_{ab}\Bigg(\delta_{ii'}\sum_j [n_j^a \bar{w}^a(i|j;t) + n_i^a \bar{w}^a(j|i;t)] - \\ &\quad - [n_{i'}^a \bar{w}^a(i|i';t) + n_i^a \bar{w}^a(i'|i;t)]\Bigg) + \\ &\quad + \sum_{j'}\sum_j [n_j^a n_{j'}^b \hat{w}_{ab}(i,i'|j,j';t) + \\ &\quad + n_i^a n_{i'}^b \hat{w}_{ab}(j,j'|i,i';t)] - \end{aligned}$$

$$
\begin{aligned}
& - \sum_{j'} \sum_{j} [n_i^a n_{j'}^b \hat{w}_{ab}(j, i'|i, j'; t) + \\
(27) \quad & + n_j^a n_{i'}^b \hat{w}_{ab}(i, j'|j, i'; t)]
\end{aligned}
$$

are *diffusion coefficients*. Equations (26) are again not closed equations. However, a first-order Taylor approximation of the drift and diffusion coefficients $m_{..}^{..}(\mathbf{n}, t)$ leads to the equations

$$
\begin{aligned}
\frac{\partial \sigma_{ij}^{ab}}{\partial t} & \approx m_{ij}^{ab}(\langle n \rangle, t) + \sum_{c,k} \left(\sigma_{ik}^{ac} \frac{\partial m_j^b(\langle \mathbf{n} \rangle, t)}{\partial \langle n_k^c \rangle} + \right. \\
(28) \quad & \left. + \sigma_{jk}^{bc} \frac{\partial m_i^a(\langle \mathbf{n} \rangle, t)}{\partial \langle n_k^c \rangle} \right)
\end{aligned}
$$

(cf. Helbing, 1992, 1992a) which are solvable together with (21). The *Approximate Covariance Equations* (28) allow the determination of the time interval during which the Approximate Mean Value Equations (21) are valid (cf. Section 4.1.5 and Figures 5a, 5b). They are also useful for the calculation of the reliability (or representativity) of descriptions made by Equations (21). Moreover, they are necessary for *corrections* of Approximate Mean Value Equations (21).

4.1.4. *Corrected Mean Value and Covariance Equations*

Equations (21) and (28) are only valid for the case

$$
(29) \quad \left| \sigma_{ij}^{ab} \right| \ll \langle n_i^a \rangle \langle n_j^b \rangle
$$

where the absolute values of the covariances σ_{ij}^{ab} are small, i.e. where the configurational distribution $P(\mathbf{n}, t)$ is sharply peaked. For increasing covariances, a better approximation of (18) and (26) should be taken. A *second-order* Taylor approximation of (18) and (26), respectively, results in the *corrected mean value equations*

$$
(30) \quad \frac{\partial \langle n_i^a \rangle}{\partial t} \approx m_i^a(\langle \mathbf{n} \rangle, t) + \frac{1}{2} \sum_{b,j} \sum_{c,k} \sigma_{jk}^{bc} \frac{\partial^2 m_i^a(\langle \mathbf{n} \rangle, t)}{\partial \langle n_j^b \rangle \partial \langle n_k^c \rangle}
$$

and the *corrected covariance equations*

$$
\begin{aligned}
\frac{d \sigma_{ij}^{ab}}{dt} & \approx m_{ij}^{ab}(\langle n \rangle, t) + \frac{1}{2} \sum_{c,k} \sum_{d,l} \sigma_{kl}^{cd} \frac{\partial^2 m_{ij}^{ab}(\langle \mathbf{n} \rangle, t)}{\partial \langle n_k^c \rangle \partial \langle n_l^d \rangle} + \\
(31) \quad & + \sum_{c,k} \left(\sigma_{ik}^{ac} \frac{\partial m_j^b(\langle \mathbf{n} \rangle, t)}{\partial \langle n_k^c \rangle} + \sigma_{jk}^{bc} \frac{\partial m_i^a(\langle \mathbf{n} \rangle, t)}{\partial \langle n_k^c \rangle} \right)
\end{aligned}
$$

(Helbing, 1992, 1992a). Note that the corrected mean value equations explicitly depend on the covariances σ_{ij}^{ab}, i.e. on the *fluctuations* due to the stochasticity of the processes described. They cannot be solved without solving the covariance equations. A comparison of (30) with (21) shows that the approximate mean value equations only agree with the corrected ones in the limit of negligible covariances σ_{ij}^{ab} (cf. also (22)). However, the calculation of the covariances is *always* to be recommended since they are a measure of the reliability (or representativity) of the mean value equations. If the covariances become large in the sense of Equation (33), this may indicate a phase transition.

