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SOPHISTICATED VOTING UNDER THE PLURALITY PROCEDURE: A TEST OF A NEW DEFINITION

Superficially, voting procedures for multicandidate elections are extraordinarily simple. Like all voting methods, however, they are open to manipulation, and once strategic considerations are introduced, their simplicity gives way to surprising complexity. Plurality voting is a case in point. Unless one assumes that voters can assign cardinal utilities to the candidates and estimate probabilities of being decisive in the vote (Merrill, 1981), only Farquharson's (1969) procedure for eliminating dominated strategies yields a foolproof way of deriving optimal voting choices. But Farquharson's procedure is seriously limited by its complexity and by the apparent frequency with which it is indeterminate (Niemi and Frank, 1982). Thus, apart from avoiding a 'wasted' vote – i.e., voting for one's second choice if one's first choice is unlikely to win – we have little understanding of what optimal plurality voting means or what would happen if voters tried to behave strategically in plurality elections.

In a previous article (Niemi and Frank, 1982), we confronted this void by proposing a new definition of sophisticated voting (with three alternatives) that appears to mirror closely the reasoning of voters about the task they confront under plurality rule. While this new definition appeared to be less complex and less often indeterminate than Farquharson's, we could not be positive of that. Similarly, we could not be certain about a variety of other comparisons with Farquharson's procedure – e.g., the tendency to pick Condorcet alternatives, the similarity of sophisticated strategies under the two definitions, and so on. Indeed, we will show below that one of our conjectures was incorrect.

In this paper we present the results of a simulation, showing that our definition is simple to apply in most situations, is much more frequently determinate than is Farquharson's, and compares very favorably with Farquharson's in selecting Condorcet alternatives. Since Farquharson-sophisticated behavior has been little explored in nonbinary situations,

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our work also advances understanding of that approach to strategic voting.

From a larger perspective, our results are significant in showing that sophisticated voting, at least using our definition, more frequently picks Condorcet alternatives than does sincere voting. Thus, paralleling conclusions about binary procedures (McKelvey and Niemi, 1978; Bjurulf and Niemi, 1982), strategic voting is in this sense actually an improvement over naive balloting.

THE NIEMI–FRANK DEFINITION OF SOPHISTICATED VOTING[1]

In order to make any headway in analyzing sophisticated plurality voting (using either definition), one must assume that voters with the same preference ordering vote as a bloc (Niemi and Frank, 1982, pp. 153–54). With this assumption, plurality voting with three alternatives can be conceived of as a noncooperative game with six or fewer players, since there are six different preference orderings, and a bloc of voters with the same preference ordering behaves as a single player. It is useful to label the voting blocs as follows:

Bloc 1: *abc*;
Bloc 2: *acb*;
Bloc 3: *bac*;
Bloc 4: *bca*;
Bloc 5: *cab*;
Bloc 6: *cba*;

Define n_i = the number of voters in bloc i, $i = 1, 2, \ldots, 6$. Then to specify a particular game we need only specify n_i, $i = 1, \ldots, 6$. Since our definition involves inequalities among sets of blocs, we define $N(i, j, \ldots)$, to be the number of voters in blocs $i, j, \ldots$ combined. Assuming no ties, we can assume, with no loss of generality, that $N(1, 2) > N(3, 4) > N(5, 6)$. In other words, a will always be the sincere winner and c will receive the fewest first place votes.

We now define sophisticated voting in terms of three steps.[2]

1. Voters consider the current situation, a situation being a description

of how all blocs vote and the outcome implied by that voting. At the beginning the current situation is sincere voting by all blocs.

2. Each bloc determines whether it can improve the outcome by altering its own vote while assuming that all other votes remain the same. Improving the outcome means changing the results from (a) one's last choice to: one's second choice; one's first choice; a tie between one's last choice and second choice; a tie between one's last choice and first choice; a tie between one's second choice and first choice; a tie between one's last, second, and first choices; or (b) one's second choice to: one's first choice; a tie between one's second and first choices.

