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A GENERAL THEOREM AND EIGHT COROLLARIES IN SEARCH OF CORRECT DECISION

ABSTRACT. The main theorem established in this study and its corollaries summarize and generalize the existing results on optimal aggregation of experts judgments under uncertain pairwise choice situations. In particular, we explicate the link between the optimal decision procedure and the decision maker's preferences and biases and the judgmental competences of his consultants. The general theorem directly clarifies under what circumstances the optimal decision rule should be the democratic simple majority rule, the elitist expert rule, an intermediate weighted simple majority rule or a biased weighted or simple qualified majority rule.

Various aspects of the basic problem of aggregating individual judgments in dichotomous choice situations have attracted considerable attention. An important question with which the literature deriving from Condorcet tradition has been concerned is, how likely are groups to reach correct judgment as a function of (1) individual decisional skills, (2) the decision rule aggregating judgments and (3) the number of individual judgments. Grofman *et al.* (1983) have recently collated thirteen results, that all, excluding the last one, deal with various facets of this question. Only Theorem XIII (Theorem 1 in Nitzan and Paroush, 1982) or the main theorem in Shapley and Grofman (1981) focuses on the central optimality issue namely, the problem of identifying the decision rule that maximizes the probability that the group will make a correct judgment.

The current study purports to generalize the classical dichotomous framework in four respects. First, by permitting heterogeneity of decisional skills. Second, by allowing asymmetry between the alternatives in the sense that, a priori, they are not necessarily equi-probable correct. Third, by taking into account the possible asymmetry between the alternatives, meaning that different payoffs might be associated with the available alternatives under different states of the world. And, finally, the classical objective of maximizing the probability of a correct judgment is substituted by the more meaningful objective of expected payoff maximization.

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Our principal result specifies the optimal decision rule as a function of decisional skills, number of individuals, the payoff matrix associated with the alternatives under the possible states of the world, and the priors of these states of the world. To facilitate exposition, our problem is interpreted as an expert resolution problem.[1] That is, we assume that a decision maker aims at optimally pooling the judgments of n consultants. Optimality is defined in terms of expected utility maximization. The decision maker is characterized by the payoff matrix representing his preferences and by the priors that the two alternatives constitute a correct choice.

Eight direct corollaries of the principal result are presented. Some of these have already been reported in the literature, while the remaining are novel. The corollaries are of two types. One specifies the optimal decision rule and provides its interpretation in certain interesting special cases, and the other establishes necessary and/or sufficient conditions for the optimality of certain commonly used particular procedures.

This study presents a simple generalized framework and a single metatheorem that shed new light on various aspects of collective decision making. This metatheorem enables a straightforward unification of various scattered results. The structural interrelationships between these theorems are clarified and their common source is exposed. This, in turn, provides a useful means for reviewing, understanding and relating them, as well as permits orderly presentation of some of the recent contributions to the literature on optimal group judgmental processes.

Consider a decision maker facing two distinct alternatives 1 and -1. Note that pairwise choice situations are very common either in natural binary contexts (e.g., choice under critical conditions that are typically characterized by the availability of only two courses of action) or in artificial binary contexts where the decision maker faces more than two alternatives, but he decomposes the decision-making process into a sequence of pairwise choices (e.g., as in a standard amendment legislative procedure). Under the uncertain dichotomous choice model two states of the world are possible: either alternative 1 or -1 is the correct choice. Correctness is defined herein in the following manner: Suppose that the benefit associated with the selection of alternative 1 in state of nature 1 is given by $B(1:1)$. Similarly, we have the three remaining benefits $B(1:-1)$, $B(-1:1)$ and $B(-1:-1)$. Given state of nature 1, alternative 1 is referred to as correct if $B(1:1) > B(-1:1)$. Given state of nature -1, alternative -1 is the correct choice if $B(-1:-1) > B(1:-1)$.[2] The decision maker is fully characterized by two elements. First, by the payoff matrix summarizing his preferences in our specific context:

$$\begin{matrix} B(1:1), & B(-1:1) \\ B(1:-1), & B(-1:-1). \end{matrix}$$

Second, by his *a priori* probabilities, α and $(1-\alpha)$, for the occurrence of state of nature 1 and -1, respectively. In other words, assuming that $B(1:1) > B(-1:1)$ and $B(-1:-1) > B(1:-1)$, α is the prior that alternative 1 is correct and $(1-\alpha)$ is the prior that alternative -1 is the correct choice.