4.1.5. *Computational Results*

A comparison of exact, approximate and corrected mean value and variance equations is given in Figures 3–5a. These show computational results corresponding to the example of Section 3.3 (cf. Figure 2). *Exact* mean values $\langle n_1 \rangle$ and variances σ_{11} are represented by solid lines, whereas approximate results according to (21), (28) are represented by dotted lines, and corrected results according to (30), (31) by broken lines.

For $\kappa \geqslant 0$ the Approximate Mean Value Equations (21) become useless since the variances are growing due to the *phase transition.* As expected, the corrected mean value equations yield better results than the approximate mean value equations and they are valid for a longer time interval. A criterion for the validity of the Approximate Equations (21), (28) and the Corrected Equations (30), (31) respectively are the *relative central moments*

$$\begin{aligned} C_m(t) &\equiv C_{i_1\cdots i_m}^{a_1\cdots a_m}(t) \\ &:= \frac{\left\langle (n_{i_1}^{a_1} - \langle n_{i_1}^{a_1} \rangle) \cdots (n_{i_m}^{a_m} - \langle n_{i_m}^{a_m} \rangle) \right\rangle}{\langle n_{i_1}^{a_1} \rangle \cdots \langle n_{i_m}^{a_m} \rangle}. \end{aligned}$$

Whereas the Approximate Equations (21), (28) already fail, if

$$(32) \quad |C_m(t)| \leqslant 0.04$$

is violated for $m = 2$ (compare to (29), (22)), the Corrected Equations (30), (31) presuppose Condition (32) only for $3 \leqslant m \leqslant l$ with a certain, well-defined value l (cf. Helbing, 1992, 1992a for details). However, even the Corrected Equations (30), (31) become useless

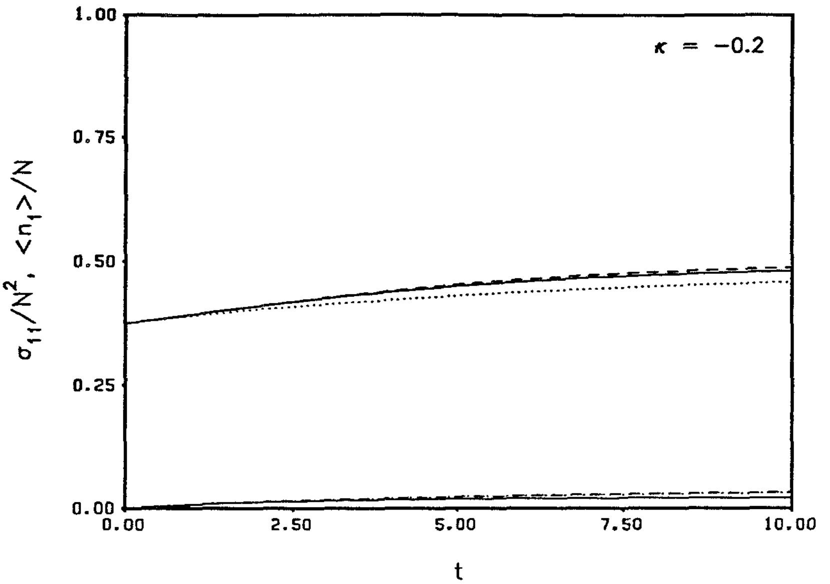

Fig. 3. Exact (—), approximate (···) and corrected (− −) mean values (upper curves) and variances (lower curves) for a *small* configurational distribution $P(\mathbf{n}, t)$: The simulation results for the approximate equations are acceptable, those for the corrected equations very well.

