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AN APPROXIMATE SOLUTION OF EIGEN-FREQUENCIES OF TRANSVERSE VIBRATION OF RECTANGULAR PLATES WITH ELASTICAL RESTRAINTS

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Abstract

This paper presents an approximate solution for calculating eigen-frequencies of transverse vibration of rectangular plates elastically restrained against rotation along edges. The formulae are not only very simple and easily programmed but also have high accuracy. Finally, some numerical results are given and compared with other results obtained.

Key words transverse vibration, eigen-frequencies, Ritz method

I. Introduction

In the fields of aeronautical, civil and naval engineerings and the like, many structures may be simplified to rectangular plates elastically restrained against rotation along edges. Thus it is of great importance to study the dynamic characteristics of such plates. For recent years, some study results about the eigen-frequencies of rectangular plates with elastical restraints along edges have been reported but these studies mainly focused attention on the approximate estimation of fundamental frequency[1-3], the calculating for high order eigen-frequencies is very limited. Laura[4,5] took the polynomial functions as the displacement function approximately to estimate the fundamental frequency of rectangular plates with elastical restraints along edges. Warburton[6] took the superposition of simply supported beam functions and clamped supported beam functions approximately to calculate the low order eigen-frequencies of such plates by using Rayleigh-Ritz method. Mukhopadhyay[7] used the semi-analysis method[8] to solve the high order eigen-frequencies of such plates, the results have good accuracy.

Ritz method is a valuable method which is widely used approximately to calculate the eigen-frequencies of structures. Its accuracy completely depends on the selection of basis functions. Considering the structural features of rectangular plates with elastical restraints along edges, this paper selects the superposition of beam functions and polynomial functions as the basis functions. Ritz method is utilized to calculate eigen-frequencies of such plates. The formulae are simple and easily programmed. The numerical results show good accuracy compared with the others obtained.

II. Mathematic Model

Known from the vibration theory of plate in rectangular coordinates, the differential

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equation of transverse vibration of plate is

$$\nabla^4 w+\frac{\rho h}{D}\frac{\partial^2 w}{\partial t^2}=0 \tag{2.1}$$

where w is the transverse displacement of plate, ρ and h are the density of material and the thickness of plate respectively, $D=\frac{Eh^3}{12(1-\mu^2)}$ is the flexural rigidity of plate, E and μ are Young's modulus and Poisson ratio of material respectively.

For the rectangular plate with elastical restraints along edges shown in Fig. 1, the boundary conditions of its transverse vibration are

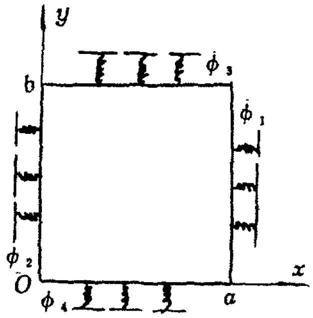

Fig. 1 The rectangular plate elastically restrained against rotation along edges

$$\left.\begin{aligned}
&w=0, \quad \frac{\partial w}{\partial x}=-\phi_1 D\left\{\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right\}, \quad x=a\\
&w=0, \quad \frac{\partial w}{\partial x}=\phi_2 D\left\{\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right\}, \quad x=0\\
&w=0, \quad \frac{\partial w}{\partial y}=-\phi_3 D\left\{\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right\}, \quad y=b\\
&w=0, \quad \frac{\partial w}{\partial y}=\phi_4 D\left\{\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right\}, \quad y=0
\end{aligned}\right\} \tag{2.2}$$

where $\phi_i(i=1, 2, 3, 4)$ are the rotational flexural coefficients of four edges of plate respectively. If $\phi_i(i=1, 2, 3, 4)$ are taken to the limit value (0 or ∞), then the simply supported edges and clamped supported edges are gained respectively.