It turns out that no more than two blocs can improve the outcome in this way.

3a. If no bloc can improve the outcome, the current situation is a Nash equilibrium, and the current situation contains the sophisticated strategies and the sophisticated outcome.

3b. If exactly one bloc can improve the outcome, it changes its vote accordingly, and the process reverts to step 1.

3c. If two blocs can improve the situation, we examine a simple two-person game. The result of the game is that only one bloc will change its vote or the situation is what Luce and Raiffa (1957, pp. 90–91) call a 'battle-of-the-sexes'. In the former case, the one bloc changes its vote and the process reverts to step 1. In the latter case, the sophisticated outcome is indeterminate.

The two-person game is one in which each bloc has two strategies – voting its first or second choice – and is based on the assumption that all other players will not change their votes. Each bloc checks for dominated strategies in the manner described earlier, and only strategies that are ultimately admissible in this game are used. Since it is a 2 × 2 game, there are no more than two simple reductions involved.

Whether a bloc can improve the outcome depends on one or more inequalities. Therefore, in practice this definition of sophisticated voting amounts to testing inequalities among sets of blocs. At a minimum there is one stage; two inequalities determine that the sincere winner is also the sophisticated winner. At a maximum there are five stages with ten inequalities to check.

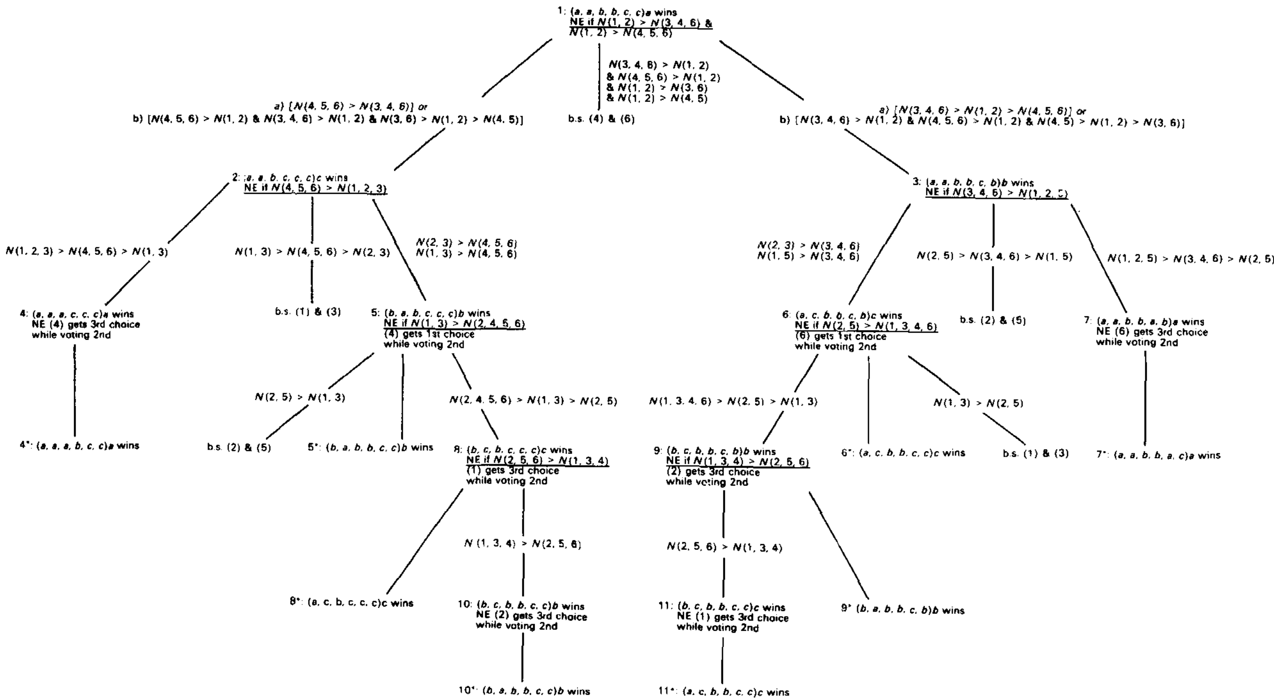

Fig. 1. Niemi-Frank Sophisticated Voting and the Associated Inequalities

Notes: (1) 'b.s.' means 'battle-of-the-sexes'. 'NE' means 'Nash equilibrium'. (2) Strategies indicated for bloc i are irrelevant if $n_i = 0$. (3) For the battle-of-the-sexes situation, the outcome is indeterminate and strategies are indeterminate for the 'battling' blocs only. (4) Situations 4*–9* are NE and do not alter the outcomes in 4–9. In 10* and 11*, blocs 1 and 2, respectively, can improve the outcome by shifting their votes to *a*. But this is the starting point – the sincere situation.

Application of the definition leads to the results shown in Figure 1. We begin with the sincere situation shown at the top. Each bloc considers voting for its first choice. Alternative *a* would win under this situation. Blocs then determine their sophisticated strategies by using steps 1–3 above. They ultimately arrive at a Nash equilibrium or determine that the situation is a battle-of-the-sexes. With fixed bloc sizes, of course, only one path would be traversed, so that the sophisticated strategies and outcome (or the discovery that there is a battle-of-the-sexes) would be determined more simply than the complete diagram suggests. Of course, just how simple or tedious it is to apply this definition depends heavily on how often the analysis stops after just a few steps, and how useful it is depends in large part on how often a determinate outcome is reached. These are the questions that we turn to first in looking at the results of our simulation.

