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DOMINATION STRUCTURES AND MULTICRITERIA PROBLEMS IN *N*-PERSON GAMES

ABSTRACT. Multiple criteria decision problems with *one decision maker* have been recognized and discussed in the recent literature in optimization theory, operations research and management science. The corresponding concept with *n-decision makers*, namely multicriteria n-person games, has not yet been extensively explored.

In this paper we first demonstrate that existing solution concepts for single criterion n-person games in both normal form and characteristic function form induce domination structures (similar to those defined and studied by Yu [39] for multicriteria single decision maker problems) in various spaces, including the payoff space, the imputation space and the coalition space. This discussion provides an understanding of some underlying assumptions of the solution concepts and provides a basis for generalizing and generating new solution concepts not yet defined. Also we illustrate that domination structures may be regarded as a measure of power held by the players.

We then illustrate that a multicriteria problem can naturally arise in decision situations involving (partial) conflict among n-persons. Using our discussion of solution concepts for single criterion games as a basis, various approaches for resolving both normal form and characteristic function form multicriteria n-person games are proposed. For multicriteria games in characteristic function form, we define a multicriteria core and show that there exists a single 'game point' whose core is equal to the multicriteria core. If we reduce a multicriteria game to a single criterion game, domination structures which are more general than 'classical' ones must be considered, otherwise some crucial information in the game may be lost. Finally, we discuss a parametrization process which, for a given multicriteria game, associates a single criterion game to each point in a parametric space. This parametrization provides a basis for the discussion of solution concepts in multicriteria n-person games.

1. INTRODUCTION

Due to the fact that many different solution concepts have been and will probably be proposed for n-person cooperative games with a single criterion, a perplexing issue continues to intrigue researchers in game theory. That issue is, for a particular n-person game, which solution concept should be used and why? A justification of the choice of a particular solution concept usually relies on both mathematical properties of the solution concept and the interpretation of the solution concept as applied in a specific situation.

In this paper we try to shed new light on the problem of understanding the relationships among solution concepts by examining a common underlying mathematical property of the concepts, namely domination structures induced by the concepts. Further, we describe how the domination structures

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are used to generate the solutions. These domination structures also indicate some natural generalizations of the solution concepts. A generalization of the core, such as the *relaxed core*, can be easily derived from the domination structure induced by the core. We also illustrate that domination structures which are more general than 'classical' structures can be used to indicate the power held by the players. The details of these ideas are presented in Section 2.

It is commonly known that the scope of applications of n-person games has not met the expectations of many early researchers in the field. In this paper we suggest that the extension of single criterion n-person games to the multicriteria case provides a more realistic model and will perhaps permit more extensive applications. We illustrate that multicriteria game models can naturally and frequently arise in situations with n-decision makers.

Based on the discussion of solution concepts for single criterion n-person games, we propose various solution approaches to multicriteria games in both normal form and characteristic function form. A new solution concept for multicriteria characteristic function form games, namely the multicriteria core, is defined. We prove that the multicriteria core is the same as the core of a particular game point. When a multicriteria game is reduced to a single criterion game point, it is shown that a more general domination structure than the classical one must be used for otherwise, some crucial information concerning the game situation may be lost. (See Section 3.2.2.) Various parametrization approaches which associate a single criterion game to each point in a parametric space are examined. Under certain conditions, the parametrized game point is a continuous function of the parameter vector. These ideas are presented in Section 3.

To conclude the introduction, it is appropriate to discuss some related literature which may be of interest to the reader. Multicriteria problems with a single decision maker have been discussed in the recent literature (see [40] for a short bibliography), but a survey of recent literature reveals that multicriteria n-person games have rarely been discussed. The following are some related papers.

In [4] Blackwell considers two-person, zero-sum matrix games with vector-valued random variable payoffs. His paper is directed toward establishing an analogous form of the minimax theorem, namely, that in an infinite sequence of plays, the 'center of gravity' of the payoff is in or 'close' to a given set S under certain conditions. The fact that the payoffs are vector-valued is equivalent to a multicriteria payoff.

Further work on two-person, zero-sum matrix games having random variables as entries in the payoff matrix can be found in [6, 7, 11a, 11b]. In our formulation when a characteristic function is derived from a parametrized normal form game, the two-person, zero-sum game matrices which arise in determining the value for the coalitions could be interpreted as matrices with random variable entries satisfying certain special conditions. (See Section 3.2.1.)

In [9a, 9b] Charnes and Granot study n-person games in stochastic characteristic function form, i.e., the values assigned to each coalition are random variables with given distributions. In some instances, the parametrized characteristic functions for multicriteria games in this paper can be viewed as stochastic characteristic functions satisfying special conditions. However, our solution approaches are entirely different from those given by Charnes and Granot.

In [8, pp. 785–797], Charnes and Cooper suggest that a multicriteria decision problem may be approached by converting it to an n-person game where each criterion is associated with a different player. They give a traffic example to illustrate this idea. Charnes and Sorensen [33] discuss the resolution of an n-goal programming problem by associating each criterion function with a player in an n-person game. These ideas are implicitly used in our consideration of the multicriteria normal form full payoff space where one can view each criterion for each player as being associated with a separate player in an enlarged player set (see Section 3.1.1).

2. SINGLE CRITERION N-PERSON GAMES

In this section we shall discuss some well-known solution concepts that have been proposed for single criterion n-person games. We first discuss a single decision maker's problem with multiple criteria. Then the concepts of ordering (complete or partial), domination structures and nondominated solutions will be briefly discussed.

For some well-known solution concepts for n-person games in both normal form and characteristic function form we discuss the underlying domination structures and related nondominated solutions. In order to avoid too much repetition we shall examine some representative solution concepts and leave the rest to the reader.

We will also illustrate that domination structures which are more general than 'classical' ones naturally occur and can be used to indicate the relative balance of power among the players.

2.1. *Ordering, Domination Structures and Nondominated Solutions in Single Player Decision Problems*

In this section we introduce the concepts of domination structures and nondominated solutions (see [38], [39] and [40]) as they have been applied to problems with a single decision maker and multiple criteria. These problems are characterized by the following elements:

(i) A set of decisions, W. An element of W is denoted by w and W is called the *decision space*.

(ii) A set of criteria represented by a vector-valued function $u = (u_1, ..., u_\ell)$ defined for each decision $w \in W$. The set of all possible outcomes is called the *outcome* or *criteria space* and is denoted by $U = u(W) = \{u(w) \mid w \in W\}$[1].

(iii) The decision maker's preference ordering (partial or complete) on the criteria (outcome) space. Given u^1 and u^2 in U, we shall write $u^1 \succ u^2$ to denote that u^1 is preferred to u^2.

A final decision should be some $w^0 \in W$ such that no other feasible u^1 is preferred to $u(w^0)$, i.e., there is no $u^1 \in U \backslash \{u(w^0)\}$ such that $u^1 \succ u(w^0)$.

Alternatively, with each point $u^0 \in U$, we can associate a set $D(u^0)$ so that $u \in u^0 + D(u^0) = \{u^0 + d \mid d \in D(u^0)\}$ and $u \neq u^0$ if and only if $u^0 \succ u$. Intuitively, we can think of the preference $u^0 \succ u$ as occurring because of a factor in $D(u^0)$.

We shall assume that $D(u)$ is a convex cone.[2] This means that if $d^1, d^2 \in D(u)$ and $\lambda_1, \lambda_2 > 0$, then $\lambda_1 d^1 + \lambda_2 d^2 \in D(u)$. For convenience $D(u)$ will be called the *domination cone for u*. Intuitively, $D(u)$ has the property that given a 'bad' factor $d \in D(u)$, then any positive multiple of d is also a 'bad' factor. Also, given two 'bad' factors, d^1 and $d^2 \in D(u)$, the sum $d^1 + d^2$ is again a 'bad' factor.

The family $\{D(u) \mid u \in U\}$, denoted simply by $D(\cdot)$, is called the *domination structure* of our decision problem.

Given a set U, a domination structure $D(\cdot)$ defined on U and $u^1, u^2 \in U$, we shall say that u^2 is *dominated* by u^1 if and only if $u^2 \in u^1 + D(u^1)$ and $u^2 \neq u^1$. A point $u^0 \in U$ is a *nondominated solution* (or *nondominated outcome*) if and only if there is no $u^1 \in U$ such that $u^1 \neq u^0$ and $u^0 \in u^1 +$

$D(u^1)$. Thus u^0 is nondominated if and only if it is not dominated by any other outcome in U.

Similarly, in the decision space W, a point $w^0 \in W$ is a *nondominated solution* (or *nondominated decision*) if and only if there is no $w^1 \in W$ such that $u(w^1) \neq u(w^0)$ and $u(w^0) \in u(w^1) + D(u(w^1))$. The set of all nondominated solutions in the decision space and the criteria space will be denoted by $N_W(D(\cdot))$ and $N_U(D(\cdot))$, respectively (or by N_W and N_U, respectively, when the domination structure is clear from the context).

It is clear that a 'good' final decision must be nondominated. In [39] and [40] it is shown that one may convert preference information into a domination structure. It is also shown that each of the existing solution concepts (such as utility construction, satisficing solutions, Pareto optimality, efficiency and compromise schemes) for single decision maker (one-player) multicriteria problems in fact induces a special domination structure. The concepts of domination structures and nondominated solutions will play an important role in our subsequent discussion.

2.2. *A Classification of Solution Concepts*

In Table I we give a brief classification of solution concepts[3] for single criterion n-person games according to (i) whether the solution concept is defined for a payoff in normal form or in characteristic function form and (ii) whether the solution concept usually yields a set (or sets) of many solution points or a unique solution point. Throughout Section 2 we will implicitly use Table I to provide a conceptual framework for the discussion.

A large amount of literature concerning solution concepts is available. There are several game theory books which provide an introduction to many solution concepts. For instance, see [24], [25] and [27]. In the brackets following each solution concept in Table I we have indicated some additional references where that solution concept has been introduced and/or discussed. Clearly, we have not provided a complete listing of the available references.

A further characteristic of the 'many point' solution concepts is the kind of stability implied by the concept. The core and stable sets are 'globally' stable while the bargaining sets and kernel have a stability which is more 'local' in nature. See [23] for a further discussion of this aspect.

Additionally, there are many categories of games which are outside the scope of this paper. These include games without side payments [3, 32], games in partition function form [35], constrained games [14, 33], differential games [5, 20] and some new forms discussed in [26]. The interested reader may consult the cited references and extend the concepts in this paper to other classes of games.

In the next sections we shall focus on some solution concepts in each category of Table I and describe the underlying domination structures. We will develop our presentation so that it can easily be extended to multicriteria n-person games.

TABLE I

Solution concepts

	Concepts (usually) yielding a set or sets of many solution points	Concepts (usually) yielding a unique solution point
Normal form	Pareto optimal set [40] Satisficing solutions [40]	Nash arbitration solution [18] Compromise solutions [16, 17, 37]
Characteristic function form	Core [12a, 23, 30, 34] (ϵ-Core [12b, 21]) Stable sets [23, 28, 36] Bargaining sets [1] (Competitive bargaining sets [19]) Kernel [15, 21] ψ-stable sets [24, 27] Core-stem solutions [13a, 13b, 33] Subsolutions [28]	Shapley value [10, 21, 31] Nucleolus [12b, 21, 22, 29] Convex nuclei [10, 12b] (ℓ_p-centers [34])

2.3 *Solution Concepts and Domination Structures for Games in Normal Form*

For games in normal form we assume that each player i, $i = 1, \ldots, n$, has a real-valued payoff function p^i which is defined over a joint decision space W. We assume that player i's preference is increasing with p^i, i.e., the greater the value of p^i, the more preferred is the payoff p^i.

In the simple matrix game, assuming cooperation among all the players, W would be set of probability distributions (mixed strategies) on the cartesian product of the pure strategy sets for the individual players. Then the *payoff space*, denoted by P, is given by $P = p(W) = \{p(w) \mid w \in W\}$. Where $p = (p^1,$

$p^2, \ldots, p^n$).[4] To illustrate games in normal form, we give a numerical example of a three-person matrix game.