Let $w=z(x,y)\sin(pt+\varphi)$, the potential energy and kinetic energy of elastical plate shown in Fig. 1 can be written as follows

$$\left.\begin{aligned}
U_{max}=&\frac{D}{2}\iint\left\{(\nabla^2 z)^2-2(1-\mu)\left[\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2}-\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2\right]\right\}dxdy\\
&+\frac{D}{2}\left\{\int_0^b\frac{\partial^2 z}{\partial x^2}\frac{\partial z}{\partial x}\bigg|_{x=0}dy-\int_0^b\frac{\partial^2 z}{\partial x^2}\frac{\partial z}{\partial x}\bigg|_{x=a}dy\right.\\
&\left.+\int_0^a\frac{\partial^2 z}{\partial y^2}\frac{\partial z}{\partial y}\bigg|_{y=0}dx-\int_0^a\frac{\partial^2 z}{\partial y^2}\frac{\partial z}{\partial y}\bigg|_{y=b}dx\right\}\\
T_{max}=&\frac{1}{2}\rho h p^2\iint z^2 dxdy
\end{aligned}\right\} \tag{2.3}$$

where p is the eigen-frequency of transverse vibration of plate, $z(x, y)$ is the transverse displacement of plate.

Form Hamiltonian principle, there is the following variation

$$\delta(U_{max}-T_{max})=0 \tag{2.4}$$

The approximate solution of $z(x,y)$ is assumed to be

$$z(x,y)=\sum_{m=1}^{q}\sum_{n=1}^{r}A_{mn}X_m(x)Y_n(y) \tag{2.5}$$

where $X_m(x)$ are called as eigen-functions of plate in x direction and $Y_n(y)$ are called as eigen-functions of plate in y direction, A_{mn} are unknown coefficients. Substituting Eq. (2.5) into Eq. (2.4) gets

$$\sum_{m=1}^{q}\sum_{n=1}^{r}\left[C_{mn}^{(ij)}-\lambda^2 m_{mn}^{(ij)}\right]A_{mn}=0 \quad \begin{pmatrix} i=1,2,\cdots,q \\ j=1,2,\cdots,r \end{pmatrix} \tag{2.6}$$

where

$$\left.\begin{aligned}
&\lambda^2=\frac{\rho h a^3 b}{D}p^2 \\
&C_{mn}^{(ij)}=a^2\iint\Big[(\nabla^2X_mY_n)(\nabla^2X_iY_i)-(1-\mu)\Big(\frac{d^2X_m}{dx^2}Y_nX_i\frac{d^2Y_i}{dy^2} \\
&\quad +X_m\frac{d^2Y_n}{dy^2}\frac{d^2X_i}{dx^2}Y_j-2\frac{dX_m}{dx}\frac{dY_n}{dy}\frac{dx_i}{dx}\frac{dY_j}{dy}\Big)\Big]dxdy \\
&\quad +\frac{a^2}{2}\Big\{\int_0^b Y_nY_j\Big(\frac{d^2X_m}{dx^2}\frac{dX_i}{dx}+\frac{d^2X_i}{dx^2}\frac{dX_m}{dx}\Big)\Big|_{x=0}dy-\int_0^b Y_nY_j\Big(\frac{d^2X_m}{dx^2}\frac{dX_i}{dx} \\
&\quad +\frac{d^2X_i}{dx^2}\frac{dX_m}{dx}\Big)\Big|_{x=a}dy+\int_0^a X_mX_i\Big(\frac{d^2Y_n}{dy^2}\frac{dY_j}{dy}+\frac{d^2Y_j}{dy^2}\frac{dY_n}{dy}\Big)\Big|_{y=0}dx \\
&\quad -\int_0^a X_mX_i\Big(\frac{d^2Y_n}{dy^2}\frac{dY_j}{dy}+\frac{d^2Y_j}{dy^2}\frac{dY_n}{dy}\Big)\Big|_{y=b}dx\Big\} \\
&m_{mn}^{(ij)}=\frac{1}{ab}\iint X_mY_nX_iY_jdxdy
\end{aligned}\right\} \tag{2.7}$$

Taking account of the structural characteristics of rectangular plates elastically restrained against rotation along edges, one can assume that

$$\left.\begin{aligned}
X_m(x)&=\sin\Big(\frac{m\pi}{a}x\Big)+\sum_{k=0}^{3}C_{mk}x^k \\
Y_n(y)&=\sin\Big(\frac{n\pi}{b}y\Big)+\sum_{k=0}^{3}D_{nk}y^k
\end{aligned}\right\} \tag{2.8}$$

where C_{mk} and D_{nk} are unknown coefficients.

