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On the first eigenvalue of the Dirac Operator on 6-dimensional manifolds

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1. Introduction

Let M^n be a compact Riemannian spin manifold with positive scalar curvature R and let R_o denote its minimum. Consider the Dirac operator $D : \Gamma(S) \longrightarrow \Gamma(S)$ acting on sections of the associated spinor bundle S. If $\lambda^{\pm}$ is the first positive or negative eigenvalue of this operator, then

$$\frac{1}{2}\sqrt{\frac{n}{n-1} R_o} \leq |\lambda^{\pm}| .$$

Furthermore, if $+\frac{1}{2}\sqrt{\frac{n}{n-1} R_o}$ or $-\frac{1}{2}\sqrt{\frac{n}{n-1} R_o}$ is an eigenvalue of the Dirac operator, then M^n must be an Einstein space (see [2])

This situation occurs quite often when n is odd. On $S^3/_\Gamma$ the lower bound is an eigenvalue of D if and only if $S^3/_\Gamma$ is homogeneous, and there are also 5-dimensional non-homogeneous spaces $S^5/_\Gamma$ of constant curvature such that both values $\pm\frac{1}{4}\sqrt{5 R_o}$

are eigenvalues of the Dirac operator (cp. [9]). Moreover, there is an Einstein metric of positive scalar curvature on the 5-dimensional Stiefel manifold $V_{4,2} = SO(4)/SO(2)$ with the same property (see [2]). Similar examples can be given for n = 7 and n = 9.

On the other hand, for n = 4, only on the sphere S^4 the lower bound is an eigenvalue of the Dirac operator (see [3]). Moreover, up to now no even dimensional spaces different from the sphere and realizing the lower bound as an eigenvalue have been known. The aim of this paper is to give such examples for n = 6.

<u>Theorem:</u> Let M^6 be either the flag manifold F(1,2) or the complex projective space $\mathbb{C}P^3$. Then M^6 admits a non-Kähler Riemannian Einstein metric of positive scalar curvature such that $\pm \frac{1}{2}\sqrt{\frac{6}{5} R_o}$ are eigenvalues of the corresponding Dirac operators.

If F(1,2) and $\mathbb{C}P^3$ are considered as twistor spaces over $\mathbb{C}P^2$ and S^4, respectively then the considered metric arises from the standard Kähler metric of these spaces by scaling in the direction of the S^2-fibres (see [4] and also [10] for a more general approach). In proving the theorem we restrict ourselves to the case M^6 = F(1,2); the remaining part can be carried out in the same way.

2. About the geometry of F(1,2)

Consider the flag manifold F(1,2), i.e. the set of pairs (l,v) where both l and v are linear subspaces of $\mathbb{C}^3$ of dimension 1 and 2, respectively, and $l \subset v$. Then the U(3)-action on $\mathbb{C}^3$ results in $F(1,2) = U(3)/U(1) \times U(1) \times U(1)$. The Lie algebras of G = U(3) and H = U(1) × U(1) × U(1) are given by

$$\underline{g} = \{ A \in M_3(\mathbb{C}) : \bar{A}^T + A = 0 \}$$
$$\underline{h} = \{ A \in \underline{g} : A \text{ is diagonal} \}.$$

We decompose $\underline{g} = \underline{h} \oplus \mathfrak{k}$ with $\mathfrak{k} = V_1 \oplus V_2 \oplus V_3$ and

$$V_1 = \left\{ \begin{pmatrix} 0 & z & 0 \\ -\bar{z} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, z \in \mathbb{C} \right\}, \quad V_2 = \left\{ \begin{pmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ -\bar{z} & 0 & 0 \end{pmatrix}, z \in \mathbb{C} \right\},$$

$$V_3 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & z \\ 0 & -\bar{z} & 0 \end{pmatrix}, z \in \mathbb{C} \right\}.$$

Consider the inner product on $\underline{g}$ given by

$$B(X,Y) = -\frac{1}{2}\operatorname{Tr}(X \circ Y), \quad X, Y \in \underline{g}.$$