THE SIMULATION

In our previous paper (p. 164) we conjectured that (a) whenever Farquharson's definition yields a determinate outcome, our definition yields the identical result; and (b) when Farquharson's definition is indeterminate and ours is determinate, our definition yields a unique set of strategies, and therefore a unique outcome, that are sophisticated in Farquharson's sense (i.e., the strategies are all ultimately admissible). We proved the conjecture analytically for one case – when the inequalities defining situation 11 in Figure 1 are satisfied. But the proof is extremely tedious. And even if it were completed for other cases, we would be left without any information on the frequency with which the two definitions are determinate, pick Condorcet winners, and so on. Therefore, we turned to a simulation similar to that used in many other analyses of voting systems.[3]

The simulation began by randomly picking six numbers between 0 and 999 to represent bloc sizes. Thus, the total size of the group could theoretically vary from 0 to 5994.[4] For a given set of blocs, we then determined the Condorcet winner and loser (or the presence of a cycle), the Farquharson winner (or that the situation was indeterminate) the Niemi–Frank winner (or indeterminacy), and the sophisticated strategies under each definition. Ten thousand sets of blocs were used, and a count was kept of the relevant features.

THE RESULTS

Simplicity

The first important question is how frequently each of the nodes in Figure 1 occurred. As noted, the reasoning leading to the nodes further down in the figure is simply repetitive. Nonetheless, the number of steps is sufficiently great that, if those nodes occur frequently, it is unlikely that Niemi–Frank sophisticated voting even comes close to representing the way voters might react to a plurality election situation.

It turns out that fully three-fourths of the time the case ends up in nodes 1–3 or in the battle-of-the-sexes just below node 1; indeed, 44 percent end in node 1 itself. The remaining cases end up almost exclusively in nodes 5–7 or in the battles-of-the-sexes just below nodes 2 and 3 (24 percent), with less than 1 percent in the nodes below that level. This means that most often the Niemi–Frank definition is very easy to apply. Three examples illustrate this with cases ending at nodes 1 and 3 and the associated battle-of-the-sexes. (Node 2 is very similar to node 3.)

In the first example, the sincere outcome cannot be upset. The analysis stops there.

