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Similarity solutions for nonlinear diffusion – further exact solutions

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Abstract. Although the nonlinear diffusion equation has been extensively studied and there exists substantial literature in many diverse areas of science and technology, the number of exact concentration profiles is nevertheless limited. In a recent article in this journal (Hill [1]) a brief review of known exact results is given, as well as an elementary integration procedure which appears to be a general device for obtaining integrals associated with similarity solutions. This paper extends the results given in [1] and for particular power law diffusivities c^m (such as $m = -1/2$, -1, $-3/2$ and -2) presents a number of new exact solutions obtained by fully integrating the ordinary differential equations derived in [1]. In addition new results are found for a general nonlinear diffusion equation which includes one-dimensional diffusion with an inhomogeneous and nonlinear diffusivity $c^m x^n$ as well as symmetric nonlinear diffusion in cylinders and spheres. Moreover by a separate and ad-hoc procedure a new solution is obtained of the travelling wave type but with a variable wave speed. Some of the new exact solutions obtained for one-dimensional nonlinear diffusion with power law diffusivities c^m are illustrated graphically.

1. Introduction

In a recent article [1] appearing in this journal, the second author has given an elementary integration device for similarity solutions,

$$c(x, t) = x^{2\lambda/m(1+\lambda)}\phi(\xi), \qquad \xi = \frac{x^{1/(1+\lambda)}}{t^{1/2}}, \tag{1.1}$$

of the nonlinear diffusion equation with power law diffusivity, namely

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(c^m \frac{\partial c}{\partial x}\right), \tag{1.2}$$

where $c(x, t)$ denotes concentration which is a function of position and time and where λ denotes an arbitrary constant such that $\lambda \neq -1$. In [1], for a number of situations, some new first integrals and a new "source-dipole" solution applicable to $m = -4/3$ are obtained. In particular, the second order nonlinear ordinary differential equation for $\phi(\xi)$ which results from (1.1) and (1.2), is shown to admit first integrals for two values of λ, namely

$$\lambda = -\frac{m}{m+2}, \qquad \lambda = -\frac{m}{m+1}, \tag{1.3}$$

and these integrals are respectively

$$\frac{\phi^m\phi'}{\xi} - \frac{2\phi^{m+1}}{(m+2)\xi^2} + \frac{2\phi}{(m+2)^2} = C_1, \tag{1.4}$$

$$\frac{\phi^m\phi'}{\xi} - \frac{(2m+3)\phi^{m+1}}{(m+1)^2\xi^2} + \frac{\phi}{2(m+1)^2} = C_1, \tag{1.5}$$

where primes denote differentiation with respect to ξ and C_1, C_2 etc. are used throughout to denote constants of integration. For C_1 zero, the well known source and dipole solutions arise from (1.4) and (1.5) respectively. The purpose of this paper is to obtain a number of new exact solutions of equations such as (1.4) and (1.5) with C_1 non-zero for special values of m. The reader is referred to [1] for a brief survey of known exact solutions for nonlinear diffusion and also for an extensive list of references to the literature. As also noted in [1], exact solutions of equations such as (1.2) are important because by making use of comparison theorems, every solution of (1.2) is a possible bound for every other solution. Moreover, exact solutions may be useful to check the accuracy of a numerical solution scheme and may be important in identifying unusual and perhaps counterintuitive physical phenomena such as the "waiting-time" phenomenon extensively discussed by Lacey, Ockendon and Tayler [3].

The general nonlinear and inhomogeneous diffusion equation

$$\frac{\partial c}{\partial t} = x^l \frac{\partial}{\partial x}\left(c^m x^n \frac{\partial c}{\partial x}\right), \tag{1.6}$$

is also considered in [1] and it is shown that the similarity solution

$$c(x, t) = x^{A/m}\phi(\xi), \qquad \xi = \frac{x^{1/(1+\lambda)}}{t^{1/2}}, \tag{1.7}$$

where the constant A is defined by

$$A = \frac{2\lambda}{1+\lambda} - (n+l), \tag{1.8}$$

admits the following first integrals

$$\frac{\phi^m\phi'}{\xi} - \frac{2(1-l)\phi^{m+1}}{[m+2-l(m+1)-n]\xi^2} + \frac{(1+\lambda)^2}{2}\phi = C_1, \tag{1.9}$$

$$\frac{\phi^m\phi'}{\xi} - \frac{2[(m+1)(l+n-2)+n-1]\phi^{m+1}}{(m+1)^2(l+n-2)\xi^2} + \frac{(1+\lambda)^2}{2}\phi = C_1, \tag{1.10}$$

which correspond respectively to the two values of λ,

$$\lambda = \frac{l(m+1)+n-m}{m+2-l(m+1)-n}, \qquad \lambda = \frac{2m-(l+n)(m+1)}{(m+1)(n+l-2)}. \tag{1.11}$$

In this paper we also present a number of new exact solutions of (1.9) and (1.10) for C_1 non-zero.

We observe that all the first-order ordinary differential equations listed above take the form

$$\frac{\phi^m\phi'}{\xi} - A_1\frac{\phi^{m+1}}{\xi^2} + B_1\phi = C_1, \tag{1.12}$$

for various constants A_1 and B_1. If C_1 is zero then we may readily deduce the generalised source-dipole solution

$$\phi(\xi) = \left(C_2\xi^{mA_1} - \frac{mB_1\xi^2}{2 - mA_1}\right)^{1/m}, \tag{1.13}$$

provided that $mA_1 \neq 2$. In the following section for C_1 non-zero we describe in general terms how special exact solutions of (1.12) can be obtained, either for special values of the index m or the constant A_1 or both. In the subsequent two sections we provide further details of the solutions appropriate to equations (1.2) and (1.6) respectively and for (1.2) we illustrate a number of solutions graphically in the final section of the paper. In Section 5 we make one or two general remarks relating to similarity solutions (1.1) of the nonlinear diffusion equation (1.2) with power law diffusivity and we present a new exact solution not emanating from the solution devices listed in the following section.