For arbitrary $c_1, c_2, c_3 \in \mathbb{R}$ with $c_i > 0$ we define a G-invariant Riemannian structure on $F(1,2) = {}^{G}/_{H}$ by

$$B_{c_1,c_2,c_3} = \sum_{i=1}^{3} c_i \; B\,|_{V_i \times V_i}.$$

According to [1] we get different Einstein metrics (up to a homothety) only for $c_1 = c_2 = c_3 = 1$ and $c_1 = c_2 = \frac{1}{2} c_3 = 1$. The former turns out to be a non-Kähler biinvariant Einstein metric of scalar curvature $R = 30$, whereas the latter (which we shall omit in our further considerations) is the standard left-invariant Kähler Einstein metric with scalar curvature $R = 24$.
Therefore, considering $c_1 = c_2 = c_3 = 1$ we introduce an orthonormal base $\{e_1, \ldots, e_6\}$ of $\underline{p}$ in the following way: Denote by $D_{ij} = (\delta_{ia} \; \delta_{jb})_{1 \le a,b \le 3}$ the usual generators of $M_3(\mathbb{R})$, consider $E_{ij} = D_{ij} - D_{ji}$ and $S_{ij} = \sqrt{-1}(D_{ij} + D_{ji})$, $1 \le i,j \le 3$, and set

$$e_1 = E_{12},\; e_2 = S_{12},\; e_3 = E_{13},\; e_4 = S_{13},\; e_5 = E_{23},\; e_6 = S_{23}.$$

As $D_{ij} \cdot D_{kl} = \delta_{jk} D_{il}$, the following commutator relations hold:

$[E_{ij}, E_{jk}] = E_{ik}$, $[S_{ij}, S_{jk}] = E_{ki}$ if $i \ne j \ne k$, and

$[E_{ij}, S_{jk}] = S_{ik}$ if $i \ne j \ne k \ne i$.

Thus $\{e_{2i-1}, e_{2i}\}$ is a base of V_i, and in addition it holds that $[V_i, V_i] \subseteq \underline{h}$, $[\underline{h}, V_i] \subseteq V_i$. We also fix a base of $\underline{h}$ by setting $H_i = \frac{1}{2} S_{ii}$, $i = 1,2,3$.

As B is Ad(G)-invariant, it follows from Wang's theorem that the induced Levi-Civita connection on F(1,2) is uniquely determined by a linear map $\Lambda : \mathfrak{R} \longrightarrow \mathrm{End}\,(\mathfrak{R})$ satisfying the conditions

$$\begin{cases} \text{(i)} \quad \Lambda(X)Y - \Lambda(Y)X = [X,Y]_{\mathfrak{R}} \\ \text{(ii)} \quad \Lambda(X) \text{ is skew-symmetric with respect to B, i.e.} \\ \qquad B(\Lambda(X)Y,Z) + B(\Lambda(X)Z,Y) = 0. \end{cases}$$

(see [7]). Here $[X,Y]_{\mathfrak{R}}$ denotes the $\mathfrak{R}$-component of $[X,Y]$. From these properties we easily derive the explicite form of Λ. In fact, for arbitrary vectors X, Y, $Z \in \mathfrak{R}$ we obtain

$$\begin{aligned} B(\Lambda(X)Y,Z) &= B(\Lambda(Y)X,Z) + B([X,Y]_{\mathfrak{R}}, Z) \\ &= -B(\Lambda(Y)Z,X) + B([X,Y], Z), \end{aligned}$$

because the decomposition $\underline{g} = \underline{h} \oplus \mathfrak{R}$ is orthogonal with respect to B. As B is Ad(G)-invariant, we also have

$$\begin{aligned} B(\Lambda(X)Y, Z) &= -B(\Lambda(X)Z,Y) = -B(\Lambda(Z)X,Y) - B([X,Z], Y) \\ &= B(\Lambda(Z)Y,X) + B([X,Y],Z). \end{aligned}$$

As $\Lambda(Y)Z, \Lambda(Z)Y \in \mathfrak{R}$, we conclude $\Lambda(Y)Z = -\Lambda(Z)Y$, and with respect to (i) it yields $\Lambda(Y)Z = \frac{1}{2}[Y,Z]_{\mathfrak{R}}$, $Y,Z \in \mathfrak{R}$. After identifying $\mathfrak{R}$ with the standard Euclidean vector space $\mathbb{R}^6$ by means of the base $\{e_1, \ldots, e_6\}$ of $\mathfrak{R}$, the image of $\Lambda : \mathbb{R}^6 \longrightarrow \mathrm{End}\,(\mathbb{R}^6)$ is actually contained in $\underline{so}\,(6)$, the Lie algebra of SO(6) spanned by $\{E_{ij} \in M_6(\mathbb{R});\ 1 \leq i < j \leq 6\}$, and we have

$$\begin{aligned} &\Lambda(e_1) = \tfrac{1}{2}(E_{35} + E_{46}) && \Lambda(e_4) = -\tfrac{1}{2}(E_{16} + E_{25}) \\ &\Lambda(e_2) = \tfrac{1}{2}(E_{45} - E_{36}) && \Lambda(e_5) = \tfrac{1}{2}(E_{13} + E_{24}) \qquad \langle 1 \rangle \\ &\Lambda(e_3) = \tfrac{1}{2}(E_{26} - E_{15}) && \Lambda(e_6) = \tfrac{1}{2}(E_{14} - E_{23}) \end{aligned}$$