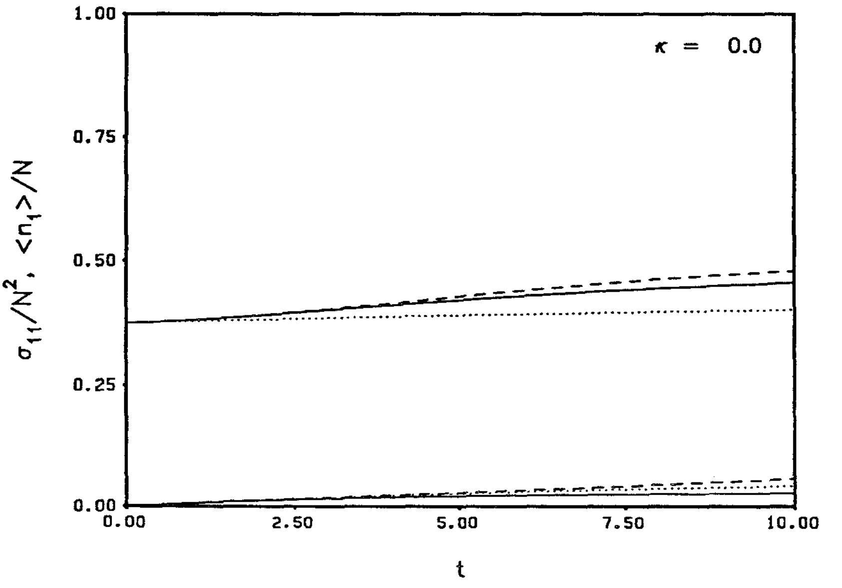

Fig. 4. As Figure 3, but for a *broad* configurational distribution: The corrected equations still yield useful results, whereas the approximate equations already fail since the variances are not negligible.

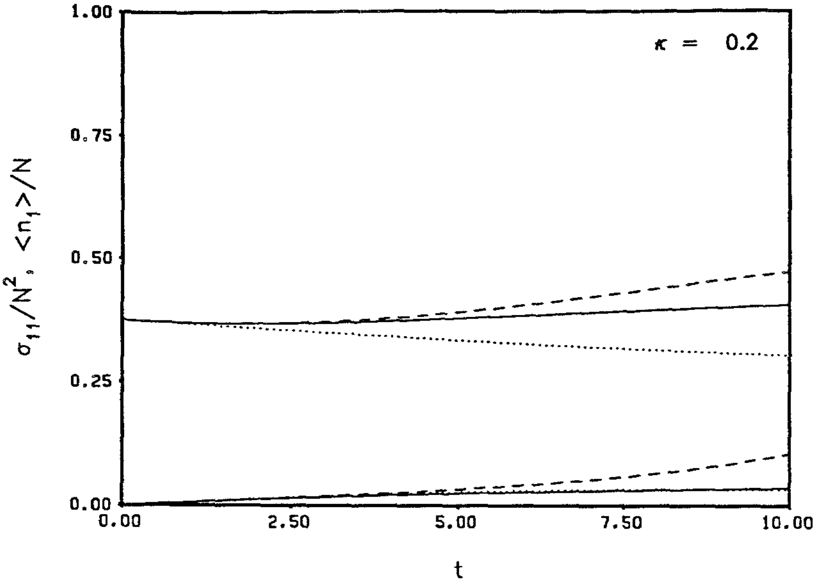

Fig. 5a. As Figure 3, but for a *multimodal* configurational distribution: not only the approximate but also the corrected equations fail after a certain time interval. However, whereas the approximate mean value and variance become unreliable already for $t > 1$, the corrected mean value and variance remain valid as long as $t \leqslant 3$.