2. General methods for obtaining special exact solutions

In this section, with reference to the general equation (1.12) with C_1 non-zero, we detail five special exact solution procedures and then in the following two sections we give the precise details which apply to (1.2) and (1.6).

(i) Solution for $m = -1/2$ via a Riccati equation

If $m = -1/2$ then the substitution $\psi = \phi^{-1/2}$ transforms (1.12) into the Riccati equation

$$\psi' = \frac{B_1\xi}{2} - \frac{A_1}{2\xi}\psi - \frac{C_1\xi}{2}\psi^2, \tag{2.1}$$

and the further substitution $\psi = 2u'/C_1\xi u$ yields

$$u'' + \frac{(A_1 - 2)u'}{2\xi} - \frac{B_1C_1}{4}\xi^2 u = 0. \tag{2.2}$$

In the usual way, this equation is reduced to a standard Bessel equation by the transformation

$$u(\xi) = \xi^\alpha v(\beta\xi^2), \qquad \alpha = 1 - A_1/4, \qquad \beta = |B_1C_1|^{1/2}/4, \tag{2.3}$$

and there are two cases to consider. If $B_1C_1 < 0$ then

$$v(z) = C_3J_{\alpha/2}(z) + C_4Y_{\alpha/2}(z), \tag{2.4}$$

while if $B_1C_1 > 0$ we have

$$v(z) = C_3I_{\alpha/2}(z) + C_4K_{\alpha/2}(z), \tag{2.5}$$

where in both cases z denotes $\beta\xi^2$ and the conventional terminology is adopted for the

Bessel functions. Thus altogether the exact solution for (1.12) for $m = -1/2$ becomes

$$\frac{1}{\phi^{1/2}} = \frac{4\beta}{C_1}\left(\frac{\alpha}{2z} + \frac{1}{v}\frac{dv}{dz}\right), \qquad z = \beta\xi^2, \tag{2.6}$$

and we note that the ratio dv/dz divided by v involves only one arbitrary constant $C_2 = C_4/C_3$.

(ii) Solution for $m = -1$ via a Bernoulli equation

If $m = -1$ then (1.12) immediately gives the Bernoulli equation

$$\phi' - \left(C_1\xi + \frac{A_1}{\xi}\right)\phi = -B_1\xi\phi^2, \tag{2.7}$$

which with the substitution $\psi = \phi^{-1}$ becomes

$$\psi' + \left(C_1\xi + \frac{A_1}{\xi}\right)\psi = B_1\xi, \tag{2.8}$$

which integrates to give

$$\frac{1}{\phi} = \xi^{-A_1} e^{-C_1\xi^2/2}\left\{B_1\int^{\xi}\eta^{A_1+1} e^{C_1\eta^2/2}\, d\eta + C_2\right\}. \tag{2.9}$$

This is the required general solution for (1.12) for $m = -1$ and for integer values of A_1, the integral can be evaluated explicitly.

(iii) Solution for A_1 zero from separation of variables

If A_1 is zero then trivially from (1.12) we have

$$\frac{\xi^2}{2} = \int^{\phi(\xi)} \frac{\eta^m\, d\eta}{(C_1 - B_1\eta)} + C_2, \tag{2.10}$$

and again depending on the value of m, this integral can be further simplified.

(iv) Solution for $A_1 = -1$ and $m = -2$ from an Abel equation

If $A_1 = -1$ and $m = -2$ then (1.12) becomes an Abel equation of the first kind, namely

$$\phi' = -\frac{\phi}{\xi} + C_1\xi\phi^2 - B_1\xi\phi^3, \tag{2.11}$$

and the substitution $\psi = (\phi\xi)^{-1}$ gives the Abel equation of the second kind,

$$\psi\psi' + C_1\psi = \frac{B_1}{\xi}, \tag{2.12}$$

which happens to be solvable by the transformation $\Psi = \psi + C_1\xi$. From (2.12) and this transformation we can readily deduce the Bernoulli equation

$$\frac{d\xi}{d\Psi} - \frac{\Psi\xi}{B_1} = -\frac{C_1}{B_1}\xi^2 , \tag{2.13}$$

which has the general solution

$$\frac{1}{\xi} = \frac{e^{-\Psi^2/2B_1}}{B_1}\left\{C_1 \int^{\Psi} e^{\eta^2/2B_1}\, d\eta + C_2\right\} , \tag{2.14}$$

which together with

$$\frac{1}{\phi} = \Psi\xi - C_1\xi^2 , \tag{2.15}$$

constitutes the general solution in parametric form of (1.12) with $A_1 = -1$ and $m = -2$.

(*v*) *Solution for $A_1 = 1$ and $m = 1$ from an Abel equation*

This solution procedure applies only for equations (1.9) and (1.10) resulting from the general nonlinear diffusion equation (1.6), because for both (1.4) and (1.5), A_1 is distinct from unity when $m = 1$. From (1.12) with $A_1 = m = 1$ we have on setting $\psi = \phi/\xi$, the Abel equation of the second kind

$$\psi' - \frac{C_1}{\xi\psi} = -B_1 , \tag{2.16}$$

which we solve by means of the transformation $\Psi = \psi + B_1\xi$ to obtain the Bernoulli equation

$$\frac{d\xi}{d\Psi} - \frac{\Psi\xi}{C_1} = -\frac{B_1\xi^2}{C_1} . \tag{2.17}$$

On solving this equation in the usual way we obtain

$$\frac{1}{\xi} = \frac{e^{-\Psi^2/2C_1}}{C_1}\left\{B_1 \int^{\Psi} e^{\eta^2/2C_1}\, d\eta + C_2\right\} , \tag{2.18}$$

which together with

$$\phi = \Psi\xi - B_1\xi^2 , \tag{2.19}$$

constitutes the general solution, in parametric form, of (1.12) with $A_1 = m = 1$.

