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Exact Analytical Solutions of Nonlinear Problems of Tsunami Wave Run-up on Slopes with Different Profiles

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Abstract. A review of papers investigating tsunami wave run-up on a beach is given and the control parameters of the problem are revealed. There are two such parameters in the case of ideal fluid: the bottom sloping angle and the breaking parameter. A stage-by-stage approach for finding run-up characteristics is formulated: the linear calculation of shoreline oscillations and the subsequent nonlinear transformation of the solution according to the Riemann method. Solution of the nonone-dimensional problems of wave run-up on a beach in the linear formulation is obtained.

Key words: Tsunami, shallow-water theory, wave run-up.

Notation

u	horizontal particle velocity
ω	vertical particle velocity
ρ	fluid density
P	pressure
t	time
η	surface displacement
g	gravity acceleration
x	distance from a shoreline
z	vertical axis
α	sloping angle to ocean surface; for small angle $\alpha = tg\alpha$;
$\tilde{t}$	dimensionless time
$\tilde{u}$	dimensionless velocity
$\tilde{\eta}$	dimensionless displacement
$\tilde{x}$	dimensionless distance

Restoration of Characteristics from Coastal Records

ω	frequency of incident wave
R	maximum displacement at shoreline
$\Phi(\sigma, \lambda)$	wave function
T	tsunami period
Br	breaking parameter
Br_*	critical breaking parameter
H_0	amplitude of a wave
λ_0	wave length

L	distance from the shoreline
h_0	basin depth at the distance L from the shoreline
φ	dimensionless amplitude spectra
δ	dimensionless phase spectra
θ	parameter
$\eta_{\text{refl.}}$	reflected wave
$\eta_{\text{inc.}}$	incident wave
$J_0(\nu)$	zero-index Bessel function
L_*	length of the shelf
$C_\pm$	characteristics
$V_\pm$	Riemann invariants

1. Introduction

When taking tsunami countermeasures much attention is paid to tsunami hazard maps specifying the most probable flooding zones. These flooding zones are calculated differently in different schemes: either using empirical formulae or the linear theory formulae. The validation and application of these formulae has, for a long time, been an open problem. Meanwhile, Carrier and Greenspan (1958) proposed an adequate method for solving the run-up problems within the framework of the nonlinear shallow-water theory. The complexity and implicitness of the transformations used by Carrier (1958) prevented the wide use of this method (some solutions can be found in papers by Carrier (1958) and Spielfogel (1976). Besides, in 1968, Le Mehaute, Koh and Hwang said in their review (Le Mehaute *et al.* (1968)) that the basic contribution in Carrier and Greenspan's paper is, rather than the run-up calculation, the demonstration that, in the nonlinear long-wave approximation, there are elevation waves propagating without breaking on a permanent sloping beach. In 1961, Keller, and in 1983 Mei, noted that solving the linear and nonlinear problems in the case of a monochomatic wave leads to the same result. The later analytical attempts to solve approximately the long-wave run-up problem were connected with the Langrangian formulation of the nonlinear shallow-water theory (Shuto, 1972; Goto and Shuto, 1978; Goto, 1979, 1984). During the last decade, there has been progress in solving the 'Eulerian' problems of wave run-up, not only in one dimension but also in two dimensions (the latter had never been considered before). As a result, control parameters of the problem have been found, new algorithms for the moving shoreline dynamics investigation have been formulated, the influence of the shape of the tsunami wave approaching a beach on the run-up characteristics elucidated, and the water elevation height on a beach in bays of various geometries calculated, (Pelinovsky, 1982; Mazova, 1984, 1985, 1988). These results are summarized in the present paper.