EXAMPLE 2.3.1. Denoting player i's pure strategy set by Q_i, let the pure strategy sets be $Q_1 = \{\alpha_1, \alpha_2\}$, $Q_2 = \{\beta_1, \beta_2\}$ and $Q_3 = \{\gamma_1, \gamma_2\}$. The payoff function for pure strategy choices is given by the following matrix.

$$(1) \qquad \begin{array}{c} \\ \alpha_1\beta_1 \\ \alpha_1\beta_2 \\ \alpha_2\beta_1 \\ \alpha_2\beta_2 \end{array} \begin{array}{c} \begin{array}{cc} \gamma_1 & \gamma_2 \end{array} \\ \begin{bmatrix} (5,1,2) & (1,1,2) \\ (3,1,4) & (1,1,3) \\ (2,3,5) & (2,3,5) \\ (5,4,1) & (1,0,5) \end{bmatrix} \end{array}$$

The ith coordinate of each three-dimensional vector in this matrix indicates a payoff for player i. For instance, if player 1 uses α_1, player 2 uses β_2 and player 3 uses γ_2, then the payoff is 1 to player 1, 1 to player 2 and 3 to player 3.

In this example W is the set of probability distributions over the eight element set $\Pi_{i=1}^3 Q_i = \{\alpha_1\beta_1\gamma_1, \alpha_1\beta_1\gamma_2, \ldots, \alpha_2\beta_2\gamma_2\}$. Thus we can write $W = \{w \in \mathbb{R}^8 \mid w \geqq 0, \ \Sigma_{i=1}^8 w_i = 1\}$. The payoff for pure strategy choices is extended to an expected payoff function on W. Thus $P = \{\Sigma_{i=1}^8 w_i p^i \mid w \in W\}$ where $p^1, \ldots, p^8$ represent the eight payoff vectors in the matrix (1).

As illustrated in this example for matrix games, W is compact and convex and p is linear implying that P is also compact and convex. These desirable properties may not be present in more general settings. In [37] Yu gives a formulation for normal form games which is more general than the matrix game. In that fomulation a general decision space W and payoff function p are required. The payoff space, P, is not necessarily compact or convex and p is not necessarily continuous.

We will now discuss the solution concepts.

2.3.1. *Concepts (usually) yielding a set of sets of many solution points.* We shall discuss the Pareto optimal set and satisficing solutions in this section (see Table I). These solution concepts involve multiple dimensional comparisons.

A payoff p^0 is *Pareto optimal* if and only if there is no $p^1 \in \mathrm{P}$ such that $p^1 \neq p^0$ and $p^1 \geqq p^0$. Further, a (joint) decision w^0 is Pareto optimal if $p(w^0)$ is a Pareto optimal payoff.

This concept induces the following natural domination structure whose nondominated solutions are the Pareto optimal points. For each $p \in P$, let $D(p) = \Lambda^{\leqq} = \{d \in \mathbb{R}^n \mid d \leqq 0\}$. Then the set of nondominated payoffs, N_P, is the set of Pareto optimal payoffs and $N_W = \{w \in W \mid p(w) \in N_P\}$ is the set of Pareto optimal decisions.

EXAMPLE 2.3.2. (Continuation of Example 2.3.1) Here P is the convex hull of $p^1, ..., p^8$ which are the eight payoff vectors in the matrix (1). By considering these eight vectors we determine that the set of nondominated extreme points of P, denoted by N_{ex}, is given by $N_{ex} = \{(5, 1, 2,), (2, 3, 5), (5, 4, 1)\}$. By Theorem 2.3. and Theorem 2.5 of [41] one can verify that the set of Pareto optimal payoffs is $N_p = \mathscr{H}(N_{ex})$ (the convex hull of N_{ex}).

REMARK 2.3.2. Throughout this paper we will use the notation introduced above, namely $\Lambda^{\leqq} = \{d \in \mathbb{R}^n \mid d \leqq 0\}$ and similarly $\Lambda^{\geqq} = \{d \in \mathbb{R}^n \mid d \geqq 0\}$.

In a satisficing model each player first establishes a satisfaction level, ℓ_i, which is a minimally acceptable payoff level for that player.[5] Given a vector of satisfaction levels, $\ell \in \mathbb{R}^n$, a payoff p^0 is a *satisficing solution* if and only if $p^0 \geqq \ell$.

Using a satisfaction level vector $\ell \in \mathbb{R}^n$, we can define a domination structure on P for each player i as follows. For each $p \in P$, let

$$D_i(p) = \begin{cases} \{d \in \mathbb{R}^n \mid d_i \leqq 0, d_j = 0 \text{ for } j \neq i\}, & \text{if } p_i \leqq \ell_i. \\ \{0\}, \text{ otherwise.} \end{cases}$$

Then the set of satisficing solutions is given by an intersection of sets of nondominated payoffs, namely $\cap_{i=1}^{m} N_p(D_i(\cdot))$.

EXAMPLE 2.3.4. To illustrate the concept of satisficing solutions, suppose the players in the game of Example 2.3.1 have established satisfaction levels given by $\ell = (4, 3, 2)$. In this case satisficing solutions exist, for example, $(14/3, 3, 2) = 1/3(3, 1, 4) + 2/3(5, 4, 1)$ is a satisficing solution.

2.3.2. *Concepts (usually) yielding a unique solution point.* For the concepts considered in this section one defines a real-valued function over the payoff space so that the solution is a payoff which maximizes or minimizes this function. Thus, a one-dimensional comparison is employed here in contrast to the multiple-dimensional comparisons used in Section 2.3.1. Such a solution process is often called an arbitration scheme and usually yields a unique solution point. We will now discuss two solution concepts, the Nash arbitration solution and compromise solutions, and their induced domination structures.

To obtain the Nash arbitration solution we first locate a 'security level', p_i^0, for each player i. An immediate approach which we follow to let p_i^0 be player i's maximin payoff, i.e., the maximum payoff he can obtain when all of the remaining players cooperate to minimize his payoff. For a different method of determining a security level, see [18] and [27].

To obtain existence and uniqueness of the Nash arbitration solution, we make the reasonable assumptions that P is compact and convex[6] and that there is some payoff p such that $p > P^0$. This last assumption implies that each player has an incentive to cooperate since there are payoffs where every player receives more than his security level.

Given a security level vector, $p^0 \in P$, let $P^0 = \{p \in P \mid p \geqq p^0\}$. Then the *Nash arbitration solution* is that payoff $\bar{p}$ which solves

$$(2) \qquad \underset{p \in P^0}{\text{maximize}} \prod_{i=1}^{n} (p_i - p_i^0).$$

To define a domination structure on P^0 yielding $\bar{p}$ as the unique nondominated solution we first observe that (2) is equivalent to

$$(3) \qquad \underset{p \in P^0}{\text{maximize}} \ln \prod_{i=1}^{n} (p_i - p_i^0) = \sum_{i=1}^{n} \ln (p_i - p_i^0).$$

Then for each $p \in P$ we let

$$(4) \qquad D(p) = \{d \in \mathbb{R}^n \mid d \cdot \nabla \left(\sum_{i=1}^{n} \ln (p_i - p_i^0) \right) \leqq 0\}.$$

$$= \{d \in \mathbb{R}^n \mid d \cdot \left(\frac{1}{p_1 - p_1^0}, \ldots, \frac{1}{p_n - p_n^0} \right) \leqq 0\}.$$

We note that the domination cone $D(p)$ is a half space which varies with p. Since the function being maximized in (3) is strictly concave, and since P^0 is convex and compact, a unique maximum point, $\bar{p}$, will exist, and $N_p(D(\cdot)) = \{\bar{P}\}$

EXAMPLE 2.3.5. For the game in Example 2.3.1 the maximin security level is $p^0 = (1, 1, 2)$. Clearly $P^0 \neq \phi$, since, for instance, the Pareto optimal extreme point, $(2, 3, 5)$, is in P^0. Therefore the assumptions yielding a unique Nash arbitration solution, $\bar{p}$, are satisfied and $\bar{p}$ is the payoff solving

$$(5) \qquad \underset{p \in P^0}{\text{maximize}}\ (p_1 - 1)(p_2 - 1)(p_3 - 2).$$

Since $\bar{p}$ is clearly Pareto optimal, we can restrict our attention to $P^0 \cap N_P$ where $N_P = \mathscr{H}(N_{ex})$ is determined in Example 2.3.2. It is easily seen that problem (5) is a standard nonlinear programming problem.

To define compromise solutions we can either translate the payoff space so that the security point p^0 is at 0 or introduce constraints so that, without loss of generality, we may assume that $p \geqq p^0$ for every $p \in P$. Following Yu [37] we then make the reasonable assumptions that W is compact and p is continuous. Since p_i is consequently a continuous function defined on a compact set, it has a maximum value p_i^* over W. The vector of these maximum payoffs, p^*, is called the *utopia point* for the game.

Next we consider the following 'regret' functions defined[7] on P:

$$R_q(p) = \left[\sum_{i=1}^{n} (p_i^* - p_i)^q\right]^{1/q} \quad \text{for} \quad q \geqq 1$$
$$R_\infty(p) = \text{maximum } \{(p_i^* - p_i) \mid i = 1, 2, \ldots, n\}.$$

A point p^q in P which minimizes $R_q(p)$ over P will be called a *compromise solution with parameter* q. For $1 < q < \infty$, $R_q(p)$ is strictly convex and hence if P is convex (see note 6), p^q is unique.

For $1 < q < \infty$ we define the following domination structure which will yield p^q as the unique nondominated solution:

$$(6) \qquad D_q(p) = \{d \in \mathbb{R}^n \mid d \cdot \nabla R_q(p) \geqq 0\}.$$

As in the case of the Nash arbitration solution we obtain a variable half space domination structure and $N_P(D_q(\cdot)) = \{p^q\}$. For $q = 1$ and $q = \infty$ a domination structure can be similarly constructed. We shall not elaborate this here. To illustrate the concept of compromise solutions we give the following.

EXAMPLE 2.3.6. For the game in Example 2.3.1 the security point is $p^0 = (1, 1, 2)$ and we use the constrained payoff space $P' = \{p \in P \mid p \geqq (1, 1, 2)\}$. The utopia point is $p^* = (5, 4, 5)$. Therefore, for $1 \leqq q < \infty$, to obtain the compromise solution with parameter q, we must minimize $R_q(p) = [(5 - p_1)^q + (4 - p_2)^q + (5 - p_3)^q]^{1/q}$ over P'. In [37] it is shown that for $1 \leqq q < \infty$, the compromise solution with parameter q is Pareto optimal, consequently to obtain p^q we can reduce our problem to miniming $R_q(p)$ over $\mathscr{H}(N_{ex}) \cap P'$ using $\mathscr{H}(N_{ex})$ from Example 2.3.2. As in Example 2.3.5, for each q (except $q = 1$ which yields a linear problem) we obtain a nonlinear programming problem. We shall not elaborate the details here. Also, compromise solutions with the parameter $q = \infty$ can similarly be found.

2.3.2. *General domination structures.* In Section 2.3.1 we saw that for Pareto optimality there is a constant domination cone, $\Lambda^{\geqq}$, which is $1/2^n$ of the entire space, while from (4) and (6) we observed that each domination cone was a half space for the solution concepts in Section 2.3.2. The difference between $1/2^n$ of the space and a half space is quite large when n is large. One may wonder whether there are domination structures lying between these extremes. In the following discussion we illustrate that such intermediate structures do occur.

Consider a game between two middle level managers in a business; suppose one is a production manager and the other a marketing manager. Suppose that for each important issue the president of the business is to make the final decision. In doing so he will ask the production and marketing managers to give their evaluation of the impact of each possible decision w on production and marketing, respectively. Denote these evaluations by $p_1(w)$ and $p_2(w)$ respectively. Note that $p_1(w)$ and $p_2(w)$ may be regarded as the payoff resulting from the decision w to the production and marketing divisions, respectively.

Now assume that the president will make his final decision may maximizing $\lambda_1 p_1(w) + \lambda_2 p_2(w)$ over all possible decisions where λ_i is the weight given to p_i. While the president may not be able to specify an exact weight vector, λ, he may be able to specify that the weight ratio λ_2/λ_1 must lie an a certain interval, say $\frac{1}{2} < \lambda_2/\lambda_1 < 2$.

Let $\Lambda = \{(\lambda_1, \lambda_2) \mid \frac{1}{2} < \lambda_2/\lambda_1 < 2\}$ and $\Lambda^* = \{(d_1, d_2) \mid d_1 + 2d_2 \leqq 0$ and $2d_1 + d_2 \leqq 0\}$ (see Figure 1). Note that Λ^* is the polar cone[8] of Λ.

For $(\lambda_1, \lambda_2) \in \Lambda$ it can be shown that a solution which maximizes $\lambda_1 p_1 + \lambda_2 p_2$ must be a nondominated solution for the domination structure $D(p) = \Lambda^*$ (for all $p \in P$). See [40]. Observe that Λ^* is much larger than $\Lambda^{\leqq}$ but is smaller than a half space.