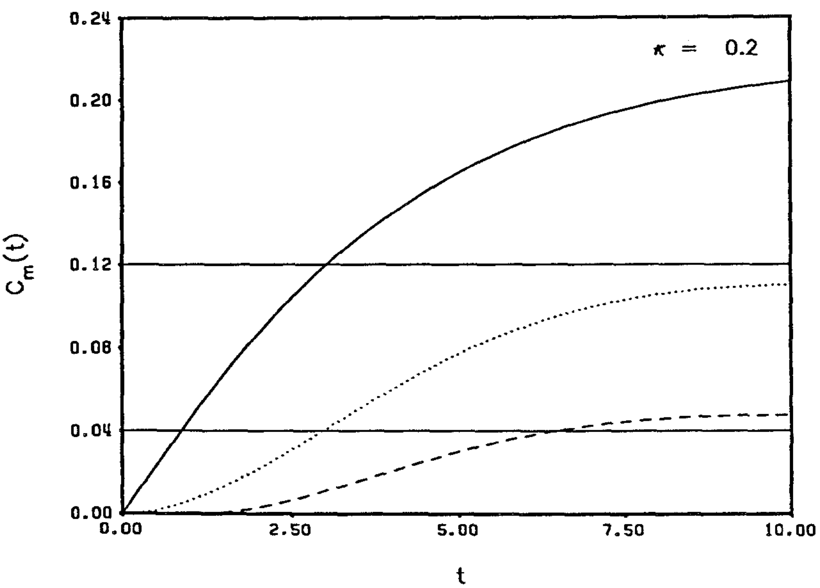

Fig. 5b. The relative central moments $C_m(t)$ are a criterion for the validity of the approximate respectively the corrected mean value and covariance equations: If $|C_2(t)|$(—) exceeds the value 0.04, the approximate equations fail. The corrected equations fail if $|C_3(t)|$(– –) or $|C_4(t)|$($\cdots$) exceed the value 0.04. This is the case if $|C_2(t)|$ becomes greater than 0.12 (indicating a phase transition).

if the probability distribution $P(\mathbf{n},t)$ becomes multimodal, i.e. if a phase transition occurs. This is the case if

$$(33) \quad |C_2(t)| = \left| \frac{\sigma_{ij}^{ab}(t)}{\langle n_i^a \rangle \langle n_j^b \rangle} \right| \leqslant 0.12$$

is violated (cf. Figure 5).

4.2. *Equations for the Most Probable Strategy Distribution*

After the transformation of Master Equation (2) into a Fokker–Planck equation by a second-order Taylor approximation, it can be reformulated in terms of a Langevin equation (1908) (cf. Weidlich and Haag, 1983; Helbing, 1992). This reads

$$(34) \quad \frac{\mathrm{d}}{\mathrm{d}t} n_i^a(t) \overset{N_a \gg 1}{=} m_i^a(\mathbf{n},t) + \textit{fluctuations}$$

and describes the temporal development of the socioconfiguration $\mathbf{n}(t)$ as it depends on process immanent fluctuations (that are determined by the diffusion coefficients m_{ij}^{ab}). As a consequence,

$$(35) \quad \frac{\mathrm{d}}{\mathrm{d}t} \hat{n}_i^a(t) \overset{N_a \gg 1}{=} m_i^a(\hat{\mathbf{n}},t)$$

are the equations governing the temporal development of the *most probable* occupation numbers $\hat{n}_i^a(t)$. Equations (35) look exactly like Approximate Mean Value Equations (21). Therefore, if $N_a \gg 1$, the *approximate* mean value equations have an interpretation even for large variances since they also describe the most probable strategy distribution.

5. THE GAME DYNAMICAL EQUATIONS

5.1. *'Macroscopic' Derivation*

Before we will connect the stochastic behavioral model to the game dynamical equations, we will discuss their derivation from a collective level of behavior. Let $E_a(i,t) := \hat{E}_a(i,\langle\mathbf{n}\rangle;t)$ be the *expected success* of strategy i for an individual of subpopulation a and

$$(36) \quad \overline{E_a}(t) := \sum E_a(i,t) P_i^a(t)$$

the *mean expected success*. If the *relative* increase

$$\frac{\mathrm{d}P_i^a/\mathrm{d}t}{P_i^a(t)}$$

of the proportion $P_i^a(t)$ is assumed to be proportional to the difference $[E_a(i,t) - \overline{E_a}(t)]$ between the expected and the mean expected success, one obtains the *game dynamical equations*

$$\text{(37)} \qquad \frac{\mathrm{d}}{\mathrm{d}t} P_i^a(t) = \nu_a P_i^a(t)[E_a(i,t) - \overline{E_a}(t)].$$