Let $X_m(x)$ and $Y_n(y)$ satisfy the corresponding boundary conditions of plate with elastical restraints, one has

$$\left.\begin{array}{lll} X_m(x)=0, & \dfrac{dX_m}{dx}=-\phi_1 D(1-\mu^2)\dfrac{d^2X_m}{dx^2} & (x=a) \\ X_m(x)=0, & \dfrac{dX_m}{dx}=\phi_2 D(1-\mu^2)\dfrac{d^2X_m}{dx^2} & (x=0) \\ Y_n(y)=0, & \dfrac{dY_n}{dy}=-\phi_3 D(1-\mu^2)\dfrac{d^2Y_n}{dy^2} & (y=b) \\ Y_n(y)=0, & \dfrac{dY_n}{dy}=\phi_4 D(1-\mu^2)\dfrac{d^2Y_n}{dy^2} & (y=0) \end{array}\right\} \tag{2.9}$$

Substituting Eqs. (2.8) into Eqs. (2.9), one can gain the unique solution of C_{mk} and D_{nk}. Moreover, because the integration of multiplication of beam functions with polynomial functions can be in the analytical formation, $C_{mn}^{(ij)}$ and $m_{mn}^{(ij)}$ can be calculated accurately without numerical integration.

III. Numerical Examples

Now considering a square plate with three edges simply supported and one edge elastically restrained against rotation shown in Fig. 2.

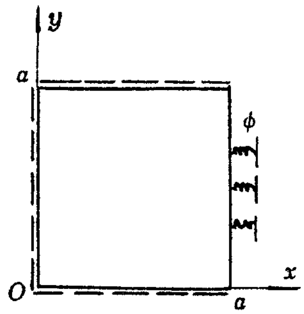

Fig. 2 The square plate with three edges simply supported and one edge elastically restrained against rotation

If let $\xi=\dfrac{x}{a}$ and $\eta=\dfrac{y}{a}$, Eqs. (2.8) can be written as

$$\left.\begin{array}{l} X_m(\xi)=\sin m\pi\xi+\sum\limits_{k=0}^{3} C_{mk}\xi^k \\ Y_n(\xi)=\sin n\pi\eta+\sum\limits_{k=0}^{3} D_{nk}\eta^k \end{array}\right\} \tag{3.1}$$

In Eqs. (2.9), let $\phi_1=\phi$, $\alpha=\dfrac{\phi D}{a}$ and $\phi_2=\phi_3=\phi_4=\infty$, one has

$$\left.\begin{array}{lll} X_m(\xi)=0, & \dfrac{dX_m}{d\xi}=-\alpha(1-\mu^2)\dfrac{d^2X_m}{d\xi^2} & (x=a) \\ X_m(\xi)=0, & \dfrac{d^2X_m}{d\xi^2}=0 & (x=0) \\ Y_n(\eta)=0, & \dfrac{d^2Y_n}{d\eta^2}=0 & (y=0,\ a) \end{array}\right\} \tag{3.2}$$

Substituting Eqs. (3.1) into Eqs. (3.2), one has

$$\left.\begin{aligned} &C_{m0}=C_{m2}=0,\ C_{m1}=-C_{m3}=\frac{(-1)^m m\pi}{2(3\alpha(1-\mu^2)+1)} \\ &D_{nk}=0 \qquad (k=0,1,2,3) \end{aligned}\right\} \tag{3.3}$$

One can easily get

$$\left.\begin{aligned} &\int_0^1 X_i(\xi)X_j(\xi)d\xi=\frac{1}{2}\delta(i-j)-\frac{6}{\pi^3}\left(\frac{(-1)^i}{i^3}C_{j1}+\frac{(-1)^j}{j^3}C_{i1}\right)+\frac{8}{105}C_{i1}C_{j1} \\ &\int_0^1 X_i'(\xi)X_j'(\xi)d\xi=\frac{1}{2}ij\pi^2\delta(i-j)-\frac{6}{\pi}\left(\frac{(-1)^i}{i}C_{j1}+\frac{(-1)^j}{j}C_{i1}\right)+\frac{4}{5}C_{i1}C_{j1} \\ &\int_0^1 X_i''(\xi)X_j''(\xi)d\xi=\frac{1}{2}i^2j^2\pi^4\delta(i-j)-6\pi(i(-1)^iC_{j1}+j(-1)^jC_{i1})+12C_{i1}C_{i1} \\ &\int_0^1 X_i''(\xi)X_j(\xi)d\xi=-\frac{1}{2}i^2\pi^2\delta(i-j)+\frac{6}{\pi}\left(\frac{(-1)^i}{i}C_{j1}+\frac{(-1)^j}{j}C_{i1}\right)-\frac{4}{5}C_{i1}C_{j1} \\ &\int_0^1 Y_i(\eta)Y_j(\eta)d\eta=\frac{1}{2}\delta(i-j),\quad \int_0^1 Y_i'(\eta)Y_j'(\eta)d\eta=\frac{1}{2}ij\pi^2\delta(i-j) \\ &\int_0^1 Y_i''(\eta)Y_j''(\eta)d\eta=\frac{1}{2}i^2j^2\pi^4\delta(i-j),\quad \int_0^1 Y_i''(\eta)Y_j(\eta)d\eta=-\frac{1}{2}i^2\pi^2\delta(i-j) \end{aligned}\right\} \tag{3.4}$$