The decision process illustrated above is not unusual and indicates that in

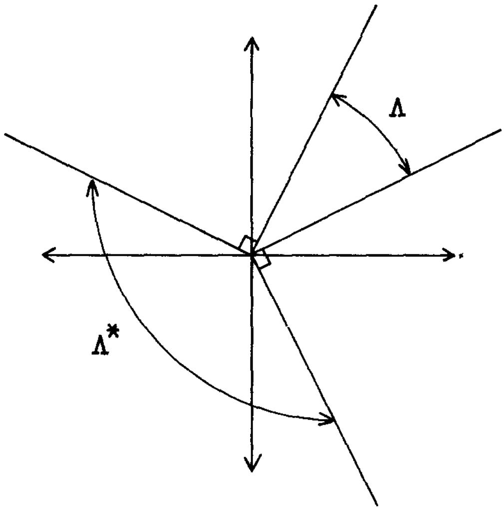


Fig. 1. The cones Λ and Λ*.

realistic game situations various general domination cones can arise. Thus we see that the cones of Sections 2.3.1 and 2.3.2 are actually special cases. Further, the above approach is equivalent to an arbitration process among the players to determine bounds (i.e., a cone Λ) for the weight vectors. Since Λ gives bounds for weights indicating the relative importance of each player's payoff, it may also be interpreted as a measurement of power among the players. Since the domination cone Λ^* and the cone Λ uniquely determine each other, we may as well regard domination cones as a tool for measuring the relative power held by the players. Methods for actually determining domination cones in specific applications can be found in [40].

2.4. *Solution Concepts and Domination Structures for Games in Characteristic Function Form*

In this section we focus on solution concepts for n-person games in characteristic function form. We shall first briefly describe basic concepts of characteristic function form games. Then we introduce 'classical' domination structures in the imputation space and in the coalition space. These domination structures will aid our intuitive understanding of the solution concepts and help to clarify relationships between the concepts. In Section 3 we will see that multicriteria n-person games induce domination structures more general than the 'classical' ones.

2.4.1. *Characteristic function form for N-person games.* An n-person game in characteristic function form is given by a set of players $N = \{1, 2, ..., n\}$ together with a real-valued function, v, (the 'characteristic function') defined on $\mathcal{N} = \{S \mid S \subseteq N\}$ such that $v(\phi) = 0$. Each element $S \in \mathcal{N}$ represents a coalition and $v(S)$ can be regarded as the total amount guaranteed to the members of S if they cooperate as a coalition.

Following von Neumann and Morgenstern [36], an n-person characteristic function form game can be induced from an n-person zero-sum normal form game as follows:
$v(S)$ is the maximum total payoff obtainable by the players in S when they cooperate, assuming the players in $N \setminus S$ are cooperating to minimize the payoff to S. This construction of a characteristic function form game has been criticized on several gounds (see [24] and [33]), nevertheless, the characteristic function form remains an important tool for analyzing cooperative n-person games. To illustrate this construction we give the following

EXAMPLE 2.4.1. The characteristic function form induced by the normal form game of Example 2.3.1 is the following[9]:

$$v(1) = 1, \quad v(2) = 1, \quad v(3) = 2,$$
$$v(12) = 4, \quad v(23) = 5, \quad v(13) = 7, \quad \text{and} \quad v(123) = 10.$$

A characteristic function form game derived from a normal form game is superadditive (i.e. $v(S) + v(T) \leqq v(S \cup T)$ for all $S, T \in \mathcal{N}$ with $S \cap T = \phi$). Of course, we may assume that a characteristic function form game is given without reference to any underlying normal form game. In these cases we will not assume superadditivity, but instead will assume the following condition:

$$(7) \qquad \sum_{i=1}^{n} v(i) < v(N)$$

The reason for this assumption will be discussed shortly.

Two games, v and v', *are strategically equivalent* if there exist real numbers $k > 0$ and a_i, $i = 1,..., n$, such that $v'(S) = kv(S) + \Sigma_{i \in S} a_i$ is satisfied for all $S \in \mathcal{N}$. A game v is *(0,1)-normalized* if $v(i) = 0$ for all $i \in N$ and $v(N) = 1$. The reason for our assumption (7) is that v is strategically equivalent to a

(0,1)-normalized game if and only if (7) is satisfied. (See [24] for instance, for the normalization procedure.) Consequently, we can study each game in this paper by passing to its strategically-equivalent (0,1)-normalized form.[10]

To illustrate (0,1)-normalization we provide the following

EXAMPLE 2.4.2. The (0,1)-normalized form of the game in Example 2.4.1 is

$$v'(i) = 0 \quad \text{for} \quad i \in N$$
$$v'(12) = 1/3, \quad v'(23) = 1/3, \quad v'(13) = 2/3$$
$$\text{and } v'(N) = 1.$$

To apply the definition of strategic equivalence to find v', we used

$$k = \frac{1}{v(S) - \sum_{i=1}^{3} v(i)} = \tfrac{1}{6}$$

and $a_i = -kv(i)$ implying that $a_i = -1/6, a_2 = -1/6$ and $a_3 = -1/3$.

An *imputation* for an n-person characteristic function form game v is a vector $x \in \mathbb{R}^n$ satisfying (i) $x_i \geqq v(i)$ for $i \in N$ and (ii) $\Sigma_{i=1}^n x_i = v(N)$.

The set of all imputations, denoted by I, will be called the *imputation space*. For (0,1)-normalized games the imputation space has obvious convenient properties and is given by $I = \{x \in \mathbb{R}^n \mid x \geqq 0 \text{ and } \Sigma_{i=1}^n x_i = 1\}$. The imputation space is empty, if $\Sigma_{i=1}^n v(i) > v(N)$, and contains one point, namely $(v(1), v(2), ..., v(n))$, if $\Sigma_{i=1}^n v(i) = v(N)$. Consequently, by assuming (7) we are assuming that the imputation space is non-trivial.

An imputation can be viewed as a potential rule ('distribution law') for distributing $v(N)$, assuming there is full cooperation among all n-players. A primary focus in characteristic function form games is to locate an imputation or set of imputations which is stable in some sense or which provides a 'fair' distribution law.

Given an imputation $x \in \mathbb{R}^n$ and a coalition $S \in \mathcal{N}$, the *coalitional excess* of S with respect to x, denoted by $e_S(x)$, is given by $e_S(x) = v(S) - x(S)$ where $x(S) = \Sigma_{i \in S} x_i$. A positive coalitional excess, $e_S(x)$, indicates that, as a coalition, the members of S do not receive their full value in the game at the imputation x. (See Proposition 2.4.4 for further discussion.)

Having presented the necessary preliminary concepts, we will now discuss solutions for characteristic function form games.

2.4.2. *Concepts (usually) yielding a set or sets of many solution points.* To illustrate 'classical' domination structures in both the imputation space and the coalition space, we will focus mainly on the core solution concept (and a related concept, the relaxed cores) in this section. This analysis can be extended to the other concepts in the lower left-hand section of Table I, but we shall not include the extension in this paper.

For a characteristic function form game v, we state the following.

DEFINITION 2.4.3. For imputations x and y and for $S \in \mathcal{N}$, *x dominates y via S*, denoted $x \succ_S y$, if and only if the following are satisfied:

(8) $x_i > y_i$ for all $i \in S$.

(9) $x(S) \leqq v(S)$.

Condition (9) implies that $x(S) (= \Sigma_{i \in S} x_i)$ is actually obtainable by the players in S. The players in S would rationally accept only imputations not dominated via S. The *core* of the game v, denoted $C(v)$, is the set of all imputations which are not dominated via *any* coalition $S \in \mathcal{N}$. We immediately obtain the useful

PROPOSITION 2.4.4. *For $S \in \mathcal{N}$, an imputation x is dominated via S if and only if $x(S) < v(S)$.*

Proof. For sufficiency, suppose $x(S) < v(S)$. Let $j \in N \setminus S$ such that $x_j > 0$. (Such a j exists since $x(S) < v(S) \leqq v(N)$ implies that $x(N \setminus S) = v(N) - x(S) > 0$.)

$$\text{Define } y = \begin{cases} x_i + x_j \backslash |S|, & \text{if } i \in S \\ 0, & \text{if } i = j \\ x_i, & \text{otherwise} \end{cases}$$

Then y is an imputation which dominates x via S.

For necessity suppose there is an imputation y such that $y \succ_S x$. Then (8) implies that $y_i > x_i$ for all $i \in s$ and hence $y(S) > x(S)$. (9) implies $v(S) \geqq y(S)$ which together with $y(S) > x(S)$ implies that $v(S) > x(S)$.

Q.E.D.

From Proposition 2.4.4 we can immediately see that domination through a single player coalition is impossible. If $S = \{i\}$, then $x(S) = x_i \geqq v(i)$ since x is an imputation. Therefore, by the Proposition, x is not dominated via S.

We can also see that an imputation x is in the core if and only if no coalitional excess in positive. That $e_S(x) = v(S) - x(S) \leqq 0$ for all $S \in \mathcal{N}$ is equivalent to $v(S) \leqq x(S)$ for all $S \in \mathcal{N}$. By Proposition 2.4.4, this is equivalent to x is not dominated via S for all $S \in \mathcal{N}$ or $x \in C(v)$. Thus we have

$$(10) \qquad C(v) = \{x \in I \mid x(S) \geqq v(S) \quad \text{for all} \quad S \in \mathcal{N}\}.$$

For the remainder of Section 2.4.2 we will assume that the games we consider are in (0,1)-normal form. Then, letting $m = 2^n - n - 2$, we fix an ordering for the proper coalitions which have more than one player in each coalition: $S_1, S_2, ..., S_m$. We will use Y to denote $\mathbb{R}^m$ indexed by the coalitions $S_1, S_2, ..., S_m$ and will refer to Y as the *coalition space*. Each vector $v \in Y$ corresponds in the obvious way to a (0,1)-normalized n-person game. Consequently, we will refer to a point $v \in Y$ as a 'game point'.

As in Spinetto [34], we define the following mapping ϕ on I into the coalition space Y which yields a one-to-one correspondence between I and $\phi(I)$. For each player i, define the game point $V_i \in Y$ by

$$V_i(S_j) = \begin{cases} 1, & \text{if } i \in S_j \\ 0, & \text{otherwise} \end{cases} \quad (\text{for } j = 1, ..., m).$$

V_i is a simple game having player i as a veto player.[11] Then for $x \in I$, define

$$(11) \qquad \phi(x) = \sum_{i=1}^{n} x_i V_i.$$

We will use G_n to denote $\phi(I)$, the image of I under ϕ. Clearly, $G_n = \mathcal{H}(\{V_1, ..., V_n\})$, the convex hull of $\{V_1, ..., V_n\}$. Also, we observe that ϕ maps the extreme points of I to the extreme points of G_n, namely $V_1, ..., V_n$. The extreme points of I are the basic unit vectors in X, $u^1, ..., u^n$

$$\left(\text{i.e., } u_j^i = \begin{cases} 1, & \text{if } j = i \\ 0, & \text{otherwise} \end{cases}\right).$$

Then the definition of ϕ in (11) implies that $\phi(u^i) = V_i$.

Further, we can see that for an imputation $x \in I$,

$$\begin{aligned}(12)\qquad \phi(x) &= \sum_{i=1}^{n} x_i V_i \\ &= (\phi_{S_1}(x), \ldots, \phi_{S_m}(x)) \\ &= (\sum_{i \in S_1} x_i, \ldots, \sum_{i \in S_m} x_i) \\ &= (x(S_1), \ldots, x(S_m))\end{aligned}$$

By combining (10) and (12), for a game $v \in Y$, we have

PROPOSITION 2.4.5. An imputation x is in $C(v)$ if and only if $\phi(x) \geqq v$.

From this Proposition we can immediately conclude the following which is Corollary 1 in Spinetto [34]:

PROPOSITION 2.4.6. $C(v) \neq \phi$ if and only if $(v + \Lambda^{\geqq}) \cap G_n \neq \phi$. Further, $C(v) = \{x \in I \mid \phi(x) \in (v + \Lambda^{\geqq})\}$.

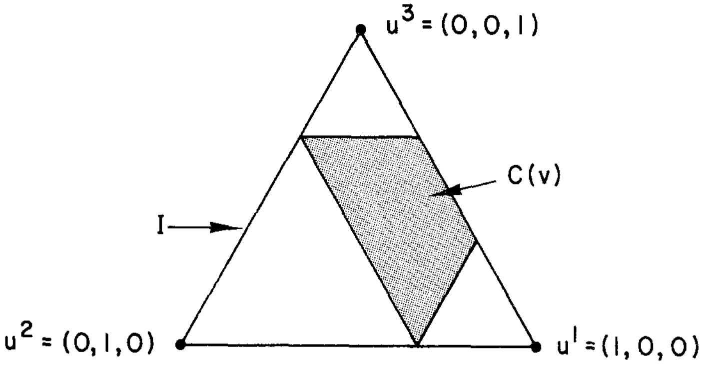


Fig. 2. $C(v)$ in the imputation space.