2. Basic Approximations

Let us consider first the classical formulation of the wave run-up problem within the ideal fluid model:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0,$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial P}{\partial z} = -g, \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

with the boundary condition on the bottom ($z = -h(x)$):

$$w + u\frac{\mathrm{d}h}{\mathrm{d}x} = 0 \tag{2}$$

and on the free surface ($z = \eta$)

$$w = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x}, \qquad P = P_{\mathrm{atm}}. \tag{3}$$

Here u is the horizontal and w is the vertical component of flow velocity, ρ is the fluid density, P is the pressure (P_{atm} is constant atmospheric pressure), g is the gravity acceleration, z is the vertical axis, and $h(x)$ is the basin depth counted from the unperturbed surface. Let the slope be a plane one: $h = -\alpha x$. Introducing characteristic physical parameters of the tsunami wave such as the height R and the frequency ω (in the case of pulse perturbation, the tsunami duration is ω^{-1}), we reduce the initial equations by disdimensionalization of arguments and functions:

$$\eta = \frac{\eta}{R}, \qquad \tilde{u} = \frac{\alpha u}{\omega R}, \qquad \tilde{w} = \frac{w}{\omega R}, \qquad \tilde{t} = \omega t, \qquad \tilde{x} = \frac{\alpha x}{R},$$
$$\tilde{z} = \frac{z}{R}, \qquad P = P_{\mathrm{atm}} + \rho g(\eta - z) + \rho\omega^2 R^2 \tilde{P} \tag{4}$$

to (tildes are omitted):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\mathrm{Br}}\frac{\partial \eta}{\partial x} + \alpha^2\frac{\partial P}{\partial x} = 0, \tag{5}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{\partial P}{\partial z} = 0, \tag{6}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{7}$$

with the boundary conditions

$$w = u, \qquad (z = x), \tag{8}$$

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = 0, \qquad (z = \eta), \tag{9}$$

$$P = 0, \qquad (z = \eta). \tag{10}$$

These equations are defined by only two mathematical parameters: the sloping angle α and the breaking parameter

$$\mathrm{Br} = \frac{\omega^2 R}{g\alpha^2}. \tag{11}$$

As a rule, tsunami waves are long. Therefore, if the beach is relatively steep (α is not small), then the coastal zone has a width much less than the tsunami wave length and can be treated as a wall. The wave remains long everywhere and the tsunami run-up on a vertical wall can be investigated within the nonlinear shallow-water theory (we shall return to this problem below). The strongest amplification of the wave is possible on a mild slope. In this case, α is small and it is the basis for various asymptotic procedures to be performed. This parameter can be neglected in the first approximation. Equation (6) can then be omitted (Kaistrenko *et al.*, 1985) and the integration of Equation (7) using (8) and (9), leads to

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}[(-x + \eta)u] = 0 \tag{12}$$

which, together with (5) at $\alpha = 0$,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\mathrm{Br}}\frac{\partial \eta}{\partial x} = 0, \tag{13}$$

forms the well-known nonlinear system of shallow-water equations determined by a single dimensionless parameter Br. It is interesting to note that the system (12)–(13) contains only one dimensionless parameter Br. It is understood that other dimensionless parameters are introduced, then other parameters (possibly more than one) can appear in this system. In particular, these equations may have no parameters; in this case, however, a parameter, equivalent to Br, can appear in the initial condition (Kaistrenko *et al.*, 1985). We emphasize that the parameter Br was used in the analysis of the partial solutions of the shallow-water system (Goto, 1974) and of the experimental data (Battjes, 1974; Bowan 1977). In this particular case, this parameter is obtained from the dimensional analysis, thus indicating its fundamental importance.