where $\delta(i-j)=\begin{cases}0 & (i\neq j)\\ 1 & (i=j)\end{cases}$. Substituting above equations into Eqs. (2.7), one can get $C_{mn}^{(ij)}$ and $m_{mn}^{(ij)}$, then eigen-frequencies can be obtained from Eq. (2.6).

The first ten order frequencies are given in Table 1 by using this paper's method and some corresponding data resulted from other methods are also listed in the table for comparison.

Table 1 The first ten order eigen-frequencies $\lambda=\left(\frac{\rho h a^4}{D}p^2\right)^{1/2}$ and $\alpha=\frac{\phi D}{a}$ of square plate with three edges simply supported and one edge elastically restrained against rotation ($\mu=0.3$, $q=r=5$)

α	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
0.0	23.647 [23.646]	51.678 [51.674]	58.647 [58.646]	86.145 [86.134]	100.284 [100.270]	113.230	133.835	140.851	168.990	187.544
0.01	23.393 (23.600)	51.459 (51.300)	57.815 (57.210)	85.353 (85.899)	100.089 (99.765)	111.531	133.092	139.196	168.810	184.684
0.05	22.636	50.879	55.543	83.366	99.626	107.323	131.392	135.345	168.419	178.261
0.1	22.037	50.486	53.960	82.118	99.351	104.757	130.432	133.149	168.211	174.800
0.5	20.600	49.719	50.843	79.917	98.890	100.460	128.911	129.675	167.901	169.680
1.0	20.222 (20.188)	49.549 (49.320)	50.156 (50.145)	79.469 (79.433)	98.799 (97.476)	99.629	128.624	129.026	167.845	168.773
5.0	19.846	49.391	49.521	79.065	98.718	98.892	128.372	128.455	167.796	167.988
10.0	19.793	49.370	49.435	79.011	98.707	98.794	128.338	128.381	167.790	167.886
100.0	19.745	49.350	49.357	78.962	98.697	98.706	128.308	128.312	167.784	167.794
1000.0	19.740 (19.460)	49.348 (49.210)	49.349 (49.348)	78.957 (79.877)	98.696 (97.214)	98.697	128.305	128.306	167.783	167.784
∞	19.739 [19.739]	49.348 [49.348]	49.348 (49.348)	78.957 [78.957]	98.696 [98.696]	98.696	128.305	128.305	167.783	167.783

Note: The data in [ ] are quoted from [9] and those in ( ) from [7].

IV. Conclusion

In this paper, the superposition of simply supported beam functions and polynomial functions is selected as the basis functions of transverse vibration of rectangular plates with elastical restraints. The simply supported beam functions are used as the main solution of basis functions, since the simply supported beam functions don't satisfy all the boundary conditions, thus the polynomial functions are added as the modified solution of basis functions in order to make basis functions satisfy all boundary conditions. It should be noted that the main solution of basis functions is not only limited to simply supported beam functions, arbitrary beam functions may also be selected as main solution but in order to enhance the accuracy, the beam functions which satisfy boundary conditions as more as possible should be selected as the main solution of basis functions. This paper selects the simply supported beam functions as main solution of basis functions, not only the calculating is very simple but also the amount of work is little, if other beam functions are selected as the main solution of basis functions then the amount of work will increase. In addition, this method can also be used to calculate eigen-frequencies of plate with arbitrary elastical supports on edges.

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