3. New solutions for nonlinear diffusion

For definiteness we assume throughout that the arbitrary constant C_1 appearing in (1.4) and (1.5) is positive. Similar formulae can readily be obtained for the case when C_1 is negative.

(i) Solutions for $m = -1/2$

For equation (1.4) with $m = -1/2$ we have $A_1 = 4/3$ and $B_1 = 8/9$ which gives

$$\alpha = \frac{2}{3}, \qquad \beta = \frac{1}{3}\left(\frac{C_1}{2}\right)^{1/2}, \tag{3.1}$$

and therefore the solution given by (2.5) and (2.6) becomes

$$\frac{1}{\phi^{1/2}} = \frac{2}{3}\left(\frac{2}{C_1}\right)^{1/2}\left\{\frac{I'_{1/3}(z) + C_2 I'_{-1/3}(z)}{I_{1/3}(z) + C_2 I_{-1/3}(z)} + \frac{1}{3z}\right\}, \qquad z = \frac{\xi^2}{3}\left(\frac{C_1}{2}\right)^{1/2}, \tag{3.2}$$

where the prime here denotes differentiation with respect to z. Equation (3.2) constitutes the general solution for (1.1) with $m = -1/2$ and $\lambda = 1/3$ and the concentration is given by

$$c(x, t) = \frac{\phi(\xi)}{x}, \qquad \xi = \frac{x^{3/4}}{t^{1/2}}. \tag{3.3}$$

Concentration curves for this solution are shown in Fig. 8 for $C_1 = 1$ and C_2 zero.

For equation (1.5) with $m = -1/2$ we have $A_1 = 8$ and $B_1 = 2$ so that

$$\alpha = -1, \qquad \beta = \frac{1}{2}\left(\frac{C_1}{2}\right)^{1/2}, \tag{3.4}$$

and in this case the solution given by (2.5) and (2.6) becomes

$$\frac{1}{\phi^{1/2}} = \left(\frac{2}{C_1}\right)^{1/2}\left\{\frac{\sinh z + C_2 \cosh z}{\cosh z + C_2 \sinh z} - \frac{1}{z}\right\}, \qquad z = \frac{\xi^2}{2}\left(\frac{C_1}{2}\right)^{1/2}, \tag{3.5}$$

and the concentration (1.1) for $m = -1/2$ and $\lambda = 1$ yields

$$c(x, t) = \frac{\phi(\xi)}{x^2}, \qquad \xi = \left(\frac{x}{t}\right)^{1/2}. \tag{3.6}$$

we observe that for t tending to zero and infinity, ξ tends to infinity and zero respectively and we have

$$\phi(\infty) = C_1/2, \qquad \phi(0) = 0. \tag{3.7}$$

(ii) Solution for $m = -1$

For the nonlinear diffusion equation (1.2) giving rise to (1.4) and (1.5), only (1.4) is meaningful for $m = -1$ and in this case we have $A_1 = B_1 = 2$ and on performing the integration in (2.9) the similarity solution (1.1) with $m = -1$ and $\lambda = 1$ becomes

$$c(x, t) = \left\{\frac{2}{C_1}\, x - \frac{4}{C_1^2}\, t + C_2 t\, e^{-C_1 x/2t}\right\}^{-1}. \tag{3.8}$$

This solution is shown graphically in Figs. 6 and 7 with $C_2 = 2$ and $C_1 = 2$ and $C_1 = -2$ respectively.

(iii) Solution for $m = -3/2$

For (1.4) and (1.5) the coefficient A_1 can only be zero for equation (1.5) in which case $m = -3/2$ and we have $B_1 = 2$ and (2.10) becomes

$$\frac{\xi^2}{2} = \int^{\phi(\xi)} \frac{d\eta}{\eta^{3/2}(C_1 - 2\eta)} + C_2 , \tag{3.9}$$

which with the substitution $\eta = C_1\tau^2/2$ yields

$$\frac{\xi^2}{2} = \left(\frac{2}{C_1}\right)^{3/2} \int^{(2\phi(\xi)/C_1)^{1/2}} \frac{d\tau}{\tau^2(1-\tau^2)} + C_2 . \tag{3.10}$$

On integration we obtain

$$\frac{\xi^2}{2} + \frac{2}{C_1}\frac{1}{\phi^{1/2}} = \left(\frac{2}{C_1}\right)^{3/2} \log\left(\frac{1+(2\phi/C_1)^{1/2}}{1-(2\phi/C_1)^{1/2}}\right)^{1/2} + C_2 , \tag{3.11}$$

where in this case the concentration given by (1.1) becomes

$$c(x, t) = \frac{\phi(\xi)}{x^2} , \qquad \xi = (xt)^{-1/2} . \tag{3.12}$$

(iv) Solution for $m = -2$

Only (1.5) is meaningful for $m = -2$ and in this case we have $A_1 = -1$ and $B_1 = 1/2$ so that the parametric solution (2.14) and (2.15) becomes

$$\xi = \frac{e^{\Psi^2}}{2}\left\{C_1 \int^{\Psi} e^{\eta^2}\, d\eta + C_2\right\}^{-1} , \qquad \phi = (\Psi\xi - C_1\xi^2)^{-1} , \tag{3.13}$$

while in this case the concentration becomes

$$c(x, t) = \frac{\phi(\xi)}{x^2} , \qquad \xi = \frac{1}{xt^{1/2}} . \tag{3.14}$$

4. New solutions for general nonlinear diffusion

In this section, with reference to (1.9) and (1.10) arising from the general nonlinear diffusion equation (1.6), we detail various new exact solutions, assuming that C_1 in (1.9) and (1.10) is non-zero. We comment that only the essential details are noted and that in general we leave all the various special solutions which follow from our results for the interested reader to evaluate explicitly for themselves.