Before reaching a final decision, the decision maker may consult a panel of n consultants. Each consultant i, $i = 1, \ldots, n$, reveals his view regarding the question which alternative constitutes the appropriate selection. Let us denote by $x_i \in \{1, -1\}$ individual i's decision. The vector $x = (x_1, \ldots, x_n)$ is the actual representation of the group members' views. Henceforth this vector is referred to as the experts decision profile. The decisional skill of an individual expert i is parametrized by the probability that he chooses correctly given that either alternative 1 or alternative -1 is the correct choice. These probabilities are denoted respectively by $p_i(1) = \Pr(1:1)$ and $p_i(-1) = \Pr(-1:-1)$. We now make the following three assumptions with respect to these probabilities:

For any consultant i, $\Pr(-1:1) = 1 - p_i(1)$ and $\Pr(1:-1) = 1 - p_i(-1)$. This assumption rules out the possibility of various courses of action such as individual i abstaining. An individual satisfying this assumption is decisively supporting either of the available alternatives and is therefore called decisive.

The second assumption requires that for any consultant $i, p_i(1) = p_i(-1) = p_i$. That is, individual decisional skills are independent of the particular state of nature and hence can be represented by the single parameter p_i. An individual satisfying this assumption is called state invariant. Finally, individual decisional skills, the p_i's are assumed to be statistically independent.

The final selection of one of the available alternatives is made by means of a decisive aggregation rule. A decisive decision rule is a function f that assigns either alternative 1 or -1 to any voting profile x in $\Omega = (1, -1)^n$. That is, f: $\Omega \to (1, -1)$. Our decision maker is assumed to select a decisive decision rule that maximizes his expected benefit. In order to formally define his objective function we need to present the conditional probabilities for making a correct choice. For that purpose, let us partition the set of all possible profiles into $X(1,f)$ and $X(-1,f)$ where $X(1,f) = (x \in \Omega: f(x) = 1)$ and $X(-1,f) = (x \in \Omega: f(x) = -1)$. For a given rule f, the decision maker chooses correctly provided that 1 or -1 is the correct alternative with probability $\pi(f:1)$ or $\pi(f:-1)$ where $\pi(f:1) = \Pr\{x \in X(1,f):1)\}$ and $\pi(f:-1) = \Pr\{x \in X(-1,f):-1\}$. Since f is a decisive decision rule,

$$\Pr\{x \in X(-1,f):1\} = 1 - \pi(f:1)$$

and

$$\Pr\{x \in X(1,f):-1\} = 1 - \pi(f:-1).$$

The general problem on which we focus is the maximization of expected benefit E over the set F of all decisive decision rules. Specifically, $\max_{f \in F} E$. Expected benefit is given by

$$\begin{aligned} E &= B(1:1)\pi(f:1)\alpha + B(-1:1)[1-\pi(f:1)]\alpha + \\ &\quad + B(-1:1)\pi(f:-1)(1-\alpha) + \\ &\quad + B(1:-1)[1-\pi(f:-1)](1-\alpha) = \\ &= B(1)\pi(f:1)\alpha + B(-1)\pi(f:-1)(1-\alpha) + \\ &\quad + [B(-1:1)\alpha + B(1:-1)(1-\alpha)], \end{aligned}$$

where $B(1) = B(1:1) - B(-1:1)$ is the alternative benefit or the net benefit associated with a correct choice given that alternative 1 is correct. Similarly, $B(-1) = B(-1:-1) - B(1:-1)$. Denoting by $\hat{f}$ a solution to our problem, we can now state the main result.

THEOREM.

$$\hat{f} = \text{sign}\left(\sum_{i=1}^{n} \beta_i x_i + \gamma + \delta\right),$$

where $\beta_i = \ln p_i/(1-p_i)$, $\gamma = \ln \alpha/(1-\alpha)$ and $\delta = \ln B(1)/B(-1)$.