Example 1:

abc	*acb*	*bac*	*bca*	*cab*	*cba*
545	825	894	354	377	71

Assuming no other vote changes, neither bloc 4 (by voting for *c*) nor bloc 6 (by voting for *b*) can overturn the sincere outcome. A shift by blocs 3 and 5 would only reinforce the sincere outcome.) Node 1.

In the second and third cases, the sincere outcome can be upset. In the second case, the contending blocs do not have a common, better strategy, so there is indeterminacy.

Example 2:

abc	*acb*	*bac*	*bca*	*cab*	*cba*
319	908	80	860	21	528

Both blocs 4 (by voting for *c*) and 6 (by voting for *b*) can overturn the sincere outcome. These blocs each prefer both *b* and *c* to *a*, but in different orders, so they are at odds with one another. Since blocs 1 and 2 together are larger than 3 and 6 combined and 4 and 5 combined, there is no resolution to the conflict.[5] Battle-of-the-sexes between blocs 4 and 6.

In the third case, the sincere outcome is overturned, but that new outcome is stable, so the analysis ends there.

Example 3:

abc	*acb*	*bac*	*bca*	*cab*	*cba*
567	919	915	515	14	895

Bloc 6 can overturn the sincere outcome but bloc 4 cannot. Neither bloc 2 (by voting for *c*) nor bloc 5 (by voting for *a*) can overturn this outcome. Node 3.

As noted, 75 percent of the cases are of the types just described. And only in rare circumstances are the bloc sizes such that a tentative outcome will be repeatedly upset by new strategic calculations. Since whenever there are six blocs, Farquharson's definition requires a time-consuming analysis (64 outcomes must be determined and numerous pairs of outcomes compared), our definition looks quite favorable in terms of its level of difficulty.

Determinacy and Similarity of Outcomes and Strategies

Analysis by hand of a limited number of examples suggests that Farquharson's definition is frequently indeterminate. To what degree is this actually true, and is the Niemi–Frank definition indeterminate any less often? Our simulation shows that Farquharson's definition yields a determinate outcome less than half of the time – in 46 percent of the cases. In contrast, the Niemi–Frank definition is determinate in 80 percent of the cases.[6] Moreover, indeterminacy under the Niemi–Frank definition is always of the battle-of-the-sexes variety. In these situations it is difficult to see how such a pattern of conflict could be resolved simply with plurality voting. Rather, something else – cooperation, side payments, a different voting system – must be brought to bear. Under Farquharson's definition, indeterminacy is not limited to battles-of-the-sexes. Indeed, an extreme example of this is found in situations in which Farquharson's method is indeterminate for all blocs even though a majority of voters have one alternative as their first choice (i.e., blocs *abc* and *acb* constitute a majority; see Niemi and Frank, 1982, p. 155).

As anticipated, there is never Farquharson determinacy and Niemi–Frank indeterminacy. And when both methods are determinant, the outcome is always identical. However, our conjecture was not correct with regard to strategies. Quite often our procedure prescribes strategies that are not ultimately admissible in Farquharson's sense. The following example shows the kind of circumstances in which this happens.

Example 4:

abc	*acb*	*bac*	*bca*	*cab*	*cba*
684	497	634	147	184	392

Using the Niemi–Frank definition, no bloc has an incentive to vote strategically since neither bloc 4 nor bloc 6 can upset the sincere outcome. It falls under node 1, and the sophisticated strategies equal the sincere strategies for all blocs. From the Farquharson perspective, however, we can easily find a circumstance in which bloc 5 is better off voting for *a*. If blocs 1–4 and 6 vote for *b*, *a*, *a*, *b*, and *b*, respectively, *a* receives 1131 votes and *b* receives 1223 votes; if bloc 5 votes sincerely (for *c*) it winds up with its last choice, but if it votes strategically for *a*, then *a* wins. This is a rather strange contingency since blocs 1, 3 and 6 are all voting for their second choices. Nevertheless, since this contingency is initially a possibility and since bloc 5 is never better off voting for *c* (possibly as well off but never better off), its Farquharson-sophisticated strategy is to vote for *a*. The contingency in which bloc 5 would vote for *a* is itself eliminated later (bloc 1, for example, ends up voting for *a*), so bloc 5's vote loses it pivotal status. Hence, the fact that bloc 5 votes for *a* under Farquharson but for *c* under Niemi-Frank makes no difference, and the two definitions yield the same outcome even though the strategies differ.[7]