(i) Solutions for $m = -1/2$

For $m = -1/2$ the two values of λ given by (1.11) become

$$\lambda = \frac{1 + l + 2n}{3 - l - 2n} , \qquad \lambda = \frac{2 + l + n}{2 - l - n} , \tag{4.1}$$

and the corresponding values of A_1 and B_1 associated with (1.9) and (1.10) become respectively

$$A_1 = \frac{4(1-l)}{3-l-2n}\,, \qquad B_1 = \frac{8}{(3-l-2n)^2}\,,$$
$$A_1 = \frac{4(4-l-3n)}{2-l-n}\,, \qquad B_1 = \frac{8}{(2-l-n)^2}\,. \tag{4.2}$$

From these equations and (2.3) we may deduce the corresponding values of α and β which apply in the general solution (2.6). These are respectively

$$\alpha = \frac{2(l-n)}{3-l-2n}\,, \qquad \beta = \left(\frac{C_1}{2}\right)^{1/2} \frac{1}{3-l-2n}\,,$$
$$\alpha = \frac{2(n-1)}{2-l-n}\,, \qquad \beta = \left(\frac{C_1}{2}\right)^{1/2} \frac{1}{2-l-n}\,. \tag{4.3}$$

Thus for these values of α and β, new exact solutions of (1.6) are given by (1.7) and (2.6) where $v(z)$ is an expression of the form (2.5), assuming that C_1 is positive.

(ii) Solution for $m = -1$

For $m = -1$, only equation (1.9) is meaningful and in this case we have $\lambda = (1+n)/(1-n)$ while A_1 and B_1 are given by

$$A_1 = 2\,\frac{1-l}{1-n}\,, \qquad B_1 = \frac{2}{(1-n)^2}\,. \tag{4.4}$$

Directly from the solution (2.9) with these values for A_1 and B_1 we obtain

$$\phi(\xi) = \xi^{2\rho}\, e^{C_1\xi^2/2} \left\{ \frac{1}{(1-n)^2} \int^{\xi^2} \tau^{\rho}\, e^{C_1\tau/2}\, d\tau + C_2 \right\}^{-1}, \tag{4.5}$$

where $\rho = (1-l)/(1-n)$. If ρ is an integer then the integral in (4.5) can be obtained explicitly. For example when $\rho = 0$ (namely $l = 1$) we have

$$\phi(\xi) = \left\{ \frac{2}{C_1(1-n)^2} + C_2\, e^{-C_1\xi^2/2} \right\}^{-1}. \tag{4.6}$$

(iii) Solutions for A_1 zero

We observe that (1.9) and (1.10) have A_1 zero if $l = 1$ and $(m+1)(l+n-2) = 1-n$ respectively and in which case the corresponding values of λ given by (1.11) become

$$\lambda = \frac{1+n}{1-n}\,, \qquad \lambda = -\frac{3-n}{1-n}\,. \tag{4.7}$$

Interestingly enough both values give rise to the same value of B_1, namely

$$B_1 = 2/(1-n)^2\,, \tag{4.8}$$

and therefore the appropriate solutions of both (1.9) and (1.10) are given by (2.10) with B_1 given by (4.8).

(iv) Solutions for $A_1 = -1$ and $m = -2$

For $m = -2$ we find that (1.9) and (1.10) have $A_1 = -1$ provided that $l = 2 - n$ and $l = n$ respectively and that the corresponding values of λ given by (1.11) become

$$\lambda = \frac{n}{1-n}, \qquad \lambda = -\frac{2-n}{1-n}, \tag{4.9}$$

and again for both (1.9) and (1.10) we obtain the same value of B_1, which is

$$B_1 = 1/2(1-n)^2 . \tag{4.10}$$

Thus for $m = -2$ and $l = 2 - n$ for (1.9) and $l = n$ for (1.10) the solutions, in parametric form, for both equations are given by (2.14) and (2.15) with the constant B_1 given by (4.10).

(v) Solutions for $A_1 = 1$ and $m = 1$

For $m = 1$ we find that both (1.9) and (1.10) have $A_1 = 1$ only if $n = 1$ and in this case the two values of λ given by (1.11) coincide and we have

$$\lambda = \frac{l}{1-l}, \qquad B_1 = \frac{1}{2(1-l)^2}, \tag{4.11}$$

so that the solution to both (1.9) and (1.10) is given in parametric form by (2.18) and (2.19) with B_1 given by $(4.11)_2$.

5. General remarks and ad-hoc new solution

In this section we make one or two general observations relating to similarity solutions (1.1) of equation (1.2), which result in new travelling wave solutions applicable to $m = -1/2$. From the general second-order nonlinear ordinary differential equation for $\phi(\xi)$ applying to arbitrary λ and which is given explicitly in [1], we can from the substitution $z = \phi^m$ readily deduce

$$zz'' + \frac{z'^2}{m} + \lambda\left(\frac{4}{m} + 3\right)\frac{zz'}{\xi} + 2\lambda\left(\frac{2\lambda}{m} + \lambda - 1\right)\frac{z^2}{\xi^2} + \frac{(\lambda+1)^2}{2}\,\xi z' = 0\,, \tag{5.1}$$

where as usual primes denote differentiation with respect to ξ. Because this equation remains invariant under the one-parameter group

$$\xi_1 = e^{\varepsilon}\xi\,, \qquad z_1 = e^{2\varepsilon}z\,, \tag{5.2}$$

we make the transformation $g = z/\xi^2$ and following Jones [2] and Lacey, Ockendon and Tayler [3] we introduce $h = z'/\xi$ so that we have

$$z' = \xi h = \xi^2 g' + 2\xi g\,, \tag{5.3}$$

and therefore

$$\log \xi = \int \frac{\mathrm{d}g}{h(g) - 2g} + \text{constant} . \tag{5.4}$$

Further, on using

$$z'' = \xi h' + h = (h - 2g) \frac{\mathrm{d}h}{\mathrm{d}g} + h , \tag{5.5}$$

it is not difficult to show that (5.1) becomes

$$\frac{\mathrm{d}h}{\mathrm{d}g} = \frac{\dfrac{h^2}{m} + \left(\dfrac{4\lambda}{m} + 3\lambda + 1\right) gh + 2\lambda\left(\dfrac{2\lambda}{m} + \lambda - 1\right) g^2 + \dfrac{(\lambda + 1)^2}{2}\, h}{g(2g - h)} . \tag{5.6}$$