Proof. For any decision profile x in Ω define a partition of the experts $1, 2, \ldots, n$ into $A(x)$ and $B(x)$ such that $i \in A(x)$ if $x_i = 1$ and $i \in B(x)$ if $x_i = -1$. Denote by $g(x:1)$ and $g(x:-1)$ the conditional probabilities to obtain x given that alternative 1 or -1 is the correct choice. That is,

$$g(x:1) = \prod_{i \in A(x)} p_i \prod_{i \in B(x)} (1-p_i)$$

and

$$g(x:-1) = \prod_{i \in B(x)} p_i \prod_{i \in A(x)} (1-p_i).$$

For a given decision rule f,

$$\pi(f:1) = \sum_{x \in X(1,f)} g(x:1)$$

and

$$\pi(f:-1) = \sum_{x \in X(-1,f)} g(x:-1).$$

By the definition of E, a sufficient condition for the optimality of the decision rule $\hat{f}$ is that

$$X(1,\hat{f}) = \{x : x \in \Omega \text{ and }$$
$$B(1) g(x:1)\alpha > B(-1) g(x:-1)(1-\alpha)\},$$

or, equivalently,

$$X(1,f) = \left\{ x : x \in \Omega \text{ and } \frac{B(1)\alpha}{B(-1)(1-\alpha)} \prod_{i \in A(x)} p_i \times \right.$$
$$\left. \times \prod_{i \in B(x)} (1-p_i) > \prod_{i \in B(x)} p_i \prod_{i \in A(x)} (1-p_i) \right\}$$
$$= \left\{ x : x \in \Omega \text{ and } \frac{B(1)\alpha}{B(-1)(1-\alpha)} \prod_{i \in A(x)} \frac{p_i}{(1-p_i)} > \right.$$
$$\left. > \prod_{i \in B(x)} \frac{p_i}{(1-p_i)} \right\}$$
$$= \left\{ x : x \in \Omega \text{ and } \sum_{i \in A(x)} \beta_i + \gamma + \delta > \sum_{i \in B(x)} \beta_i \right\}$$
$$= \left\{ x : x \in \Omega \text{ and } \sum_i \beta_i x_i + \gamma + \delta > 0 \right\}.$$

$$X(-1,f) = \Omega - X(1,f) =$$

$$= \left\{ x : x \in \Omega \text{ and } \sum_i \beta_i x_i + \gamma + \delta < 0 \right\}.$$

So that

$$\hat{f} = \text{sign}\left(\sum_i \beta_i x_i + \gamma + \delta \right).$$

Q.E.D.

The optimal decision rule turns out to be a weighted qualified majority rule. The optimal experts' weights are proportional to the log odds of their decisional competences, the β_i's. The particular qualified majority required depends on the log odds of the decision maker's priors, γ, and on the log ratio between the net benefits under the two possible states of the world, δ.

The following four direct corollaries specify the optimal decision rule under four particular cases:

COROLLARY 1. If $B(1) = B(-1)$, then $\hat{f} = \text{sign}\ (\Sigma_{i=1}^{n} \beta_i x_i + \gamma)$.

(See Corollary 3 to Theorem 1 in Nitzan and Paroush, 1982.) This partially symmetric case is not very common as Type 1 and Type 2 errors are usually associated with different payoffs. For example, symmetry in the consequences of mistaken action and inaction might not be plausible in the criminal jury context; an unwarranted conviction might be considered worse than an objectionable acquital.

COROLLARY 2. If $\alpha = \frac{1}{2}$, then $\hat{f} = \text{sign}\ (\Sigma_{i=1}^{n} \beta_i x_i + \delta)$.

Under this partially symmetric case the *a priori* probabilities for the occurrence of the two possible states of nature are identical. This case is not a bizarrely unlikely one. In particular, under complete ignorance conditions, the decision maker's priors would be equal.

COROLLARY 3. If $\alpha B(1) = (1 - \alpha) B(-1)$, then $\hat{f} = \text{sign}\ (\Sigma_{i=1}^{n} \beta_i x_i)$.