In Figure 2 we illustrate the core of the game in Example 2.4.2 in the imputation space, I. In the coalition space Y in Figure 3, we have located the game point v corresponding to this example and have illustrated $\phi(C(v))$, the image of the core in G_n. In order to obtain an intuitive understanding, in Figure 4 we picture Y as a two-dimensional space with the interpretation that the set of coalitions is abstractly represented by two coalitions. The n-dimensional simplex, G_n, in the original space Y is now represented by a line segment. Then $\phi(C(v))$ is given by the shaded portion of G_n.

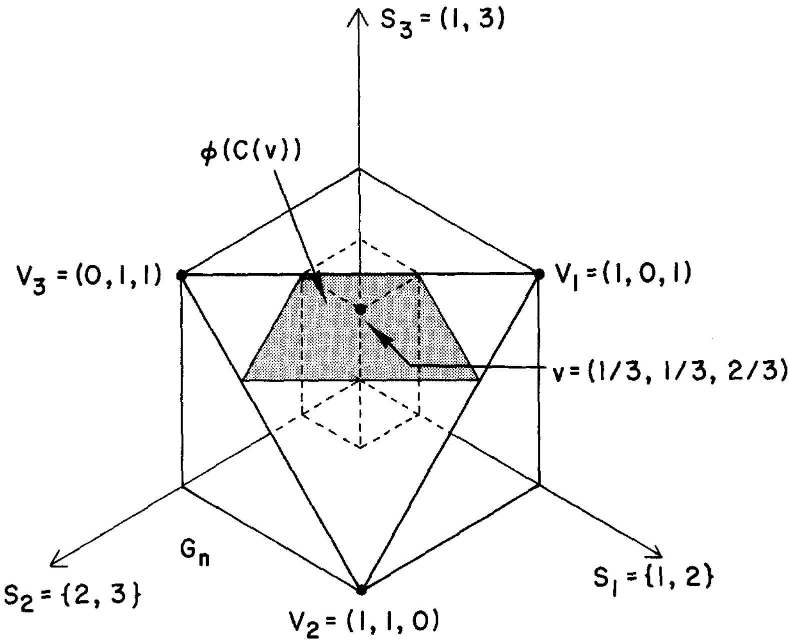


Fig. 3. $\phi(C(v))$ in the coalition space.

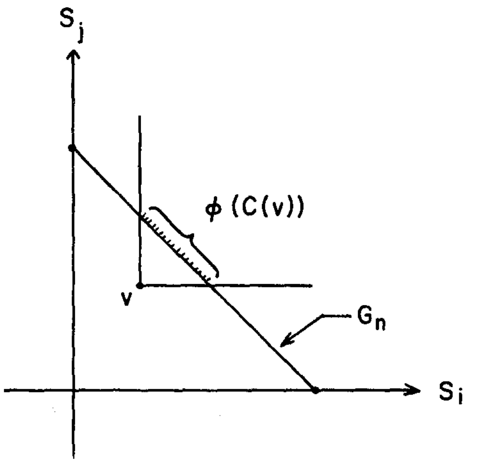


Fig. 4. A representation of Y and $\phi(C(v))$.

Both Figures 3 and 4 illustrate the intuitive idea that $C(v) \neq \phi$ if and only if v lies 'on or below' the simplex G_n.

We now define domination structures induced by the core concept. For $S \in \mathcal{N}$ we define the cone

$$\Lambda_S^< = \{d \in \mathbb{R}^n \mid d_i < 0 \quad \text{for all} \quad i \in S\} \cup \{0\}$$

Then corresponding to each coalition $S \in \mathcal{N}$ we define a domination structure on I as follows:

$$D_S(x) = \begin{cases} \Lambda_S^<, & \text{if} \quad x(S) \leqq v(S) \\ \{0\}, & \text{otherwise} \end{cases} \tag{13}$$

This domination structure is closely related to the concept of domination in Definition 2.4.3. We see that for imputations x and y, $x \succ_S y$ if and only if $y \in x + D_S(x)$ and $y \neq x$. For simplicity we let $N(D_S)$ denote nondominated solutions in I under the domination structure, $D_S(\cdot)$ (in the notation of Section 2.1, $N(D_S) = N_I(D_S(\cdot))$). We immediately obtain

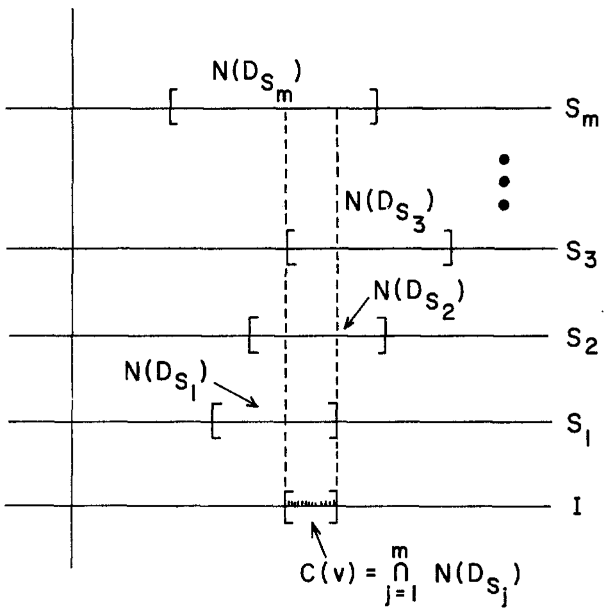


Fig. 5. 'Levels' of domination structures.

PROPOSITION 2.4.7. For an n-person game, v, $N(D_S) = \{x \in I \mid x(S) \geqq v(S)\}$ and $C(v) = \cap_{j=1}^{m} N(D_{S_j})$.

One can regard the domination structures in (13) as defined on parallel copies or 'levels' of I indexed by $S_1, \ldots, S_m$ (see Figure 5) such that for each $j = 1, \ldots, m$, $D_{S_j}(x)$ indicates the domination directions on level S_j. Thus we can think of locating the core of a game by projecting the nondominated points for each level down to a single copy of I and forming the intersection. This conceptualization will be useful when we discuss multicriteria n-person games. In that occasion, corresponding to each coalition and each criterion we have a domination cone as in (13) (see Section 3.2).

We will refer to the domination structures in (13) as the 'classical' domination structures on I. Other domination structures will be discussed in the following sections.

This approach provides two immediate ways for generalizing the concept of the core. First, for an index set $K \subseteq \{1, 2, \ldots, m\}$ we can define the *relaxed core indexed by* K, denoted $C_K(v)$, by

$$C_K(v) = \bigcap_{j \in K} N(D_{S_j}).$$

One may think of the coalitions indexed by K as the coalitions which are permitted or able to form; the imputations in $C_K(v)$ take into account objections from only these coalitions. The concept of a relaxed core is closely related to core-stem solutions (see [13b] and [33]) and ψ-stability (see [24] and [27]). For further discussion, see Sections 3.2.2 and 3.2.4.

Another method for generalizing the core is to replace $\Lambda_S^{\leqq}$ in (13) by an arbitrary cone Λ_S. This would generalize the binary relation $\succ_S$ to a reasonable class of binary relations; von Neumann and Morgenstern [36] point out the desirability of generalizing $\succ_S$ to new classes of relations.

We now define a domination structure in the coalition space, Y, which also will yield the core of the game. For each $v \in Y$, define the constant domination cone

$$D(v) = \{d = (d_{S_1}, \ldots, d_{S_m}) \mid \text{at least one } d_{S_j} < 0\} \cup \{0\} = (Y \setminus \Lambda^{\geqq}) \cup \{0\}.$$

(Note that $D(v)$ is not convex in contrast to our usual assumption for domination structures. See Figure 6 where Y is represented abstractly as a two-dimensional space.)

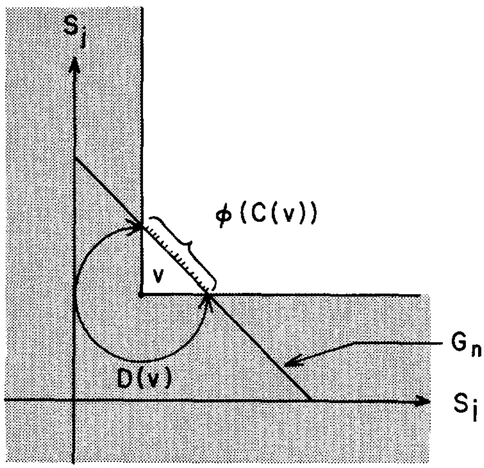


Fig. 6. The domination cone, $D(v)$.

Then given a game $v \in Y$, for each game $v' \in [v + D(v)] \setminus \{v\}$) the set of game points 'dominated' by v), at least one coalition has a smaller value than it has at the current point v. Thus given an imputation $x \in I$, if the induced game point $\phi(x)$ is in $[v + D(v)] \setminus \{v\}$, then some coalition, S, will object to playing the game $\phi(x)$ instead of v (because $\phi_S(x) < v(S)$ and $\phi_S(x) = x(S)$ by (12), x is dominated via the coalition S according to Proposition 2.4.4).

Thus we can think of the complementary cone of $D(v)$, namely $\Lambda^{\geqq}$, as representing the set of directions which are not objectionable to any coalition. Given that the game v is being played, no coalition could effectively object to instead playing any game, v', in $v + \Lambda^{\geqq}$ (because the value of every coalition at v' is at least as great as its value at v).

Then any imputation, x, such that its induced game point $\phi(x)$ is not dominated by v under $D(\cdot)$ (i.e., $\phi(x) \in v + \Lambda^{\geqq}$), will be in the core of v. Thus we can formally write $C(v) = \{x \in I \mid \phi(x)$ is not dominated by v under $D(\cdot)\} = \{x \in I \mid \phi(x) \in v + \Lambda^{\geqq}\}$ (cf. Proposition 2.4.6).

The domination structure $D(\cdot)$ will be referred to as the 'classical' domination structure in Y. Similar to the classical structures, domination structures can be defined on I and Y for the other solution concepts in the lower left-hand section of Table I, but we shall not stop to do so. One should note that the stability conditions imposed by each concepts must be taken into consideration when using the underlying domination structure to induce that solution concept.

2.4.3. *Concepts (usually) yielding a unique solution point*. In this section we shall show that the Shapley value and convex nuclei are in fact arbitration schemes over the coalitional excess space. Consequently these solution concepts will induce domination structures similar to those described in Section 2.3.2. For the nucleolus we obtain a sequence of domination structures which yield the nucleolus as a nondominated solution.

The concept of convex nuclei was first proposed by Charnes and Kortanek [12b] as a generalization of the nucleolus. Given a game v, consider the function, $G(x)$, defined on I by $G(x) = \Sigma_{S\in\mathcal{N}'} g_S(e_S(x))$ where $\mathcal{N}' = \mathcal{N} \setminus \{\phi, N\}$ and where, for each $S \in \mathcal{N}'$, g_S is a strictly convex function[12] of the coalitional excess, $e_S(x)$ (implying that $G(x)$ is strictly convex). For our discussion we will also assume that G is differentiable although more general cases are discussed in [12b]. Then imputations which minimize $G(x)$ over I are called *convex nuclei*.

Let $m' = 2^n - 2$. For the discussion concerning the convex nuclei of a game v, we fix an ordering on the coalitions in $\mathcal{N}'$, $S_1,\ldots,S_{m'}$, and let the coalitional excess space Y', be $\mathbb{R}^{m'}$ indexed by $S_1, \ldots, S_{m'}$. Thus Y' can be viewed as the m'-dimensional coalitional space translated so that v is at the origin. We then define a mapping, ψ, on I into Y' by $\psi(x) = (e_{S_1}(x), \ldots, e_{S_{m'}}$ (x)) for all $x \in I$. For each $y = (y_{S_1}, \ldots, y_{S_{m'}}) \in \psi(I)$, we define the domination cone[13]

$$(15) \qquad D(y) = \{d \mid d \cdot \nabla(\sum_{j=1}^{m'} g_{S_j}(y_{S_j})) \geqq 0\}.$$

Then $N_{\psi(I)}$, the set of nondominated points of $\psi(I)$ with respect to $D(\cdot)$, in fact contains a unique point. The convex nuclei with respect to the function G are the points in the inverse image of $N_{\psi(I)}$ (i.e., $\psi^{-1}(N_{\psi(I)})$). Under certain linear independence conditions it can be shown that this inverse image contains a unique point (for details, see [12b]).