By postulating simple forms for $h(g)$ it is possible to deduce exact solutions. For example suppose that $h(g) = a$ where a is a constant then we find that this is a valid solution of (5.6) provided $m = -3/2$, $\lambda = -3$ and $a = 3$ and in which case equation (5.4) and $z = \xi^2 g$ yield

$$z = \frac{3}{2}\, \xi^2 + \text{constant} , \tag{5.7}$$

which is just the dipole solution for $m = -3/2$. Similarly, if we assume $h(g) = a + bg$ where a and b are constants then we find from (5.6)

$$a = -\frac{m(\lambda + 1)^2}{2} , \qquad b = -\frac{ma}{(m + 1)} , \tag{5.8}$$

while λ must take on either of the values (1.3). Thus in this case, this particular assumed form simply generates the standard source and dipole solutions. In order to deduce a new solution by this process we assume $h(g) = ag^n$ and from (5.6) we readily obtain $a = 12$ and $n = 2$ and valid only for $m = -1/2$ and $\lambda = -3/5$. From (5.4) we have

$$g = (6 - C\xi^2)^{-1} , \tag{5.9}$$

where C is a constant while from $\phi^m = \xi^2 g$ we obtain

$$\phi(\xi) = (6/\xi^2 - C)^2 , \tag{5.10}$$

and finally from (1.1) with $m = -1/2$ and $\lambda = -3/5$ we may deduce

$$c(x, t) = \frac{(6t - Cx^5)^2}{x^4} . \tag{5.11}$$

This new travelling wave solution with a variable speed can be further generalized by directly looking for a solution of (1.2) of the form

$$c(x, t) = F(x)[t - G(x)]^n . \tag{5.12}$$

From (1.2) and (5.12) we may deduce

$$nF\zeta^{n-1} = (F^m F')'\zeta^{(m+1)n} - n[F^{m+1}G'' + 2(m+1)F^m F' G']\zeta^{(m+1)n-1} + n[(m+1)n - 1]F^{m+1}G'^2\zeta^{(m+1)n-2}, \tag{5.13}$$

where $\zeta = t - G(x)$ and primes here denote differentiation with respect to x. From (5.13) it is apparent that $n = (m+1)^{-1}$ and $n = 2$ so we require $m = -1/2$ and we have

$$(F^{-1/2}F')' = 2F, \qquad G' = C_1/F, \tag{5.14}$$

where C_1 is a constant. On introducing $H = F^{1/2}$, $(5.14)_1$ and its first integral become

$$H'' = H^2, \qquad H'^2 = \frac{2}{3}H^3 + C_2, \tag{5.15}$$

and therefore

$$\int^{H(x)} \frac{\mathrm{d}H}{(2H^3/3 + C_2)^{1/2}} = \pm(x - x_0), \tag{5.16}$$

where x_0 is a constant and the integral can, if necessary, be expressed in terms of elliptic functions. The solution (5.11) emerges from $(5.15)_2$ by taking C_2 zero, thus

$$H' = \pm\left(\frac{2}{3}\right)^{1/2} H^{3/2}, \tag{5.17}$$

from which we may readily deduce

$$F(x) = \frac{36}{(x - x_0)^4}, \qquad G(x) = \frac{C_1(x - x_0)^5}{180} + C_3, \tag{5.18}$$

which can be reconciled with (5.11) in a straightforward manner.

6. Numerical results

In this section we briefly illustrate concentration profiles for some of the new solutions. First, however, we note that all the new solutions obtained apply for negative m and in general the behaviour of solutions of (1.2) for m positive and negative is quite distinct. To illustrate this point Figs. 1 and 2 show the source solution

$$c(x, t) = \frac{1}{x}\left(C\xi^{2m/(m+2)} - \frac{m\xi^2}{2(m+2)}\right)^{1/m}, \tag{6.1}$$

where $\xi = x^{(m+2)/2}/t^{1/2}$, for $C = 1$ and $m = 1$ and $m = -1$ respectively. Clearly for $m = 1$ and in general m positive there is a free boundary moving with finite velocity while for $m = -1$ there is an instantaneous spread of concentration and no free boundary. This behaviour occurs only for $-2 < m < 0$ since the solution is not defined for $m = -2$ and has singularities

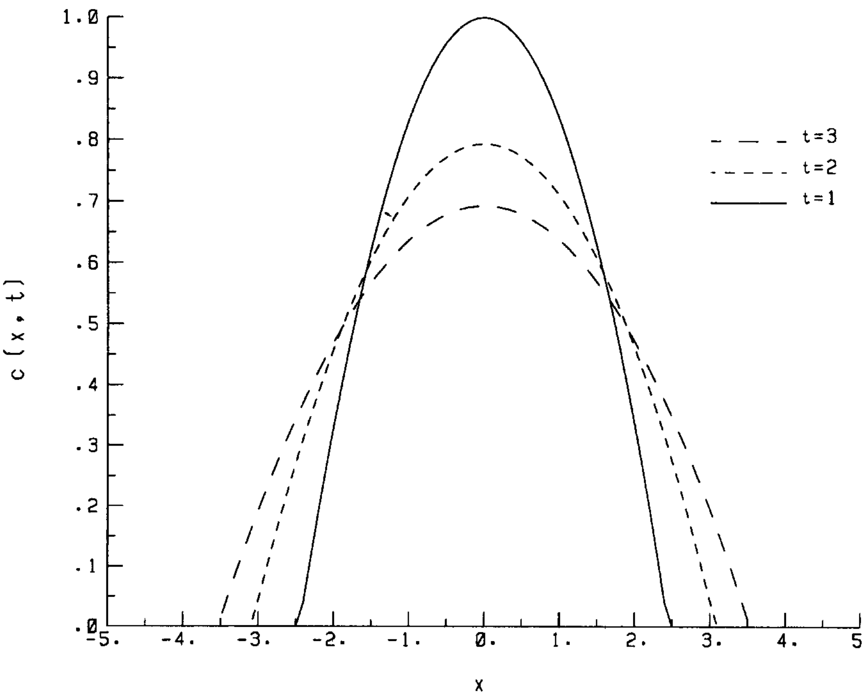

Fig. 1. Behaviour of classical source solution (equation (6.1)) for $C = 1$ and $m = 1$.