Under this case the two types of asymmetry are balanced and the resulting optimal decision rule is a weighted majority rule. Notice that the fully symmetric case whereby, first, the prior odds as to which of the two alternatives is the correct one are even and, second, the benefit incurred by a correct (incorrect) choice is assumed to be the same, regardless of the particular alternative correctly (mistakenly) chosen, is a special case covered by this corollary. In such a case, requiring the neutrality of f as an additional constraint in our central problem does not alter the solution $\hat{f}$.[3] Furthermore, since the requirement of neutrality implies that $\pi(f:1) = \pi(f:-1) = \pi$, maximization of expected benefit E is equivalent to the maximization of π since $E = B(1)\pi + B(1:-1)$. In such a case, then, the optimal weighted majority rule specified in Corollary 3 is the solution to the problem $\max_{f \in F} \pi$ (see Nitzan and Paroush, 1982, Theorem 1; and Grofman *et al.*, 1983, Theorem XIII).

If the experts are equally skilled, the optimal decision rule is a simple qualified majority rule. The definition of this common rule is as follows: A decision rule f_k is a qualified majority rule if for any decision profile x in Ω,

$$f_k(x) = \begin{cases} -1 & N(-1) \geqslant kn \\ 1 & \text{Otherwise,} \end{cases}$$

where $N(-1)$ is the number of experts supporting alternative -1.

We can now state

COROLLARY 4. The optimal rule $\hat{f}$ for equally skilled individuals, $p_i = p$ for $i = 1, \ldots, n$, is a qualified majority rule, $f_{\hat{k}}$, where

$$\hat{k} = \frac{1}{2}\left[1 + \frac{\delta + \gamma}{n\beta}\right] \quad \text{and} \quad \beta = \ln \frac{p_i}{1 - p_i} = \ln \frac{p}{(1-p)}.$$ [4]

The partially and fully symmetric cases of the preceding corollaries together with the homogeneity assumption yield the following particular cases. If $p_i = p$ for $i = 1, \ldots, n$ and $\alpha = \frac{1}{2}$, then the optimal decision rule is a qualified majority rule $f_{\hat{k}}$ with $\hat{k} = \frac{1}{2}[1 + (\delta/n\beta)]$. Similarly, if $p_i = p$ for $i = 1, \ldots, n$ and $B(1) = B(-1)$, then the optimal rule is a qualified majority rule $f_{\hat{k}}$ with $\hat{k} = \frac{1}{2}[1 + (\gamma/n\beta)]$. This special case is analyzed in detail in Nitzan and Paroush (1984). Lastly, if $p_i = p$ for $i = 1, \ldots, n$, $\alpha = 1/2$, and $B(1) = B(-1)$, then the optimal rule is a qualified majority rule $f_{\hat{k}}$ with $\hat{k} = \frac{1}{2}\{1 + [(\gamma + \delta)/n\beta]\} = \frac{1}{2}$. That is, $f_{\hat{k}}$ is, in fact, a simple majority rule.[5]

The main theorem provides direct insights into the issue of general democratic *vs.* specific elitist consultation strategies, or, more generally, the issue of democracy *vs.* rule by the select few. For instance, it directly reveals under what circumstances the decision should be made by a subgroup of the available experts.

COROLLARY 5. Let $\beta^*(m) = \min_{x_1,\ldots,x_n} \Sigma_{i=1}^{m} \beta_i x_i : \Sigma\, \beta_i x_i \geqslant \delta + \gamma)$. Then $\hat{f} = \text{sign}\; \Sigma_{i=1}^{\bar{m}} \beta_i x_i + \gamma + \delta$ where $\bar{m}$ is the smallest m satisfying $\beta^*(m) > \Sigma_{i=m+1}^{n} \beta_i$.[6]

Corollary 5 implies that whenever $\bar{m} < n$, the optimal decision rule entirely ignores the views of the $(n - \bar{m})$ least skilled experts. That is, the optimal decision rule in such cases is, in fact, an $\bar{m}$ experts weighted qualified majority rule. In particular, it might be beneficial to ignore all experts. Specifically,

COROLLARY 6. If $\gamma + \delta > \Sigma_{i=1}^{n} \beta_i$ then $\hat{f} = \text{sign}\,(\delta + \gamma)$.