According to these equations, the proportions of strategies with an expected success that exceeds the *average* $\overline{E_a}(t)$ are growing, whereas the proportions of the remaining strategies are falling. For the expected success $E_a(i,t)$ one often takes the form

$$\text{(38)} \qquad E_a(i,t) := \sum_b \sum_j A_{ab}(i,j) P_j^b(t)$$

where the quantities $A_{ab}(i,j)$ have the meaning of *payoffs* which are exogeneously determined. Consequently, the matrices

$$\mathbf{A}_{ab} := (A_{ab}(i,j))$$

are called *payoff matrices*. Inserting (36) and (38) into (37), one obtains the explicit form

$$\text{(39)} \qquad \begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} P_i^a(t) = \nu_a P_i^a(t) \Bigg[&\sum_{b,j} A_{ab}(i,j) P_j^b(t) - \\ &- \sum_{i'} \sum_{b,j} P_{i'}^a(t) A_{ab}(i',j) P_j^b(t) \Bigg] \end{aligned}$$

of the game dynamical equations. Equations of this kind are very useful for the investigation and understanding of the competition or cooperation of individuals (cf. e.g. Hofbauer and Sigmund, 1988; Schuster *et al.*, 1981). Due to their nonlinearity they may have a complex dynamical solution, e.g. an *oscillatory* one (Hofbauer *et al.*, 1980; Hofbauer and Sigmund, 1988) or even a *chaotic* one (Schnabl *et al.*, 1991).

A slightly generalized form of (37),

$$\text{(40a)}\quad \frac{\mathrm{d}}{\mathrm{d}t}P_i^a(t) = \sum_{i'}[\hat{w}_a(i|i';t)P_{i'}^a(t) - \hat{w}_a(i'|i;t)P_i^a(t)] +$$

$$\text{(40b)}\quad +\nu_a P_i^a(t)[E_a(i,t) - \overline{E_a}(t)],$$

is also known as the *selection mutation equation* (Hofbauer and Sigmund, 1988): (40b) can be understood as an effect of a *selection* (if $E_a(i,t)$ is interpreted as *fitness* of strategy i), and (40a) can be understood as an effect of *mutations*. Equation (40) is a powerful tool in evolutionary biology (cf. Eigen, 1971; Fisher, 1930; Eigen and Schuster, 1979; Hofbauer and Sigmund, 1988; Feistel and Ebeling, 1989). In game theory, the mutation term could be used for the description of *trial and error behavior* or accidental variations of the strategy.

5.2. *Derivation from the Stochastic Behavioral Model*

In this section we look for a connection between the stochastic behavioral model of Section 3 and the game dynamical equations. For this purpose we compare the approximate mean value equations of this stochastic behavioral model, i.e. the Boltzmann-Like Equations (23), (25) with the Game Dynamical Equations (40). Both equations will be identical only if

$$\nu_{ab} f_{ab} = \nu_a \delta_{ab}.$$

This condition corresponds to (12) if

$$\nu_a = \nu_{aa}.$$

The insertion of Assumptions (10)–(12) into the Boltzmann-Like Equations (23), (25) gives the Game Dynamical Equations (40) as a result. We have only to introduce the identities

$$A_{ab}(i,j) := r_{ab}E_{ab}(i,j),$$

$$E_a(i,t) := \hat{E}_a(i,\langle \mathbf{n}\rangle;t) = \sum_b \sum_j r_{ab}E_{ab}(i,j)P_j^b(t),$$

and to apply the relation

$$\max(E_a(i,t) - E_a(j,t), 0) - \max(E_a(j,t) - E_a(i,t), 0) = E_a(i,t) - E_a(j,t).$$

The game dynamical equations (including their properties and generalizations) are more explicitly discussed elsewhere (Helbing, 1992). An interesting application to a case with two subpopulations can be found in the book by Hofbauer and Sigmund (1988: pp. 137–146). In the following, we will again examine the example of Section 3.3, where we have one subpopulation and two equivalent strategies. The Game Dynamical Equations (40) corresponding to (13) and (14) then have the explicit form

$$\text{(41)} \quad \frac{\mathrm{d}}{\mathrm{d}t}P_i(t) = -2\left(P_i(t) - \frac{1}{2}\right)[W + \nu C P_i(t)(P_i(t) - 1)].$$

According to (41), $P_i = 1/2$ is a stationary solution. This solution is stable for

$$\kappa = 1 - \frac{4W}{\nu C} < 0,$$

i.e. if spontaneous strategy changes are dominant and, therefore, prevent a self-organization process.