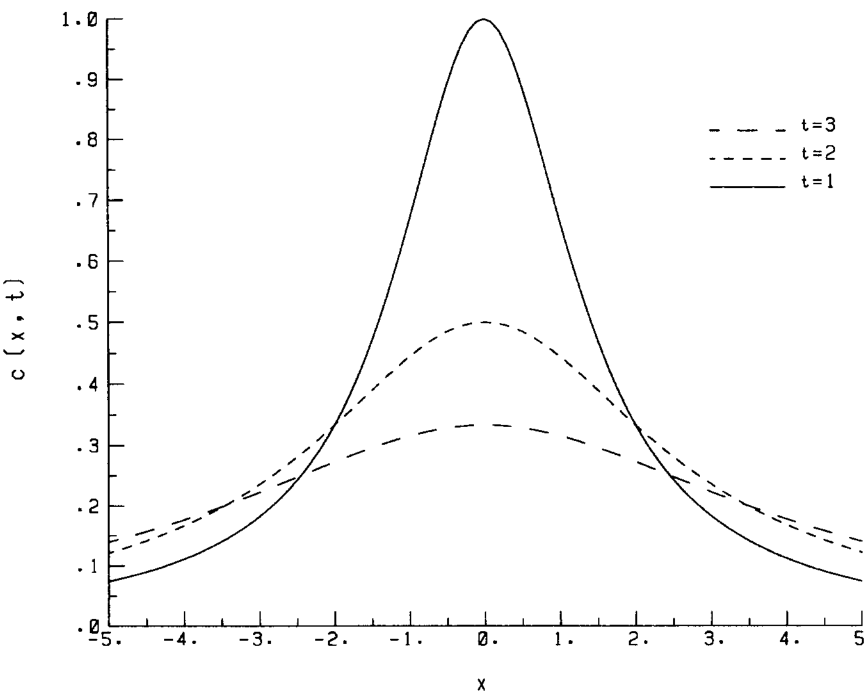

Fig. 2. Behaviour of classical source solution (equation (6.1)) for $C = 1$ and $m = -1$.

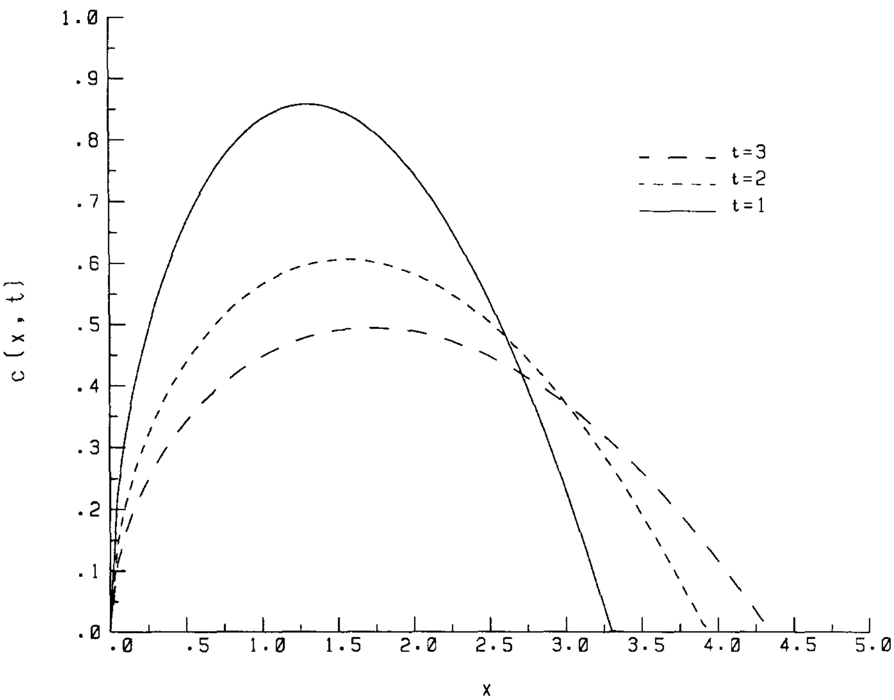

Fig. 3. Behaviour of classical dipole solution (equation (6.2)) for $C = 1$ and $m = 1$.