A domination structure on the imputation, I, yielding the convex nuclei can be similarly constructed using the linear transformation ψ. We will not include the details here.

The *Shapley value* is a unique imputation satisfying three axions given by Shapley [31]. The Shapley value for player i, $\theta_i(v)$, is given by

$$\theta_i(v) = \sum_{S \in \mathcal{N}} \frac{(|S|-1)!(n-|S|)!}{n!} [v(S) - v(S \setminus \{i\})]$$

where $|S|$ denotes the number of players in S. Charnes and Keane [10] show that the Shapley value may be obtained as a convex nucleus for a particular function $G(x)$. Therefore, we can use the domination structure analysis for convex nuclei to obtain the Shapley value.

The ℓ_p-center [34] for $p > 1$ is a unique imputation which minimizes a strictly convex function (associated with the ℓ_p-norm) of the coalitional excesses for the proper coalitions of more than one player. Thus ℓ_p-centers induce domination structures (in a coalition excess space of dimension $2^n - n - 2$) which are similar to those induced by convex nuclei. We shall not repeat them here.

For the nucleolus we first define, for each imputation x, a vector $\theta(x) = (e_{S_1}(x), ..., e_{S_{2^n}}(x))$ where $e_{S_j}(x)$ is the coalitional excess for S_j and such that for each x, the coalitions are ordered so that $e_{S_k}(x) \geqq e_{S_j}(x)$ whenever $k < j$. Intuitively, $\theta(x)$ gives the coalitional excesses in decreasing order for all the coalitions. The *nucleolus*, defined in [29], is the imputation x^N such that $\theta(x^N)$ is minimal under the lexicographic ordering on $\theta(I) = \{\theta(x) \mid x \in I\}$.

For each $\theta(x) \in \theta(I)$ and each $j = 1, ..., 2^n$ we define the constant domination cone $D_j(\theta(x)) = \{d \in R^{2^n} \mid d_j \leqq 0, d_k = 0 \text{ if } k \neq j\}$. Let N_1 be the set of all nondominated solutions of $\theta(I)$ with respect to the domination structure $D_1(\cdot)$. Inductively, define N_{k+1} to be the set of all nondominated solutions of N_k with respect to the domination structure $D_{k+1}(\cdot)$ (restricting the domination structure to N_k). The final set of nondominated points, N_{2^n}, thus will be sequentially derived. Note that N_{2^n} contains the unique lexicographically minimal element of $\theta(I)$. The nucleolus is the unique inverse image of N_{2^n} (i.e. $\theta^{-1}(N_{2^n})$). (The uniqueness was proven in [29].)

This concludes the discussion of solution concepts for single criterion games. Multicriteria games will be discussed in the next section.

3. MULTICRITERIA N-PERSON GAMES

Frequently in situations involving (partial) conflict among n-persons (or groups), the 'players' will use several criteria to assess the results of their decisions. In order to more accurately reflect and understand such situations, n-person games can be naturally extended to multicriteria n-person games. Using our discussion of solution concepts and their domination structures for single criterion n-person games as a basis, in this section we examine various solution approaches for multicriteria n-person games in both normal form and characteristic function form.

To illustrate the concept of multicriteria games we consider some situations where multicriteria game models naturally arise. The members of most deliberative decision-making bodies, such as committees or legislatures, frequently consider several noncomensurable aspects of the consequences of their decisions. For example, a faculty committee, when evaluating candidates for a position will probably consider each candidate's potential contribution to research, teaching and the University as criteria. In such situations one may wonder whether the committee members can agree on the relative importance of each of the criteria or whether each committee member will weigh the criteria in an independent manner.

An interesting type of multicriteria games arises when the payoffs for some single criterion game depend on a future uncertain event. For example, construction companies may bid for a government project for which funding and awarding of the contract depend on the outcome of the next election. Then the payoff under each possible outcome of the election can be viewed as one criterion of a multicriteria game. We will consider a natural linear parametrization of the criteria which, in this type of game, can be viewed as a probability distribution over the possible outcomes of the future uncertain event.

We finally note that multicriteria game models may be useful in understanding and resolving conflict/cooperation situations which have recently received extensive media coverage. For example, the decision to build a nuclear power plant or a new highway will often generate concern among conflicting groups involving criteria such as cost, convenience and environmental impact. Also, complicated and interrelated global problems involving food, energy, natural resources, population growth and economic growth clearly generate many multicriteria situations where choices leading to cooperation or conflict face the nations of the world.

We first describe multicriteria n-person games in normal form and then briefly sketch several possible solution approaches.

3.1. *Multicriteria N-Person Games in Normal Form*

For multicriteria n-person games in normal form we assume that each player has a vector-valued payoff function defined on a joint payoff space W. In the most general formulation, player i has a set of criteria indexed by $1, \ldots, \ell_i$ and his payoff function, p^i, is defined on W into $\mathbb{R}^{\ell_i}$. Each of the other players may share some, none or all of player i's criteria.

The payoff space for player i, denoted by P^i, is given by $P^i = p^i(W)$. In our notation P^i has dimension ℓ_i. The full payoff space, denoted by P^F, is the space of dimension $\Sigma_{i=1}^{n} \ell_i$ given by

$$P^F = p(W) = (p^1, \ldots, p^n)(W) = \{[(p^1(w)), (p^2(w)), \ldots, (p^n(w))] \mid w \in W\}.$$

EXAMPLE 3.1.1. In this example the notation for the players' pure strategy sets is the same as in Example 2.3.1. For each choice of pure strategies in $\Pi_{i=1}^{3} Q_i$, the matrix (14) gives a six-dimensional payoff vector where the first two coordinates represent the payoff to player 1, the next two coordinates represent the payoff to player 2 and the last two coordinates represent player 3's payoff.

Entries in the matrix may be interpreted as payoffs which depend on the outcome of a future event. More specifically, the values in columns labeled A may be regarded as the payoff if party A wins forthcoming election and the values in columns B may be viewed as the payoff when party B is victorious. Note that in this example, each player has the same two criteria. The full payoff space has dimension six and is given by $P^F = \{\Sigma_{i=1}^{8} w_i p^i \mid w \in W\}$ where $p^1, \ldots, p^8$ represent the eight six-dimensional payoff vectors in the matrix (14).

	γ_1			γ_2		
	$A\ B$	$A\ B$	$A\ B$	$A\ B$	$A\ B$	$A\ B$
$\alpha_1 \beta_1$	[(5,1),	(1,2),	(2,4)]	[(1,1),	(1,2),	(2,4)]
$\alpha_1 \beta_2$	[(3,2),	(1,1),	(4,4)]	[(1,1),	(1,2),	(3,6)]
$\alpha_2 \beta_1$	[(2,0),	(3,3),	(5,5)]	[(2,3),	(2,1),	(5,5)]
$\alpha_2 \beta_2$	[(5,2),	(4,1),	(1,3)]	[(1,0),	(0,0),	(5,4)]

In the next section we discuss the application of the solution concepts in Section 2.3 to the full payoff space. Next we consider solution possibilities when each player, i, limits his attention to his own payoff space, P^i. Finally we discuss solutions when each player reduces his multicriteria payoff to a single criterion.

3.1.1. *Solution concepts in the full payoff space.* The players may agree to apply a solution concept (such as those discussed in Sections 2.3.1 and 2.3.2) to the full payoff space, P^F. In doing so, each player implicitly respects and considers all of the criteria for all of the other players in the solution process. This approach is equivalent to viewing P^F as the payoff space for a single criterion normal form game with $\Sigma_{i=1}^n \ell_i$ players (where ℓ_i is the number of criteria for player i). In other words, the payoff for each criterion is interpreted as the payoff of a single player.

EXAMPLE 3.1.2. To locate the compromise solution with parameter q (see Section 2.3.2) for the full payoff space of the multicriteria game of Example 3.1.1, we need to use the utopia point $p^* = [(5,3), (4,3), (5,6)]$. Then for $1 \leqq q < \infty$, we would minimize

$$R_q(p) = [(5-p_1)^q + (3-p_2)^q + (4-p_3)^q + (3-p_4)^q + (5-p_5)^q + (6-p_6)^q]^{1/q}$$

over P^F (as in Section 2.3.2 we assume that for all $p \in P^F$, $p \geqq p^0$ [14]).

Also, P^F can be viewed as the criteria space for a single-decision maker (possibly an outside arbiter for the game) problem with $\Sigma_{i=1}^n \ell_i$ criteria. Hence, any solution concept used for single player multicriteria decision problems (see [40]) could be applied to P^F.

In general, the players may agree to use a particular domination structure in P^F (such as those discussed in Section 2.3.1 which generate Pareto optimal solutions and satisficing solutions or more general structures as illustrated in Section 2.3.3) and select the final payoff from among the nondominated solutions with respect to that domination structure. As in Section 2.3.3 a domination structure used by agreement among the players in this way gives an implicit measurement of the power held by the players.

3.1.2. *Solution concepts using each player's payoff space.* In contrast to the solution approaches discussed in the preceding section, suppose that each player i determines a domination structure, $D_i(\cdot)$, in his own payoff space, P^i. Then let W_0 be the set of decisions such that for each $w \in W_0$, $p^i(w)$ is nondominated with respect to $D_i(\cdot)$ for all $i = 1, \ldots, n$. Any decision in W_0 permits every player to 'claim victory' (i.e. achieve a nondominated payoff) and would be an acceptable solution for the game.

Clearly W_0 may be empty and, at the same time, there may exist payoffs, p, such that $p > p^0$ (see note 14) so that the players still have an incentive to cooperate. In this case, suppose that the nondominated set, N_i, in each player's payoff space is convex. (For example, if p^i is convex, domination structures for some satisficing solution concepts will yield such nondominated sets.) Then we can treat the set N_i as a 'utopia set' for each player and obtain the payoffs which minimize the sum of the distances to each player's utopia set. We shall not elaborate the details here.

Other alternative approaches which could be used when W_0 is empty are discussed in Section 3.1.1 and 3.1.3.

3.1.3. *The reduction of each player's payoff to a single criterion.* Suppose that each player i defines a real-valued (possibly nonlinear) 'utility' function u_i on his payoff space, P^i. Then, by using the function $u_i \circ p^i$ as player i's payoff function, the multicriteria normal form game is reduced to a single criterion normal form game and the solution concepts of Section 2.3 can be applied.

In general, it is very difficult to determine such a utility function. However each player may be able to specify a vector λ^i (a probability vector of dimension ℓ_i) of weights for his criteria. In the case where the criteria represent payoffs for different outcomes of a future event, such a weight vector can be naturally seen as the player's estimate of the probability distribution for the outcomes of the future event. In this case player i's payoff function is $\Sigma_{k=1}^{\ell_i} \lambda_k^i p_k^i$ (which is the expected payoff for the probability distribution λ^i) and the multicriteria game is reduced to a single criterion game, allowing the solution concepts of Section 2.3 to be applied.

It would be unusual for a player to be able to specify an exact weight vector. More realistically, each player might specify a set of weight vectors, say the vectors in a cone Λ_i. Then each choice of n weight vectors[15] $\lambda_i, \ldots, \lambda_n$ where $\lambda_i \in \Lambda_i$ for $i = 1, \ldots, n$ yields a single criterion normal form game. When a solution concept from Section 2.3 is applied to the set of games generated in this way, a set of solution points would be determined.

The case where all of the players have the same criteria, $1, \ldots, \ell$, provides possibilities for a simpler reduction. The players might agree to simultaneously use the same ℓ-dimensional weight vector λ or a set of such weight vectors given by Λ. Then for each weight vector $\lambda \in \Lambda$, a single criterion game is derived by using $\Sigma_{k=1}^{\ell} \lambda_k p_k^i$ as player i's payoff. Again, by applying each solution concept to the set of such induced single criterion games, a set of solution points is derived.