3. Method of Solution

For the solution of system (12)–(13), it is efficient to use the tangential Legendre transformation (its use for the nonlinear system of shallow-water equations was first described by Carrier and Greenspan (1958));

$$u = \frac{1}{\mathrm{Br}}\frac{2}{\sigma}\frac{\partial \Phi}{\partial \sigma},$$

$$\eta = \frac{1}{\mathrm{Br}}\left[\frac{\partial \Phi}{\partial \lambda} - \frac{2}{\sigma^2}\left(\frac{\partial \Phi}{\partial \sigma}\right)^2\right],$$

$$x = \frac{1}{\mathrm{Br}}\left[\frac{\partial \Phi}{\partial \lambda} - \frac{\sigma^2}{4} - \frac{2}{\sigma^2}\left(\frac{\partial \Phi}{\partial \sigma}\right)^2\right], \tag{14}$$

$$t = \lambda - \frac{2}{\sigma}\frac{\partial \Phi}{\partial \sigma},$$

using which, system (12)–(13) reduces to a linear wave equation:

$$\frac{\partial^2 \Phi}{\partial \lambda^2} - \frac{\partial^2 \Phi}{\partial \sigma^2} - \frac{1}{\sigma}\frac{\partial \Phi}{\partial \sigma} = 0, \tag{15}$$

where the new variable σ is proportional to the total basin width:

$$\sigma^2 = 4\,\mathrm{Br}(\eta - x).$$

Equation (15) is solved on a fixed semi-axis $0 \leqslant \sigma < \infty$ ($\sigma = 0$ corresponds to a moving run-up boundary) unlike the variable region for the initial system.

Together with the nonlinear shallow-water equations, we shall consider the linear system

$$\frac{\partial u}{\partial t} + \frac{1}{\mathrm{Br}}\frac{\partial \eta}{\partial x} = 0,$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(-xu) = 0. \tag{16}$$

Using the linear version of the Carrier–Greenspan transformation in system (16)

$$u = \frac{2}{\sigma^2 \mathrm{Br}} \frac{\partial \Phi_0}{\partial \sigma_0},$$

$$\eta = \frac{1}{\mathrm{Br}} \frac{\partial \Phi_0}{\partial \lambda_0}, \tag{17}$$

$$x = -\frac{\sigma_0^2}{4\,\mathrm{Br}}, \quad t = \lambda_0$$

we reduce Equations (17) to linear wave equations

$$\frac{\partial^2 \Phi_0}{\partial \lambda_0^2} - \frac{\partial^2 \Phi_0}{\partial \sigma_0^2} - \frac{1}{\sigma_0} \frac{\partial \Phi_0}{\partial \sigma_0} = 0, \tag{18}$$

where $\sigma_0 = 0$ corresponds to the shoreline.

As will be shown below, a comparison of the linear and nonlinear problems is useful for the tsunami run-up calculation. Let the tsunami be generated in the open ocean far from the shore. In this case, the wave is linear and the asymptotic forms of the functions $\Phi(\sigma, \lambda)$ and $\Phi_0(\sigma_0, \lambda_0)$ appear to be identical and correspond to the same initial and boundary conditions. Because of the identity of Equations (15) and (18), the functions $\Phi(\sigma, \lambda)$ and $\Phi_0(\sigma_0, \lambda_0)$ will be the same in the whole variation range of their arguments and, therefore, the functions $\Phi(0, \lambda)$ and $\Phi_0(0, \lambda_0)$ and their maxima will be the same. The first function $\Phi(0, \lambda)$ describes the moving shoreline oscillations in the real problem, while the second one, $\Phi_0(0, \lambda)$, describes the water-level oscillations on the fixed shoreline ($x = 0$) in the equivalent linear problem. Analogous results are obtained for the flow velocity. Consequently, by solving the linear problem and determining the maximum wave height and flow velocity on the shoreline, we find the maximum water elevation on the shore and the flow velocity within the framework of the nonlinear equations. This is exactly what makes the linear approach useful for the calculation of extreme run-up characteristics. Such a conclusion was drawn by Keller (1961) from the analysis of the partial solutions of Equations (15) and (18) but has, in fact, a general meaning. Meanwhile, the detailed characteristics such as the time-dependence of run-up and flow, run-up and run-down time, etc., are related to the nonlinearity of the wave approaching a beach. For their calculation, however, it is not necessary to completely solve the nonlinear equations again; if the solution of the linear problem is known, then the required characteristics are obtained from the linear solution with the aid of its Riemann transformation. In fact, the first and fourth relations of system (14) at $\sigma = 0$ are combined as

$$u = U_{\mathrm{lin}}(t + \mathrm{Br} \cdot u), \tag{19}$$

where the function U_{lin} has a clear physical meaning; it describes the time-depen-