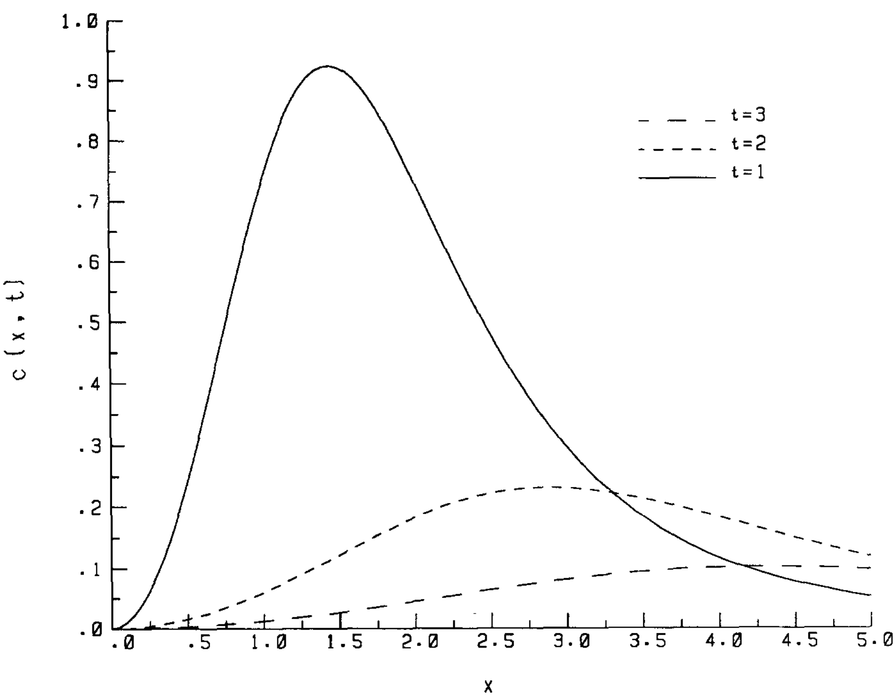

Fig. 4. Behaviour of classical dipole solution (equation (6.2)) for $C = 1$ and $m = -1/2$.

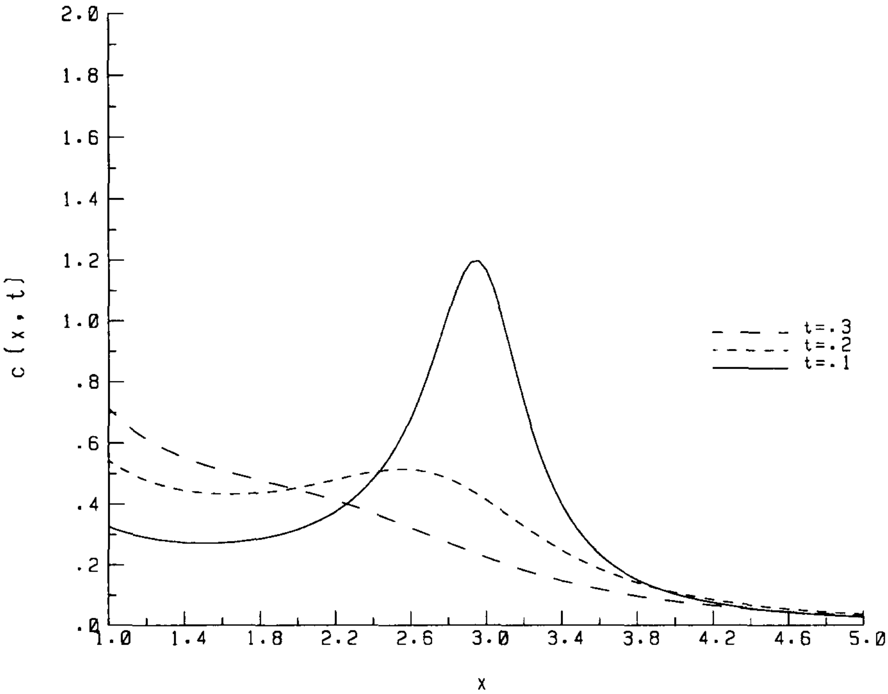

Fig. 5. Behaviour of "source-dipole" solution (equation (6.3)) for $C = 1$ and applying for $m = -4/3$

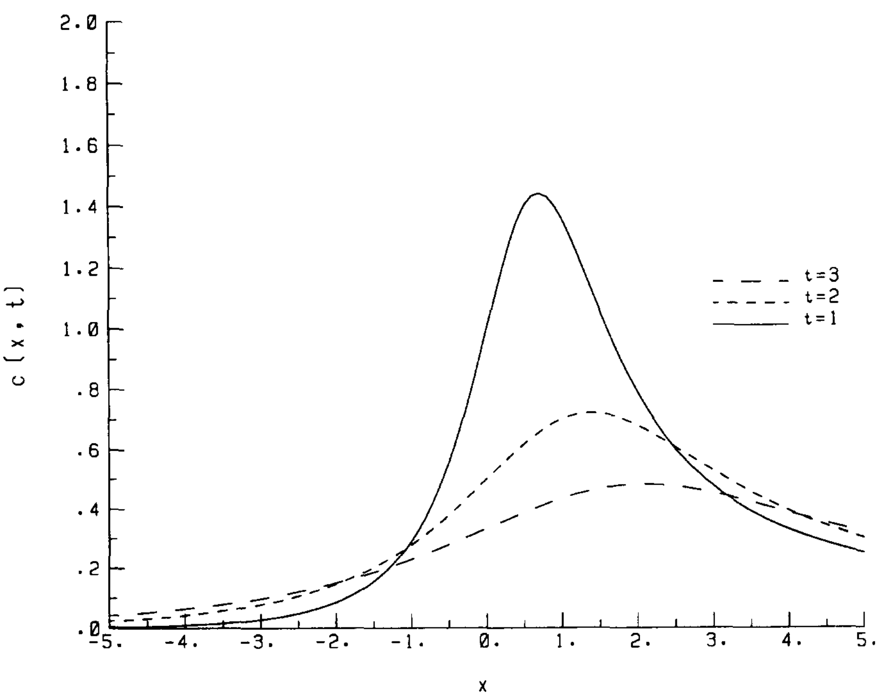

Fig. 6. Behaviour of new solution (equation (3.8)) for $C_1 = C_2 = 2$ and applying for $m = -1$.