At the *critical point* $\kappa = 0$ *symmetry breaking* appears: For $\kappa > 0$ the stationary solution $P_i = 1/2$ is unstable and the Game Dynamical Equations (41) can be rewritten in the form

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}P_i(t) &= -2\left(P_i(t) - \frac{1}{2}\right)\left(P_i(t) - \frac{1+\sqrt{\kappa}}{2}\right) \\ \text{(42)} \qquad &\quad \times \left(P_i(t) - \frac{1-\sqrt{\kappa}}{2}\right). \end{aligned}$$

This means that, for $\kappa > 0$, we have two additional stationary solutions, $P_i = (1 + \sqrt{\kappa})/2$ and $P_i = (1 - \sqrt{\kappa})/2$, which are stable. Depending on initial fluctuations, one strategy will win a majority of $100 \cdot \sqrt{\kappa}$ percent. This majority is greater the smaller the rate W of spontaneous strategy changes is.

6. MODIFIED GAME DYNAMICAL EQUATIONS

At first glance the pleat of $P(n_1, N - n_1; t)$ at $n_1 = N/2 = n_2$ in the illustrations of Figure 2 appears somewhat surprising. A mathematical analysis shows that this is a consequence of the pleat of the

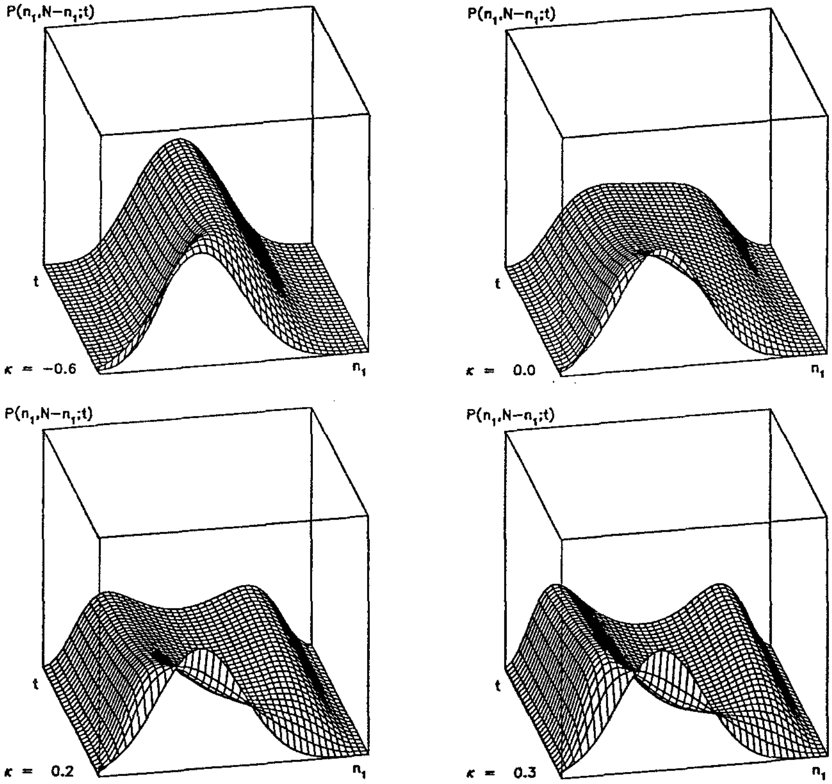

Fig. 6. Probability distribution $P(\mathbf{n}, t) \equiv P(n_1, N - n_1; t)$ of the socioconfiguration $\mathbf{n}$ according to the *modified* stochastic game dynamical equations. The results are similar to those in Figure 2. For $\kappa < 0$ both strategies are used by about one half of the individuals, for $\kappa > 0$ very probably one of the strategies will be preferred after some time. Again, for $\kappa = 0$ a phase transition occurs.

function $\hat{R}_a(j|i;t) = \max(\hat{E}_a(j,t) - \hat{E}_a(i,t), 0)$. It can be avoided by using the modified approach

$$\hat{R}_a(j|i;t) := \frac{1}{2}\exp[\hat{E}_a(j,t) - \hat{E}_a(i,t)]. \tag{43}$$

(43) also leads to a phase transition for $\kappa = 0$ (cf. Figure 6) and very similar results for the approximate mean value equations since the game dynamical equations result as Taylor approximation of those. According to (43), imitative strategy changes from i to j

will again occur more frequent the greater is the expected increase $\Delta_{ji}\hat{E}_a = \hat{E}_a(j,t) - \hat{E}_a(i,t)$ of success.