While this difference in prescribed strategies is an interesting sidelight, the major point is that whenever strategic voting is determinate under both definitions, the outcome is the same. The strategies differ because of analyses in early stages that become moot at a later stage. While the Farquharson reasoning can be defended on the grounds that voters have considered *all possible* contingencies and eliminated only those that are dominated in such comparisons, the result of the simpler, perhaps empirically more relevant Niemi–Frank model are the same except that the latter yields a single outcome for a considerably larger proportion of the cases. But are these good outcomes? That is the last question to which we turn.

Plurality, Condorcet, Farquharson, and Niemi–Frank Winners

Under binary voting procedures, sophisticated voting guarantees the selection of a Condorcet winner if one exists (McKelvey and Niemi, 1978). Not so with plurality voting. Neither sincere nor Condorcet winners are guaranteed victory under the Farquharson or the Niemi–Frank definition of sophisticated voting. Indeed, even a Condorcet winner that is also the sincere winner may lose. However, under the assumptions of our model, when a Condorcet alternative exists, it is chosen only about three-fourths of the time when voting is sincere (75.7 percent in out simulation). Therefore, it may still be the case that sophisticated voting chooses the Condorcet winner more often than does sincere voting.

If one looks only at the cases in which sophisticated voting is determinate, then by either definition, it chooses Condorcet winners much more frequently than does sincere voting. Of more that 4000 cases in which the outcome was Farquharson – (and therefore Niemi–Frank) determinate and there was a Condorcet winner, sophisticated voting picked the Condorcet winner 99 percent of the time. Overall, the Niemi–Frank definition chose the Condorcet winner 97 percent of the time in the 7300 cases in which it was determinate and there was a Condorcet winner. When the Condorcet winner coincided with the sincere winner, sophisticated voting selected that alternative almost without failure – 99.9% for Farquharson and 99.5 percent for Niemi–Frank.

These numbers are impressive, but a fair comparison with sincere voting needs to take into account the indeterminacy of sophisticated voting. We can do so relatively easily with the Niemi–Frank definition since the indeterminacy always takes the form of a battle-of-the-sexes between two blocs. When the battle-of-the-sexes is between blocs 4 and 6, for example, if the blocs vote differently (probability assumed to equal 0.5), than a will win; if the blocs vote the same, then either b or c will win. From the simulation we can calculate the probability that each alternative is the Condorcet winner for each battle-of-the-sexes case. This allows us to calculate the probability that the Niemi–Frank winner will be a Condorcet winner. When there is a battle-of-the-sexes, a Condorcet winner is chosen only about 37 percent of the time, barely more than would be expected if everyone were voting randomly. These cases occur infrequently enough, however, that overall some 85 percent of the winners are the Condorcet alternative. Thus, even if we continue to assume noncooperative behavior and random voting by the battling blocs, Niemi–Frank sophisticated voting yields a Condorcet winner a very high proportion of the time and more frequently than sincere voting.[8]

A final note concerns Condorcet losers (alternatives that lose to both other alternatives). In our previous work (1982, p. 165) we showed that a Condorcet loser cannot win under Niemi–Frank sophisticated voting. Our simulation shows that the same is apparently true under Farquharson's definition. Under sincere voting, however, it is possible for such alternative to win. Nonetheless, this point is not too significant, since sincere voting picked the Condorcet loser only 0.5 percent of the time.