Under the same identification one can also compute the isotropy representation $Ad : H \longrightarrow SO(\mathfrak{p}) = SO(6)$ of $G/_H$. For an arbitrary element

$$h = \begin{pmatrix} e^{it} & 0 & 0 \\ 0 & e^{is} & 0 \\ 0 & 0 & e^{it} \end{pmatrix} \in H, \quad t,r,s \in \mathbb{R},$$

it is defined by $Ad(h)e_j = h \cdot e_j h^{-1}$ and given by the matrix

$$Ad(h) = \begin{pmatrix} \cos(t-s) & -\sin(t-s) & 0 & 0 & 0 & 0 \\ \sin(t-s) & \cos(t-s) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(t-r) & -\sin(t-r) & 0 & 0 \\ 0 & 0 & \sin(t-r) & \cos(t-r) & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(s-r) & -\sin(s-r) \\ 0 & 0 & 0 & 0 & \sin(s-r) & \cos(s-r) \end{pmatrix}$$

It follows that the differential $Ad_* : \underline{h} \longrightarrow \underline{so}(6)$ is then given by

$$\begin{aligned} Ad_*(H_1) &= -E_{12} - E_{34} \\ Ad_*(H_2) &= E_{12} - E_{56} \qquad \qquad \langle 2 \rangle \\ Ad_*(H_3) &= E_{34} + E_{56}. \end{aligned}$$

3. Proof of the theorem

Let q : Spin(6) $\longrightarrow$ SO(6) be the 2-fold covering of SO(6). Then the induced map q_* : $\underline{\text{spin}}(6) \longrightarrow \underline{\text{so}}(6)$ on the Lie algebras is an isomorphism.

Lemma: There exists a lifting homomorphism $\widetilde{Ad} : H \longrightarrow Spin(6)$ of Ad, hence $q\widetilde{Ad} = Ad$ holds.

Proof: We have to show only that $Ad^*(\pi_1(H)) \subset q^*(\pi_1(Spin(6))) = 0$, or equivalently, that each generator of $\pi_1(H)$ vanishes under the superposition with Ad. Knowing $\pi_1(SO(6)) = Z_2$, the assertion follows immediately from the formula of Ad(h) given above and the additional remark, that for

$$A(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} : [0,\ 2\pi] \longrightarrow SO(2),$$ the matrices

$$\begin{pmatrix} A(t) & 0 & 0 \\ 0 & A(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A(2t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

describe the same element of $\pi_1(SO(6))$. □

The map $\widetilde{Ad} : H \longrightarrow Spin(6)$ gives rise to a natural spin-structure $P = G \times_{\widetilde{Ad}} Spin(6)$ over G/H. However, this is the only possible one, since $H^1(F(1,2); Z_2) = 0$ (see [8]). Then the spinor bundle S is a vector bundle over G/H, which is associated to P by the map $\varkappa : Spin(6) \longrightarrow GL(\Delta_6)$; here $\varkappa$ means the restriction of $K : Cliff^c(\mathbb{R}^6) \longrightarrow End(\Delta_6)$ to $Spin(6)$. The $\mathbb{C}$-algebra-isomorphism K can be realized as follows (a general reference is [5]): In $\Delta_6 \cong \mathbb{C}^8$ we choose the base obtained from

$u_{+1} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $u_{-1} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ by setting $u(\varepsilon_1, \varepsilon_2\ \varepsilon_3) = u_{\varepsilon_1} \otimes u_{\varepsilon_2} \otimes u_{\varepsilon_3}$, $\varepsilon_j \in \{\pm 1\}$. Using the notations

$$g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},\quad T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

we define

$$K(e_1) = E \otimes E \otimes g_1, \qquad K(e_2) = E \otimes E \otimes g_2,$$
$$K(e_3) = E \otimes g_1 \otimes T, \qquad K(e_4) = E \otimes g_2 \otimes T,$$
$$K(e_5) = g_1 \otimes T \otimes T, \qquad K(e_6) = g_2 \otimes T \otimes T.$$

and since $Cliff^c(\mathbb{R}^6)$ is generated by the vectors $e_1, \ldots, e_6 \in \mathbb{R}^6$, the isomorphism K is completely described.