Approach (43) originally stems from physics, where the exponential function for the transition probability is due to the need to obtain the Boltzmann distribution (1964) as stationary distribution. Its application to behavioral changes was suggested by Weidlich (1971, 1972) in connection with a Ising-like (1925) opinion formation model. Meanwhile, related models were also proposed for economic systems (Haag *et al.*, 1993; Weidlich and Braun, 1992; Durlauf, 1989, 1991). In contrast to this, Orléan (1992, 1993) and Orléan and Robin (1992) prefer a transition probability which has the form of a polynomial of degree two and is based on a Bayesian rationale.

The advantage of (43) is that it guarantees the non-negativity of $\hat{R}_a(j|i;t)$. Moreover, the exponential approach factorizes into a *pull term* $\exp[\hat{E}_a(j,t)]$ and a *push term* $\exp[-\hat{E}_a(i,t)]$. For strategy changes it is not the *absolute* success $\hat{E}_a(j,t)$ of an available strategy j that is relevant, but that the *relative* success $[\hat{E}_a(j,t) - \hat{E}_a(i,t)]$ with respect to the pursued strategy i.

Furthermore, Approach (43) can be related to a decision theoretical model for *choice under risk*. For this let us assume that the *utility* of a strategy change from i to j is given by a known part

$$U_a(j|i;t) := [\hat{E}_a(j,t) - \hat{E}_a(i,t)]$$

and an unknown part ϵ_j (i.e. an error term) which comes from the *uncertainty* about the exact value of $[\hat{E}_a(j,t) - \hat{E}_a(i,t)]$ (since $\hat{E}_a(i,t)$, like n_j^b, is subject to fluctuations). If the individual choice behavior is the result of a maximization process (i.e. if an individual chooses the alternative j for which $U_a(j|i;t) + \epsilon_j > U_a(i'|i;t) + \epsilon_{i'}$ holds in comparison with all other available alternatives i') and if the error terms are identically and independently Weibull distributed, the choice probabilities $p_a(j|i;t)$ are given by the well-known *multinomial logit model* (Domencich and McFadden, 1975). This reads

$$p_a(j|i;t) = \frac{\exp[\hat{E}_a(j,t) - \hat{E}_a(i,t)]}{\sum_{i'} \exp[\hat{E}_a(i',t) - \hat{E}_a(i,t)]}.$$

(For a more detailed discussion cf. Helbing, 1992.)

Approach (43) can also be derived by *entropy maximization* (Helbing, 1992) or from the *law of relative effect* in combination with Fechner's *law of psychophysics* (Luce, 1959; Helbing, 1992).

7. SUMMARY AND OUTLOOK

A quite general model for changes of behavioral strategies has been developed which takes into account spontaneous changes and changes due to pair interactions. Three kinds of pair interactions can be distinguished: imitative, avoidance, and comprising processes. The game dynamical equations result for a special case of imitative processes. They can be interpreted as equations for the most probable strategy distribution or as approximate mean value equations of a stochastic version of evolutionary game theory. In order to calculate correction terms for the game dynamical equations as well as to determine the reliability or the time period of validity of game dynamical descriptions, one has to evaluate the corresponding covariance equations. Therefore, covariance equations have been derived for a very general class of master equations.

The model can be extended in a way that takes into account the expectations about the future temporal evolution of the expected successes $E_a(i,t)$ (the '*shadow of the future*'). For this purpose, in (40) $E_a(i,t)$ must be replaced by a quantity $E_a^*(i,t)$ which represents the expectations about the future success of strategy i on the basis of its success $E_a(i,t')$ at past times $t' \leqslant t$. Different ways of mathematically specifying the future expectations $E_a^*(i,t)$ were discussed by Topol (1991), Glance and Huberman (1992) as well as Helbing (1992).

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