EXAMPLE 3.1.3. For $\lambda \in L = \{\lambda = (\lambda_1, \lambda_2) \mid \lambda \geqq 0 \text{ and } \lambda_1 + \lambda_2 = 1\}$ we can set $\lambda_2 = 1 - \lambda_1$ and obtain the following parametrized normal form of the payoff matrix (14):

$$\begin{array}{c} \\ \alpha_1\beta_1 \\ \alpha_1\beta_2 \\ \alpha_2\beta_1 \\ \alpha_2\beta_2 \end{array} \begin{array}{c} \begin{array}{cc} \qquad\gamma_1 \qquad\qquad\qquad & \qquad\gamma_2 \end{array} \\ \left[\begin{array}{ll} (1+4\lambda_1, 2-\lambda_1, 4-2\lambda_1) & (1, 2-\lambda_1, 4-2\lambda_1) \\ (2+\lambda_1, 1, 4) & (1, 2-\lambda_1, 6-3\lambda_1) \\ (2\lambda_1, 3, 5) & (3-\lambda_1, 1+\lambda_1, 5) \\ (2+3\lambda_1, 1+3\lambda_1, 3-2\lambda_1) & (\lambda_1, 0, 4+\lambda_1) \end{array}\right] \end{array}$$

3.2. *Multicriteria N-Person Games in Characteristic Function Form*

A multicriteria n-person game in characteristic function form consists of the player set, N, and a vector-valued characteristic function, $v = (v_1, \ldots, v_\ell)$. Each coordinate function, v_k, $k = 1, \ldots, \ell$ is a characteristic function in the sense of Section 2.4.1 and may or may not have been derived from an underlying normal form game. By separately (0,1)-normalizing each coordinate function v_k, $k = 1, \ldots, \ell$, we obtain the multicriteria characteristic function $v' = (v_1', \ldots, v_\ell')$. To illustrate the concept of a multicriteria characteristic function, we give the following

EXAMPLE 3.2.1. We use the normal form game of Example 3.1.1 to induce a multicriteria characteristic function. Using the payoffs for the first criterion we induce (as discussed in Section 2.4.1) a single criterion characteristic function, v_1. Using the interpretation given in Example 3.1.1, we can view v_1 as giving the value for each coalition, should party A win the election. Similarly we use the payoff for the second criterion to induce a single criterion characteristic function, v_2, which gives coalitional values in the event party B in the winner. Thus we derive the multicriteria characteristic function $v = (v_1, v_2)$ given by:

$$v(1)=(1,1),\ v(2)=(1,1),\ v(3)=(2,4)\ v(12)=(4,3),\ v(23)=(5,20/3),\ v(13)=(7,6) \text{ and } v(123)=(10,9).$$

Then by (0,1)-normalizing each criterion separately, we obtain:

$$v'(1)=v'(2)=v'(3)=(0,0),\ v'(12)=(1/3,\ 1/3),\ v'(23)=(1/3,\ 5/9),\ v'(13)=(1/3,\ 1/3) \text{ and } v'(123)=(1,1),$$

For most of the remainder of Section 3.2 we will assume, as is the case in this example, that all of the players have the same set of criteria. When this assumption holds for a normal form multicriteria game, as in Example 3.1.1, we see that a multicriteria characteristic function form game can naturally be induced. In Section 3.2.4, however, we relax this assumption and treat more general cases.

With every multicriteria characteristic function $v=(v_1, \ldots, v_\ell)$ and its associated (0,1)-normalized form, $v'=(v_1', \ldots, v_\ell')$, we can naturally associate the following single criterion characteristic functions:

(i) $\bar{v}$ defined by $\bar{v}(S) = \max\limits_{1 \leqq k \leqq \ell} v_k(S)$ for all $S \in \mathcal{N}$.

(ii) the (0,1) normalization of $\bar{v}$, denoted $(\bar{v})'$

(iii) $\overline{(v')}$ defined by $\overline{(v')}(S) = \max\limits_{1 \leqq k \leqq \ell} v_k'(S)$ for all $S \in \mathcal{N}$

(iv) $\underline{v}$ defined by $\underline{v}(S) = \min\limits_{1 \leqq k \leqq \ell} v_k(S)$ for all $S \in \mathcal{N}$

(v) the (0,1) normalization of $\underline{v}$, denoted $(\underline{v})'$

(vi) $(\underline{v'})$ defined by $(\underline{v'})(S) = \min\limits_{1 \leqq k \leqq \ell} v_k'(S)$ for all $S \in \mathcal{N}$.

In general, $(\bar{v})' \neq \overline{(v')}$ and $(\underline{v})' \neq (\underline{v'})$. The games (i)–(vi) above will be useful reference points for our later discussion of solution approaches.

3.2.1. *Parametrization of multicriteria characteristic functions*. To reduce the multicriteria characteristic function to a single criterion characteristic function, the players might be able to cooperatively determine a real-valued 'utility' function defined on $v(\mathcal{N})$ (cf. Section 3.1.3). However, such an approach would be very difficult.

It is much more likely that the players might agree on a probability weight vector, λ, or a set of such weight vectors[16], $\Omega \subseteq L = \{\lambda \in \mathbb{R}^{\ell} \mid \lambda \geqq 0$ and $\Sigma_{k=1}^{\ell} = 1\}$, which would then be used to parametrize the characteristic function and thus obtain a single criterion characteristic function. We will discuss two different methods for parametrizing a characteristic function $v = (v_1, ..., v_{\ell})$. One method is a direct parametrization of the characteristic function and the other is through a parametrization of the underlying normal form game (see Section 3.1.3). In the following sections we will discuss solution approaches involving these derived parametrized games.

Given $\lambda \in L$, we can define the parametrized game v_{λ}^{c} (the superscript c denotes a parametrization directly on the characteristic function) by $v_{\lambda}^{c}(S) = \lambda \cdot v(S) = \Sigma_{k=1}^{\ell} \lambda_k v_k(S)$ for all $S \in \mathcal{N}$. We will denote the (0,1)-normalization of v_{λ}^{c} by $(v_{\lambda}^{c})'$. On the other hand, we might first (0,1)-normalize v to obtain $v' = (v', ..., v_{\ell}')$ and then parametrize as above to obtain $(v')_{\lambda}^{c}$. Clearly, $(v')_{\lambda}^{c}$ is in (0,1)-normalized form.

EXAMPLE 3.2.2. Using the game in Example 3.2.1 we illustrate parametrization directly on the characteristic function form. In this case $L = \{\lambda \in \mathbb{R}^2 \mid \lambda \geqq 0$ and $\lambda_1 + \lambda_2 = 1\}$. Setting $\lambda_2 = 1 - \lambda_1$ we obtain

$$
\begin{array}{lll}
v_{\lambda}^{c}(1) = 1 & v_{\lambda}^{c}(2) = 1 & v_{\lambda}^{c}(3) = 4 - 2\lambda_1 \\
v_{\lambda}^{c}(12) = 3 + \lambda_1 & v_{\lambda}^{c}(23) = 20/3 - (5/3)\lambda_1 & \\
v_{\lambda}^{c}(13) = 6 + \lambda_1 & v_{\lambda}^{c}(123) = 9 + \lambda_1 &
\end{array}
$$

By (0,1)-normalizing v_{λ}^{c}, we obtain

$$
(v_{\lambda}^{c})'(12) = 1/3 \qquad (v_{\lambda}^{c})'(23) = \frac{5 + \lambda_1}{9(1+\lambda_1)}
$$

$$
(v_{\lambda}^{c})'(13) = \frac{1 + 3\lambda_1}{3(1+\lambda_1)}
$$

On the other hand, parametrizing the (0,1)-normalized form, v', in Example 3.2.1 yields the following game:

$$
\begin{array}{ll}
(v_{\lambda}')^{c}(12) = 1/3 & (v_{\lambda}')^{c}(23) = 5/9 - 2/9\lambda_1 \\
(v_{\lambda}')^{c}(13) = 1/3. &
\end{array}
$$

In this example, we can see that in general, $(v_\lambda^c)' \neq (v_\lambda')^c$. For instance, letting $\lambda_0 = (1/2, 1/2)$ yields $(v_{\lambda_0}^c)'(23) = 11/27$ and $(v')_{\lambda_0}^c(23) = 4/9$.

In the parametrization approach through the underlying normal form, we first use a given weight vector λ to parametrize the underlying normal form (as in Section 3.1.3) and then induce a single criterion characteristic function (as in Section 2.4.1). Specifically, given $\lambda \in L$, we parametrize player i's normal form payoff function p^i to obtain the real-valued function $\Sigma_{k=1}^{\varrho} \lambda_k p_k^i$ as his payoff. We denote the characteristic function derived from this parametrized normal form by v_λ^N (the superscript N denotes a parametrization through the underlying normal form).[17]

The fact that for each $S \in \mathcal{N}$, $v_\lambda^N(S)$ is defined as the maximin of an aggregated parametrized payoff function where the maximum and minimum are taken over (assumed) compact sets of mixed strategies allows us to show that v_λ^N is a continuous function of λ.

With each multicriteria characteristic function form game $v = (v_1, ..., v_\varrho)$ derived from a normal form game, using the parametrization process just described, we define the single criterion characteristic function v^* by $v^*(S) = \text{maximum}_{\lambda \in L}\, v_\lambda^N(S)$ for each coalition $S \in \mathcal{N}$.[18]

In Example 3.2.4 we illustrate that it is possible to have $v^*(S) > \overline{v}(S)$ for some $S \in \mathcal{N}$. Consistent with earlier notation, we will denote the (0,1)-normalization of v^* by $(v^*)'$.

For each $\lambda \in L$ we could also (0,1)-normalize v_λ^N to obain $(v_\lambda^N)'$, if $\Sigma_{i=1}^n v_\lambda^N(i) < v_\lambda^N(N)$. (See note 10.) Then we similarly define v^{**} by $v^{**}(S) = \text{maximum}_{\lambda \in L}\, (v_\lambda^N)'(S)$. ($v^{**}$ is in (0,1)-normal form.) It is evident that $(v^*)'$ may not equal v^{**}. The relationships between these games and between v and $(v_\lambda^N)'$ are currently under investigation.

EXAMPLE 3.2.3. We now parametrize the game in Example 3.2.1 using the parametrized underlying normal form in Example 3.1.3. Table II gives the values for v_λ^N as $\lambda_1 = 1 - \lambda_2$ varies over various subintervals of [0,1]. The particular parameter value $\lambda_0 = (23 - \sqrt{337})/32 \simeq 0.145$ is an irrational 'change point' for $v_\lambda^N(13)$. We note that in general, on each of possibly many different regions of L, $v_\lambda^N(S)$ will be represented by a rational function in several variables. In Figure 7 we sketch the graphs of $v_\lambda^N(12)$, $v_\lambda^N(23)$ and $v_\lambda^N(13)$. We observe that for all λ, the value for coalition {1,2} is non-decreasing and smaller than the values for the other two player coalitions. Coalition {1,3} has a maximum value when $\lambda_1 = 1$ while the value for coali-

TABLE II

Parametrized characteristic function

Intervals for λ_1	$[0, \lambda_0]$	$[\lambda_0, 1/2]$	$[1/2, 3/4]$	$[3/4, 1]$
$v_\lambda^N(1)$	1	1	1	1
$v_\lambda^N(2)$	$1+\lambda_1$	$1+\lambda_1$	$2-\lambda_1$	$2-\lambda_1$
$v_\lambda^N(3)$	$4-2\lambda_1$	$4-2\lambda_1$	$4-2\lambda_1$	$4-2\lambda_1$
$v_\lambda^N(12)$	$\frac{-2\lambda_1^2+21\lambda_1+12}{4+3\lambda_1}$	$\frac{-2\lambda_1^2+21\lambda_1+12}{4+3\lambda_1}$	4	4
$v_\lambda^N(23)$	$\frac{3\lambda_1^2-26\lambda_1+40}{6-2\lambda_1}$	$\frac{3\lambda_1^2-26\lambda_1+40}{6-2\lambda_1}$	$\frac{3\lambda_1^2-26\lambda_1+40}{6-2\lambda_1}$	$\frac{3\lambda_1^2+6\lambda_1+16}{3+2\lambda_1}$
$v_\lambda^N(13)$	$\frac{7\lambda_1^2-33\lambda_1+36}{6-4\lambda_1}$	$\frac{-5\lambda_1^2-16\lambda_1+28}{5-4\lambda_1}$	$\frac{-5\lambda_1^2-16\lambda_1+28}{5-4\lambda_1}$	$\frac{-5\lambda_1^2-16\lambda_1+28}{5-4\lambda_1}$
$v_\lambda^N(123)$	9	9	$8+2\lambda_1$	$8+2\lambda_1$

tion {2,3} is a maximum at $\lambda_1 = 0$. The function v_λ^N for coalition {2,3} is decreasing over the interval [0, 3/4] and increasing over [3/4, 1] while coalition {1,3} has a decreasing value over $[0, \lambda_0]$ and an increasing value over $[\lambda_0, 1]$.