That is, if $\gamma + \delta > \Sigma_{i=1}^{n} \beta_i$, the decision-maker should make a decision without resorting to his consultants' services. The necessary and sufficient condition for the optimality of the expert rule are directly derived from Corollary 5. With no loss of generality let $\gamma + \delta \geqslant 0$ and $p_1 > p_j > 1/2, j = 2, \ldots, n$. The expert (individual 1) rule is denoted f^e where $f^e = x_1$ for any decision profile x in Ω.

COROLLARY 7. $\hat{f} = f^e$ if and only if $\beta_1 > \Sigma_{j=2}^{n} \beta_j + \gamma + \delta$.

This corollary directly reveals the necessary and sufficient condition for the superiority of the expert rule under the fully symmetric case where $\delta = \gamma = 0$. Namely, whenever $\alpha = (1 - \alpha)$ and $B(1) = B(-1)$, $\hat{f} = f^e$ if and only if $\beta_1 > \Sigma_{j=2}^{n} \beta_j$. (See Corollary 1 to Theorem 1 in Nitzan and Paroush, 1982.)

The superiority of the expert rule is certainly eliminated if the experts are sufficiently homogenous in their abilities. Under the extreme case of equally skilled individuals the optimal rule is the very common simple majority rule, provided that $\delta + \gamma = 0$. Simple majority rule is denoted f^m where

$$f^m = \operatorname{sign}\left(\sum_{i=1}^{n} x_i\right) \text{ for any decision profile } x \text{ in } \Omega.$$

COROLLARY 8. $\hat{f} = f^m$ if $p_i = p, i = 1, \ldots, n$ and $\gamma + \delta = 0$.

The special case where individuals are equally skilled and $\gamma + \delta = 0$ is mentioned in Nitzan and Paroush (1982). In fact, Condorcet (1785) classical jury theorem is chiefly concerned with the relationship between the number of individuals n and the probability of correct decision under the special case where $\alpha = 1/2$, individuals are equally skilled and the payoffs $B(.:.)$ are ignored. Under such circumstances f^m is indeed the optimal rule maximizing π or, more generally, maximizing E, provided $B(1) = B(-1)$, as implied by Corollary 8.

The theorem established in this study summarizes and generalizes the existing results on optimal aggregation of experts' judgments under uncertain pairwise choice situations. In particular, we explicate the link between the optimal decision procedure and the decision maker's preferences and biases and the judgmental competences of his consultants. The general theorem directly clarifies under what circumstances the optimal decision rule should be the democratic simple majority rule, the elitist expert rule, an intermediate weighted simple majority rule or a biased weighted, or simple, qualified majority rule.

NOTES

[1] Our problem can be alternatively interpreted as one of a group comprised of subjects whose individual interests are identical and whose problem is to attain the decision that will optimally utilize their decisional resources.

[2] Consider the following examples:

(i) A decision maker has to decide on whether to pack or not an umbrella for his Sunday countryside walk. There are two possible states of nature: either it will rain on Sunday or not. The correct decision under the two possible states of nature is self evident.

(ii) A student facing an arithmetic problem has to select one of two distinct solutions. In this case, there is one meaningful state of nature where the rules of logic are valid. Obviously, only one solution is correct in terms of these rules. This problem can be artificially presented as a pairwise choice situation in which the two meaninful states of nature are, in fact, identical.

(iii) A decision maker, in evaluating two final stage beauty contest candidates, has to make a choice between them. Under the two relevant states of nature, either of the two candidates will gain the support of the majority of the decision makers. The meaning of choosing correctly differs here in comparison with the two previous examples; however, formally this example is perfectly tractable within the suggested framework.

[3] A decision rule f is neutral if for any decision profile x in $\Omega, f(-x) = -f(x)$.

[4] The proof of Corollary 4 is similar to that of the main result in Nitzan and Paroush (1984).

[5] An alternative definition for simple majority rule, f^m, is introduced below.

[6] One can readily verify that sign $(\Sigma_{i=1}^{n} \beta_i x_i + \delta + \gamma) =$ sign $(\Sigma_{i=1}^{m} \beta_i x_i + \delta + \gamma)$.

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