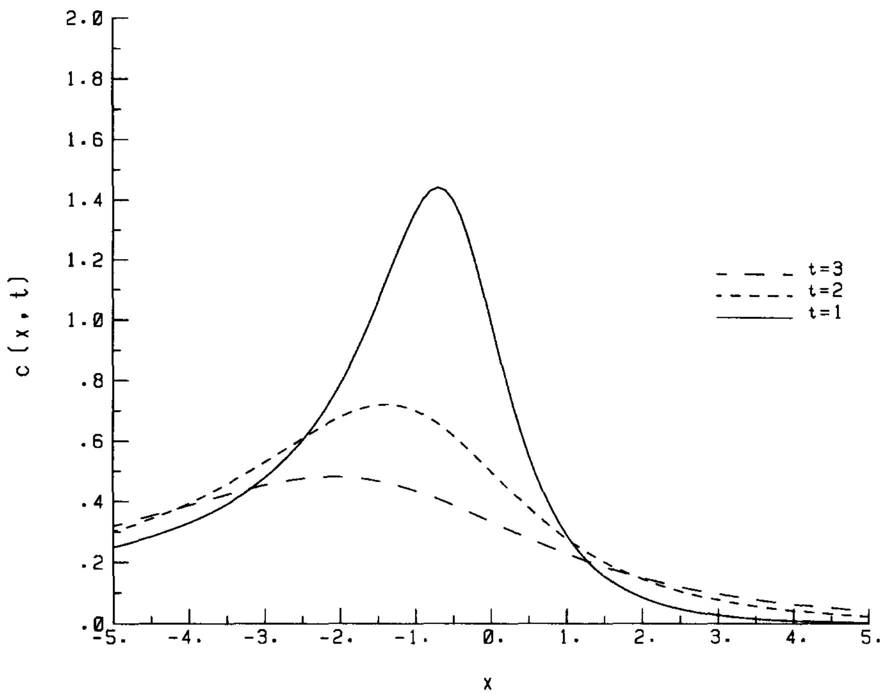

Fig. 7. Behaviour of new solution (equation (3.8)) for $C_1 = -2$ and $C_2 = 2$ and applying for $m = -1$.

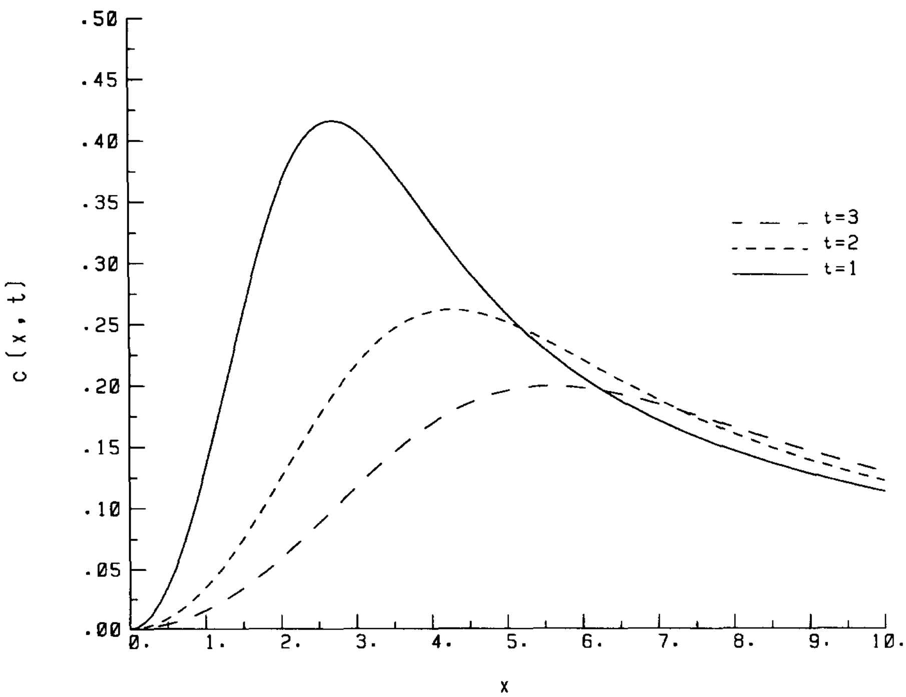

Fig. 8. Behaviour of new solution (equation (3.2)) for $C_1 = 1$ and C_2 zero and appıying for $m = -1/2$.

for $m < -2$. Similarly Figs. 3 and 4 show the dipole solution

$$c(x, t) = \frac{1}{x^2}\left(C\xi^{m(3+2m)/(m+1)^2} - \frac{m\xi^2}{2(m+2)}\right)^{1/m}, \tag{6.2}$$

where $\xi = x^{(m+1)}/t^{1/2}$, for $C = 1$ and $m = 1$ and $m = -1/2$ respectively. Similar comments apply here except that the behaviour shown in Fig. 4 for $m = -1/2$ generally applies for m such that $-1 < m < 0$. The new solution derived in [1] and applying to $m = -4/3$ is shown in Fig. 5. This solution is a combined "source-dipole" solution and the various values for constants used in Fig. 5 are chosen to locate the dipole at the origin and the source at $x = 3$. In the notation of [1] we have $\alpha = 1$, $\mu = 1$ and $\lambda = -4$ so that $x_1 = 3$ and $x_2 = 0$ and the new solution becomes

$$c(x, t) = \frac{1}{x^{3/2}}\left(\frac{9t}{(x-3)^2 + 9Ct^3x^2}\right)^{3/4}, \tag{6.3}$$

and the curves in Fig. 5 are for $C = 1$.

The new solution (3.8) which applies for $m = -1$ is shown in Figs. 6 and 7 for $C_2 = 2$ and $C_1 = 2$ and $C_1 = -2$ respectively. This new solution of (1.4) applies for $\lambda = -m/(m+2)$ which is also the value of λ appropriate to the source solution (6.1). It is therefore not surprising that the new solution has source solution like characteristics, except that the maximum concentration is not fixed at the origin but moves with time such that if C_1 is positive, it moves to the right while if C_1 is negative, it moves to the left. This behaviour is shown in Figs. 6 and 7. We note that for the new solution (3.8) the problem of singularities is avoided provided the constants C_1 and C_2 are such that $C_1^2C_2 \geqslant 4$. The new solution (3.2) which applies for $m = -1/2$ is shown in Fig. 8 for $C_1 = 1$ and C_2 zero. This solution also originates from (1.4) and exhibits source like behaviour away from the origin and has the property that the concentration is zero at the origin for all the time.

Acknowledgement

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