EXAMPLE 3.2.4. Suppose a two criteria normal form game yields the following two person zero-sum game matrices for a particular coalition S:

$$A^1 = \begin{bmatrix} 5 & 8 \\ 0 & 1 \end{bmatrix} \quad A^2 = \begin{bmatrix} 8 & 5 \\ 1 & 8 \end{bmatrix}$$

Then $A^\lambda = \lambda_1 A^1 + (1-\lambda_1)A^2 =$

$$\begin{bmatrix} 8 - 3\lambda_1 & 5 + 3\lambda_1 \\ 1 - \lambda_1 & 8 - 7\lambda_1 \end{bmatrix}$$

Solving for $v_\lambda^N(S)$, we obtain

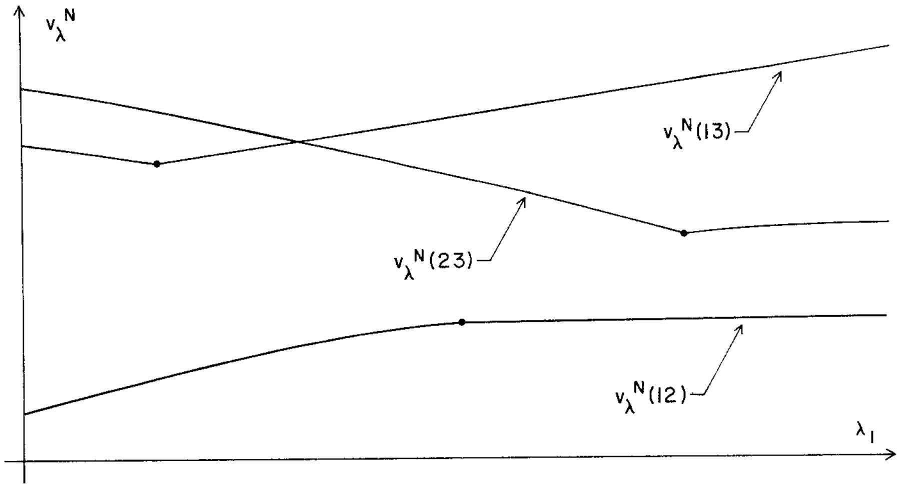


Fig. 7. Graphs of the coordinate functions of the parametrized characteristic function, v_λ^N.

$$v_\lambda^N(S) = \begin{cases} \dfrac{24\lambda_1^2 - 78\lambda_1 + 59}{10 - 12\lambda_1}, & \text{if } \lambda_1 \in [0, 3/10] \\ 5 + 3\lambda_1, & \text{if } \lambda_1 \in [3/10, 1/2] \\ 8 - 3\lambda_1, & \text{if } \lambda_1 \in [1/2, 1] \end{cases}$$

We see that $v_1(S) = 5$ and $v_2(S) = 59/10$ implying that $\overline{v}(S) = 59/10$. But $v^*(S) = 13/2$ (the maximum occurs when $\lambda_1 = 1/2$). This illustrates that we may have $v^*(S) > \overline{v}(S)$.

3.2.2. *The use of solution concepts (usually) yielding a set or sets of many solution points.* Section 2.4.2 was devoted primarily to an analysis of the core concept. Similarly in this section we focus primarily on an extension of the core concept to multicriteria games which we call the multicriteria core. Other solution concepts in the lower left-hand section of Table I can be extended to multicriteria games but we shall not elaborate here.

Suppose that each criteria v'_k, of a multicriteria (0,1)-normalized characteristic function form game, $v' = (v'_1, ..., v'_\ell)$, represents the coalition values for a different outcome of some future uncertain event. Given any outcome of this future event no coalition will object to an imputation in the core of *every* characteristic function. Consequently, we define the multicriteria core of the game v', denoted $MC(v')$, to be the intersection of the cores of the coordinate functions, i.e., $MC(v') = \cap_{k=1}^{\ell} C(v'_k)$. In the next Proposition, we see that (v') has a close relationship to the multicriteria core.

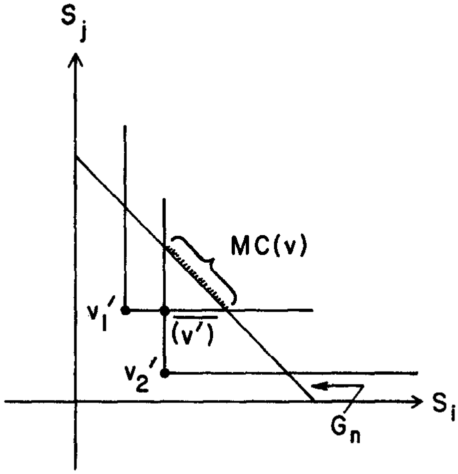


Fig. 8. $C(v'_1) \neq \phi$, $C(v'_2) \neq \phi$, $MC(v') \neq \phi$.

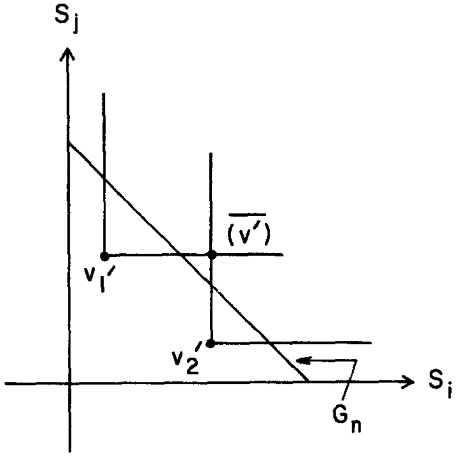


Fig. 9. $C(v'_1) \neq \phi$, $C(v'_2) \neq \phi$, $MC(v') = \phi$.

PROPOSITION 3.2.5. *For a* (0,1)-*normalized characteristic function form game* $v' = (v_1', \ldots, v_\ell')$, $MC(v') = C(\overline{(v')})$.

Proof. We see that $x \in MC(v') = \cap_{k=1}^{\ell} C(v_k')$ if and only if $x(N) = 1$ and $x(S) \geqq v_k'(S)$ for $k = 1, \ldots, \ell$ and for all $S \in \mathcal{N}$. This is equivalent to $x(N) = 1$ and $x(S) \geqq \text{maximum}_{1<k<\ell} v_k'(S) = \overline{(v')}(S)$ for all $S \in \mathcal{N}$ which is equivalent to $x \in C(\overline{(v')})$.

Q.E.D.

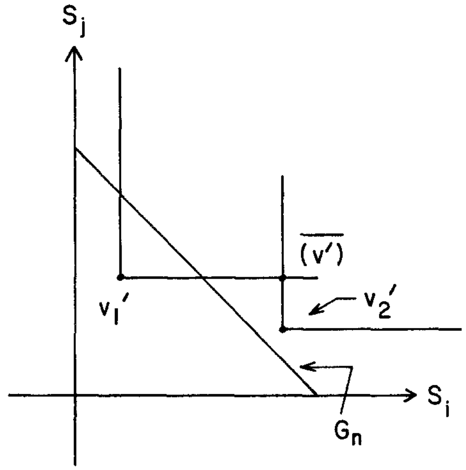


Fig. 10. $C(v_1') \neq \phi$, $C(v_2') = \phi$, $MC(v') = \phi$.

In Figures 8, 9 and 10 for the case of two criteria we illustrate three possible locations of the game points v_1', v_2' in Y with respect to G_n. As in Figure 4 we have represented Y as a two-dimensional space. In case $MC(v') = \phi$ (as in Figures 9 and 10), another solution approach would be required.

EXAMPLE 3.2.6. We give the multicriteria core for v' in Example 3.2.1. $MC(v') = \{x \in \mathbb{R}^3 \mid x \geqq 0,\ \Sigma_{i=1}^3 x_i = 1, x_1 + x_2 \geqq 1/3,\ x_2 + x_3 \geqq 5/9$ and $x_1 + x_3 \geqq 1/3\}$. In this case $MC(v') \neq \phi$.

Proposition 3.2.5 provides a way to define a domination structure on I for a (0,1)-normalized game $v' = (v_1', \ldots, v_\ell')$ which yields the multicriteria core. Using the notation of the domination structure (13), and given a coalition $S \in \mathcal{N}$ we define the following cone for each $x \in I$:

$$D_S(x) = \begin{cases} \Lambda_S^{\leq}, & \text{if } x(S) \leqq \max\limits_{1 \leqq k \leqq \ell} \{v_k'(S)\} \\ \{0\}, & \text{otherwise} \end{cases}$$

Then similar to Proposition 2.4.7, we obtain $MC(v') = \cap_{j=1}^{m} N(D_{S_j})$.

We now illustrate that the multicriteria core naturally induces general domination structures in the coalition space Y. Given the (0,1)-normalized multicriteria game $v' = (v_1', v_2')$ we treat v_1' as a reference game point and enlarge the cone $D(v_1') = (Y \setminus \Lambda^{\geqq}) \cup \{0\}$ (see Figure 6) to form the cone $D(v_1'; v_2')$ as follows: To $D(v_1')$ we add rays induced by imputations which are objectionable to any coalition at the other criteria (game point), v_2'. More specifically, given an imputation x, if $x(S) < v_2'(S)$ for some coalition S, then we adjoin the ray $\{\alpha(\phi(x) - v_1') \mid \alpha \geqq 0\}$ to $D(v')$ (ϕ is defined in Section 2.4.2). See Figure 11.

As is clear from Figure 11, $MC(v') = \{x \in I \mid \phi(x)$ is not dominated by v_1' under $D(v_1'; v_2') = \{x \in I \mid \phi(x) \in v_1' + D(v_1'; v_2')^C\}$. ($D(v_1'; v_2')^C$ is the complementary cone of $D(v_1'; v_2')$). Clearly, if we reduced the multicriteria game to the single criterion game v_1' and used the classical domination structure, we would lose some crucial information from the game point v_2'.

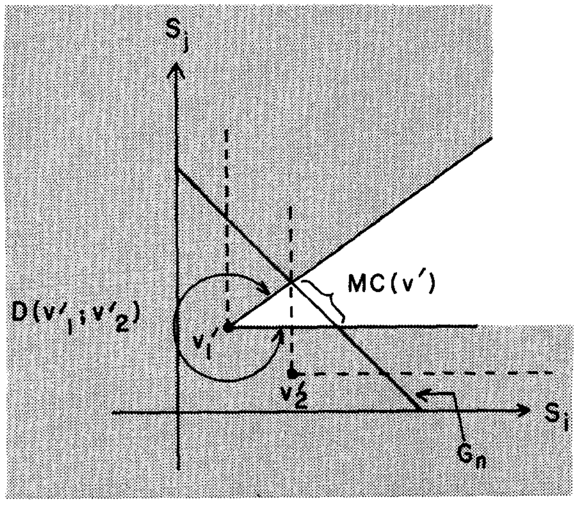


Fig. 11. The domination structure $D(v_1'; v_2')$.

Consequently, to avoid this loss of information we must use the more general cone $D(v_1' ; v_2')$.

In general, given a (0,1)-normalized multicriteria game $v' = (v_1', ..., v_\ell')$, for each $k = 1, ..., \ell$, we define the following cone:

$$D(v_k'; v_1', ..., v_{k-1}', v_{k+1}', ..., v_\ell') =$$
$$(Y \setminus \Lambda^{\geqq}) \cup \{0\} \cup \{\alpha(\phi(x) - v_k') \mid \alpha > 0, x \in I \text{ such that } x(S) < v_j(S) \text{ for some } S \in \mathcal{N} \text{ and some } j = 1, ..., \ell, j \neq k\}.$$

As above this domination cone can be used to generate $MC(v')$. Similarly for each weight vector λ we can define the domination structure $D((v_\lambda^N)'; v_1', ..., v_\ell')$ which generates $MC(v')$ using $(v_\lambda^N)'$ as a reference point.

Referring to Example 3.1.1, if all of the players agree on a probability distribution, λ, over the outcomes of the future election, it would be reasonable to apply the core and other solution concepts to $(v_\lambda^N)'$. By the continuity of v_λ^N and of the (0,1)-normalization process, one can show that if $C((v_\lambda^N)') \neq \phi$, then $C((v_\lambda^N)')$ is a *continuously varying* set function of λ. That is, for each $\lambda_0 \in L$ and each $\in > 0$, there is a neighborhood about λ_0, $N(\lambda_0)$, such that if $\lambda \in N(\lambda_0)$, then $C((v_\lambda^N)') \subseteq [C((v_{\lambda_0}^N)') + N_\in(0)]$. ($N_\in(0)$ denotes the open ball about 0 with radius $\in$.) In fact, this continuity property holds over the parameter space for v_λ^N (not necessarily (0,1)-normalized) and for v_λ^C.