CONCLUSION

The Niemi–Frank definition of sophisticated voting can now be evaluated on two grounds. First, we can compare our definition to Farquharson's. For the most part, the two definitions yield identical outcomes. Both pick Condorcet winners a very high proportion of the time and prevent the selection of Condorcet losers. The major differences are in the logic underlying the two definitions and in the rate of determinacy of outcomes. Here there is a tradeoff. The logic underlying the Farquharson model is especially persuasive, although it is our feeling that the Niemi–Frank definition comes closer to mirroring the way in which voters might actually analyze a plurality situation. In any case, the price paid by the Farquharson definition for its ironclad logic is a much higher rate of indeterminacy. In over half of the cases, the Farquharson logic fails to lead to any conclusion whatsoever. The Niemi–Frank definition yields many more determinate situations, with mostly Condorcet winners and with strategies that make good, if not completely unassailable sense.

A second way of evaluating the Niemi–Frank definition is in comparison with sincere voting. A commonly-cited shortcoming of plurality voting is that often fails to choose a Condorcet winner. As we noted earlier, sophisticated plurality voting, unlike binary voting, is imperfect in this respect. Nonetheless, even taking account of the indeterminacy that remains in the Niemi–Frank definition, sophisticated voting picked a Condorcet winner about 10 percent more frequently than did sincere voting as well as eliminating the possibility of a Condorcet loser being chosen. By this measure, the Niemi–Frank definition is not only acceptable but suggests that this form of strategic behavior actually leads to better outcomes.

By proposing and now by testing a new definition of sophisticated voting under plurality rule, we have begun to make some headway on understanding strategic behavior and its effects in an outwardly simple yet deceptively complex voting system. We are, of course, far from finished. Most significantly, our definition applies to only three alternatives, and Farquharson's (even if one is willing to live with its high indeterminacy) becomes extraordinarily cumbersome with more than three alternatives.[9] In any event, the results of this foray into sophisticated nonbinary voting suggests once again that strategic behavior, rather than making things worse, improves the chances that the outcome will be the one most favored by the majority criterion.

NOTES

[1] We assume familiarity with Farquharson's definition. A brief explanation, along with a simpler display procedure than Farquharson's, is given in Niemi and Frank (1982).

[2] Described abstractly, the definition seems complicated. Results and examples in the next section show that it is most often very simple to use. The definition bears some resemblence to the decisionmaking sequences in sequential games and especially to the notion of staying power in such games (Brams and Hessel, 1983).

[3] A good, recent summary that compares the frequency with which various voting procedures pick Condorcet winners (when voting is sincere) is Fishburn and Gehrlein (1982).

[4] This is not equivalent to assuming that all preference orders are equally likely. If we did that, while using a fairly large number of preference orders, the blocs would tend to be equally sized. That would lead to an inordinately large number of cycles and would not yield very meaningful results.

[5] The conflict between 4 and 6 is best seen as a two-person game. Blocs 1 and 2 do not switch from voting for *a* because they will (tentatively) obtain their first choice. Blocs 3 and 5 cannot improve the outcome by switching their votes (because *a* is their second choice). Hence, the votes of blocs 1, 2, 3, and 5 are for *a*, *a*, *b*, and *c*, respectively. The two-person game is then visualized:

		Bloc 6 (*cba*)	
		c	*b*
Bloc 4 (*cba*)	*b*	*a*	*b*
	c	*c*	*a*

Neither bloc has a dominant strategy, and as along as the game remains noncooperative, there is direct conflict.

[6] The percentages are very similar whether there is a Condorcet winner (46% vs. 80%) or a cycle (48% vs. 80%).
[7] Nonindentical strategies are also found when Niemi–Frank is determinate but Farquharson is not.
[8] We did not calculate the number of times the Farquharson definition would yield a Condorcet winner when taking into account the indeterminate cases. If we did, however, Farquharson sophisticated voting would yield a Condorcet winner less often than sincere voting because there are so many indeterminate situations.
[9] It is possible, of course, that sophisticated voting in situations of four or more alternatives simply becomes too complex to analyze systematically in all but the simplest cases.

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