Noting that

$$g_1(u_{+1}) = iu_{-1}, \qquad g_1(u_{-1}) = iu_{+1},$$
$$g_2(u_{+1}) = u_{-1}, \qquad g_2(u_{-1}) = -u_{+1}$$
$$T(u_{+1}) = -u_{+1}, \qquad T(u_{-1}) = u_{-1},$$

the action of K on the $u(\ldots)$ can easily be computed. The restriction $\varkappa$ of K splits into two irreducible 4-dimensional resprentations

$\varkappa = \varkappa^{+} \oplus \varkappa^{-}$ with $\varkappa^{\pm}(e_j): \Delta_6^{\pm} \longrightarrow \Delta_6^{\pm}$, $1 \leq j \leq 6$; here we set

$$\Delta_6^{+} = \mathrm{Lin}\left\{u(1,1,1),\ u(1,-1,-1),\ u(-1,1,-1),\ u(-1,-1,1)\right\},$$

$$\Delta_6^{-} = \mathrm{Lin}\left\{u(1,1,-1), u(1,-1,1),\ u(-1,1,1),\ u(-1,-1,-1)\right\}.$$

Using this order of the $u(\ldots)$, the $\varkappa^{\pm}(e_j)$ are given by the matrices

$$\varkappa^{+}(e_1) = \varkappa^{-}(e_1) = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \quad \varkappa^{\pm}(e_2) = \pm \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\varkappa^{\pm}(e_3) = \pm \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \qquad \varkappa^{\pm}(e_4) = \pm \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \langle 3 \rangle$$

$$\varkappa^{\pm}(e_5) = \pm \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \qquad \varkappa^{\pm}(e_6) = \pm \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

In this way we also described the differential of $\varkappa$ acting on the Lie algebra $\underline{\mathrm{spin}}(6)$, which we denote by $\varkappa$ too.
Together with $\mathfrak{q}_*^{-1}(E_{ij}) = \frac{1}{2} e_i e_j$ $(1 \leq i < j \leq 6)$ $\langle 4 \rangle$ the equations $\langle 2 \rangle$ determine already the map $\widetilde{\mathrm{Ad}}_* = \mathfrak{q}_*^{-1} \mathrm{Ad}_* : \underline{h} \rightarrow$ $\rightarrow \underline{\mathrm{spin}}(6)$. Combining this with $\langle 3 \rangle$ we note that every $H_i \in \underline{h}$ annihilates both vectors $u(-1,1,-1)$ and $u(1,-1,1)$. Therefore, for arbitrary $z_1, z_2 \in \mathbb{C}$ and all $X \in \underline{h}$

$\varkappa \widetilde{\mathrm{Ad}}_*(X)(z_1 u(-1,1,-1) + z_2 u(1,-1,1)) = 0$ holds, and because H is connected, we conclude

$$\varkappa\,\widetilde{Ad}(h)(z_1u(-1,1,-1) + z_2u(1,-1,1)) =$$
$$= z_1(u(-1,1,-1) + z_2u(1,-1,1)).$$

Thus, the constant function $\psi \equiv z_1u(-1,1,-1) + z_2u(1,-1,1)$ on G defines a section in the spinor bundle S, the sections of $S = G \times_{\varkappa \widetilde{Ad}} \Delta_6$ being as usual identified with functions $\varphi : G \longrightarrow \Delta_6$ satisfying $\varphi(g\,h) = \varkappa\,\widetilde{Ad}(h^{-1})\varphi(g)$ for all $g \in G$, $h \in H$.
Now we are going to prove the theorem.
Given a section φ of the spinor bundle, the action of the Dirac operator D on φ is given by the formula
$D\varphi = \sum_{j=1}^{6} \varkappa(e_j)(e_j(\varphi) + \varkappa\tilde{\Lambda}(e_j)\varphi)$, as stated in [6]. On the constant section ψ this expression simplifies to
$D\psi = \sum_{j=1}^{6} \varkappa(e_j)\cdot \varkappa\,\tilde{\Lambda}(e_j)\psi$; here $\tilde{\Lambda} = q_*^{-1}\cdot\Lambda$ is given by <1> and <4>. Thus from <3> we obtain

$$D\psi = 3\,(-iz_2u(-1,1,-1) + i\,z_1u(1,-1,1)),$$

and by $z_1 = 1$, $z_2 = i$ and $z_1 = i$, $z_2 = 1$ eigenspinors of D are obtained with eigenvalues ± 3. Because of $R = 30$ we finally have
$\frac{1}{2}\sqrt{\frac{6R}{5}} = |\lambda^{\pm}|$.

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