Instead of agreeing on a unique distribution it is more likely that the players would agree on an interval estimate of the probabilities for each outcome of the election. Thus the players might agree to use a set, Ω, of probability weight vectors. Then each imputation in $\cap_{\lambda \in \Omega} C((v_\lambda^N)')$ has the property that no coalition will object no matter which $\lambda \in \Omega$ is the actual distribution for the future event. We might also use $\cap_{\lambda \in \Omega} C((v')_\lambda^C)$ or $\cap_{\lambda \in \Omega} C((v_\lambda^C)')$ depending on the particular application. On the other hand, the players might want to consider imputations in $\cup_{\lambda \in \Omega} C((v_\lambda^N)')$ which gives all nondominated imputations under each possible probability distribution in the estimate set Ω.

Further, the core and other solution concepts could be applied to the associated single criterion games $(\bar{v})'$, $(v^*)'$ or v^{**}. These games indicate the best that a coalition can do under particular circumstances. Thus the coalitions may agree to use one of these game points in determining the final solution.

An additional solution approach can be obtained by using the full payoff space of the underlying normal form game. Assuming that each player has the same ℓ criteria, we can view the normal form game as a single criterion game with $\ell \cdot n$ players. In other words, each criterion for each player is associated with a different player in a game with an enlarged player set. Then the characteristic function for the game with $n \cdot \ell$ players could be derived. A reasonable solution would be the relaxed core (see Section 2.4.2) where the only permitted coalitions are those which do not split up each player's criteria set in the original game. This approach implicitly assumes that for each imputation each player in the original game receives the sum of the coordinates associated with criteria under his control. Other solution concepts could be applied to this characteristic function derived from the full playoff space. Of course, consideration must be given to the interpretation and rationale for such solutions.

3.2.3. *The use of solution concepts (usually) yielding a unique solution point.* Arbitration schemes such as the Shapley value, nucleolus and convex nuclei can be applied to all of the single criterion game points associated with a multicriteria game $v = (v_1, ..., v_\ell)$. For example, as in the preceding section, $(\overline{v})'$, $(v^*)'$ or v^{**} would be reasonable game points that the players might agree to use.

If the players can agree on a set of probability weight vectors, Ω, (this includes the possibility that the players might agree on a unique vector λ) then each arbitration scheme generates the set of solution points for the games parametrized by all $\lambda \in \Omega$. In view of the continuity of v_λ^N and v_λ^C and continuity properties of the arbitration schemes, one can show that the Shapley value, nucleolus and convex neclei are continuous functions of the parameter λ.

As in the immediately preceding section we can consider the characteristic function induced by the full payoff space of the underlying normal form. Then arbitration schemes can be applied to this single criterion characteristic function. Again, the assumption is that at the final solution point, each player receives the sum of the coordinates associated with criteria under his control.

We can also define some appealing new arbitration schemes, using the parametrization process. For example, if $C(\overline{(v')}) = \phi$ or $C(v^{**}) = \phi$, the players may treat $\overline{(v')}$ or v^{**} as a kind of 'utopia' game point. Both $\overline{(v')}$ and v^{**} represent the best value a coalition can have under certain conditions. Therefore the players may agree to use the imputation x whose game point image, $\phi(x)$, (see Section 2.4.2) best approximates $(\overline{v'})$ or v^{**} in the sense of some distance measure, such as an ℓ_p • norm.

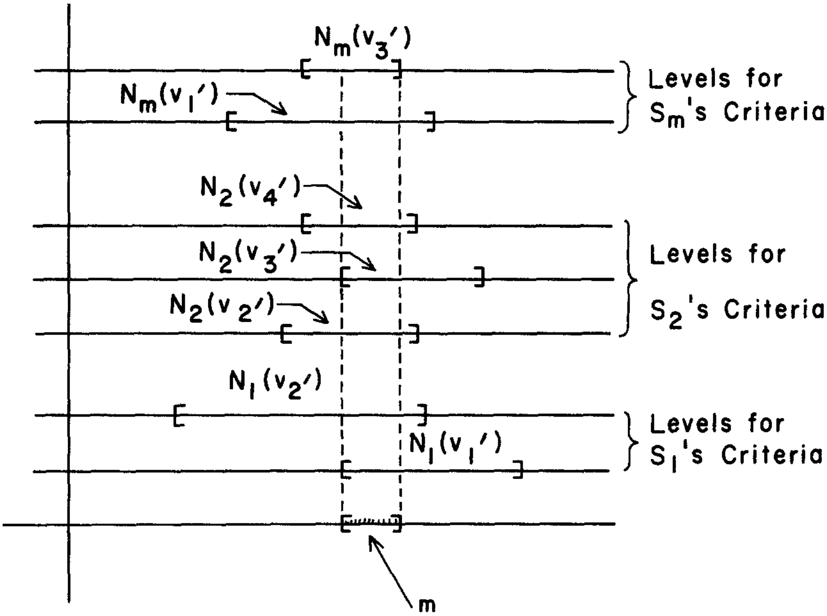
A second arbitration scheme using the parametrization process involves using the parametrized games $(v_\lambda^N)'$ and $(v')_\lambda^C$ to approximate v^{**} and $(\overline{v'})$ respectively. In other words, the players would agree to use the parameter λ_0 for wich the distance $d((v_\lambda^N)', v^{**})$ or the distance $d((v')_\lambda^C, (\overline{v'}))$ is minimal. Then after λ_0 had been located, solution concepts from Section 2.4 could be applied to $(v_{\lambda_0}^N)'$ or $(v')_{\lambda_0}^C$.

3.2.4. *The case where the players may have different criteria.* In this section we discuss several formulations and resolution approaches for characteristic function form games where the players may have different criteria. For the first approach we consider a (0,1)-normalized multicriteria game $v' = (v_1', ..., v_\ell')$ where each proper coalition of more than one player, S_j, $j = 1, ..., m$, determines an index set $C_j \subseteq \{1, ..., \ell\}$ containing the indices of the criteria which are of concern to S_j as a coalition. In this case we would expect that each coalition, S_j, will object to potential distribution laws only on the basis of criteria indexed by C_j.

A natural solution approach would then be to use a relaxed multicriteria core, namely $\cap_{j=1}^m \cap_{k \in C_j} N_j(v_k')$, the imputations which are nondominated under the classical domination structure for any coalition S_j with respect to any criterion indexed in C_j. (We use $N_j(v_k')$ to denote the imputations which are nondominated via coalition S_j on criteria v_k'.)

In Figure 12, which is an expansion of Figure 5, there is a set of 'levels' (copies of I) for each coalition. Given coalition S_j, there is one level for each criterion indexed in C_j. As in Figure 5 we project all of the levels onto a single copy of I and take the intersection to obtain the relaxed multicriteria core.

Secondly, given a normal form multicriteria game where not all of the players necessarily have the same criteria, we consider methods for inducing a characteristic function form game. Each coalition will have as its criteria set the criteria which are of concern to at least one of the members of the coalition. Intuitively, an individual player would not join a coalition unless the coalition pays some attention to all of that player's criteria. For each criterion for a given coalition, we could compute the maximin value by ignoring all of the other criteria. Clearly a coalition could not necessarily obtain the maximin values thus derived on all of its criteria simultaneously, but these values could serve as a basis for bargaining or arbitration. If a given



$$\bigcap_{j=1}^{m} \bigcap_{k \in C_j} N_j(v_k')$$

Fig. 12. 'Levels' of domination structures yielding a relaxed multicriteria core.

coalition is not concerned about a particular criterion, we would assign that coalition a value of zero on that criterion. In this way we construct a characteristic function where each coalition has a value for every criterion in the game. We then take the (0,1)-normalization and procede to use appropriate solution approaches as described in Sections 3.2.2 and 3.2.3. A similar but less intuitively appealing approach is to assume that each coalition considers only those criteria which concern all of its members. In this case we would assume that there is at least one criterion common to all players.

Alternatively we could induce a characteristic function from the full payoff space by viewing that space as the payoff space for a single criterion game with an enlarged player set (for a given player i, there is one 'fictitious' player controlling each of player i's criteria). As in Section 3.2.2 we would place restrictions on the coalitions that are permitted to form. No group of fictitious players associated with the criteria set of one player in the original game could be split among more than one coalition in the fictitious game.

4. CONCLUSIONS

The underlying domination structures of well-known solution concepts for cooperative n-person games and various solution approaches for multicriteria games have been discussed. It is hoped that this discussion will enhance the understanding of solution concepts in various game situations which in turn would provide help in applying the concepts.

A variety of problems remain to be resolved. For instance, one such problem is to extend domination structures and the multicriteria concept to various categories of games which are not discussed in this paper such as games in partition function form [35], games without side payments. [3, 32], constrained games [14, 33] and differential games [5, 20]. Some remaining issues in the parametrization of multicriteria games are the following: (i) How should the parametrized game points and other associated single criterion games be interpreted? (ii) In the game situation how should a particular parametrized game point or set of such game points be used to determine a final decision? These questions are currently under investigation. We shall report any significant results.

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NOTES

[1] We will use the symbol u to denote both the criteria function and a point in the criteria space U. The intended interpretation will be clear from the context.

[2] One can relax the assumption that $D(u)$ is a convex cone and instead assume only that $D(u)$ is a convex set. For instance, see [2].

[3] In Table I we have not attempted to list every proposed solution concept; however, many of the well-known concepts in each category have been included.

[4] Consistent with the convention in note 1, we will use the symbol p to denote both the payoff function and a point in the payoff space P. Again the intended interpretation should be clear from the context.

[5] There is another type of satisficing model which is equivalent to imposing lower bound constraints on the decision space. For details see [40].

[6] Since the Nash arbitration solution and compromise solutions with parameter $1 \leq q < \infty$ (which are discussed later) are Pareto optimal we can relax the convexity assumption on P and require only that P be $\Lambda^{\leqq}$-convex. This means that $P + \Lambda^{\leqq}$ is convex (See [39]).

[7] These functions use the ℓ_p and ℓ_∞ norms as a distance measure. Since we use the letter p for a point in the payoff space, to avoid confusion we have used q to index the regret functions.

[8] For a general cone $\Lambda \subseteq \mathbb{R}^n$, the polar cone of Λ, denoted Λ^*, is given by $\Lambda^* = \{\lambda \in \mathbb{R}^n \mid \lambda \cdot d \leqq 0 \text{ for all } d \in \Lambda\}$.

[9] Throughout this paper we will use the notation $v(12)$ to represent the functional value of v at the coalition $\{1,2\}$ and we will use similar notation for other coalitions.

[10] A rare exception to this may occur in Section 3.2 where a derived parametrized game v_λ may satisfy $\Sigma_{i=1}^{n} v_\lambda(i) = v_\lambda(N)$. In this case v_λ cannot be (0,1)-normalized and we would use the unique imputation as the solution for v_λ.

[11] v is a simple game if and only if $v(s) = 0$ or 1 for all $S \in \mathcal{N}$. Player i is a veto player for the simple game v if and only if $i \notin S$ implies $v(S) = 0$.

[12] We assume strict convexity here to simplify the domination structure. As Charnes and Kortanek [12b] point out, strict convexity may be relaxed to convexity if uniqueness may be waived. Similar domination structures are induced in the case where g_S is a convex function for all $S \in \mathcal{N}'$.

[13] If we relax the strict convexity on G, the domination cone will satisfy $\{d \mid d \cdot \nabla (\Sigma_{S \in \mathcal{N}'} g_S(y_S)) > 0\} \subseteq D(y) \subseteq \{d \mid d \cdot \nabla (\Sigma_{S \in \mathcal{N}'} g_S(y_S)) \geqq 0\}$.

[14] Here the vector p^0 represents minimally acceptable payoff levels. Assuming that full cooperation will not occur unless each player obtains at least this established minimal level on each criterion, we restrict our attention to the payoffs satisfying $p \geqq p^0$.

[15] To permit a realistic comparison among the payoffs, the weight vectors may have to satisfy certain conditions, for instance, equal length.

[16] This is equivalent to using all of the weight vectors in the cone generated by Ω. For different vectors along the same ray the maximin method applied to the underlying normal form will yield parametrized characteristic function values having a different scale. However, such different characteristic functions induce equal (0,1)-normalized games. The situation is similar when the parametrization is performed directly on the multicriteria characteristic function.

[17] A more complicated normal form parametrization process could be used where each player i determines a set of weight vectors, Ω_i, used to parametrize his normal form payoff. Then for each choice of weight vectors $(\lambda^1, ..., \lambda^n) \in \Pi^n \Omega_i$, a normal form parametrization and the induced characteristic function form game $v^N_{(\lambda^1, ..., \lambda^n)}$ could be obtained. For simplicity we have confined our discussion to the case where the players agree to apply the weight vectors λ in a set Ω simultaneously to their normal form payoffs.

[18] Clearly $\underline{v}^*$ is not as interesting if we replace v^N_λ by v^c_λ since in that case we would have $v^* = \overline{v}$.

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