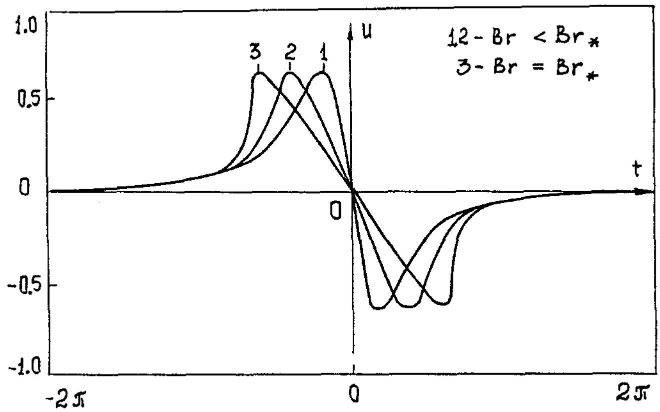

Fig. 1. Moving shoreline velocity.

dence of the moving shoreline velocity in the linear problem (Br = 0) (Figure 1). So, using (19), it is possible to describe the time-dependence of the running-up wave tongue at any amplitude if the solution of the linear problem is known.

Through the velocity u, it is easy to also find the time-dependence of the water elevation on the shore. From the definition $u = \mathrm{d}\eta/\mathrm{d}t$ (in dimensionless variables), we find, using (19),

$$\eta(t) = \eta_{\text{lin}}(t + \text{Br} \cdot u) - \tfrac{1}{2}\text{Br} \cdot u^2, \tag{20}$$

where $\eta_{\text{lin}}(t) = \int U_{\text{lin}}(t)\,\mathrm{d}t$ is the water elevation in the linear approximation. The functions $u(t)$ and $\eta(t)$ are implicit, but they can be easily constructed graphically. Note that from (19) and (20) it follows that the nonlinearity does not influence the extrema of both functions (such a conclusion was cited above), but influences the asymmetry of shoreline oscillations leading to increased steepness of the front slope of the wave, increased flow velocity, and a sharp peak in the water-level oscillogram.

These solutions, based on the Legendre transformations, describe the dynamics of the run-up of nonbreaking tsunami waves. The breaking criterion is usually obtained from the condition of unambiguous solvability of the Legendre transformations, and only for a monochromatic wave. From formula (19), this criterion is easily obtained for an arbitrary form of the wave as the condition of unambiguity of (19) or boundedness of $\partial u/\partial t$. The wave unboundedness (gradient catastrophy, or breaking) occurs at the following value of Br:

$$\text{Br}_* = \frac{1}{\text{Max}\,\dfrac{\mathrm{d}U_{\text{lin}}}{\mathrm{d}t}} = \frac{1}{\text{Max}\,\dfrac{\mathrm{d}^2\eta_{\text{lin}}}{\mathrm{d}t^2}}, \tag{21}$$

that is why Br is called the breaking parameter. Because of its importance, Equation (21) is rewritten in a dimensional form:

$$\mathrm{Max}\,\frac{d^2\eta_{\mathrm{lin}}}{dt^2} = g\alpha^2. \tag{22}$$

The left-hand side of (22) determines the vertical acceleration of fluid particles; therefore the breaking corresponds to a rather large value of vertical acceleration in the wave, equal to $g\alpha^2$. Taking into account (16) and the condition $u = 0$ by $x = 0$, relation (22) can be rewritten as

$$\mathrm{Max}\,\frac{\partial\eta_{\mathrm{lin}}}{\partial x}\bigg|_{x=0} = \alpha. \tag{23}$$

It is interesting to note that condition (22), proposed by Miche long ago (Mei, 1983), is thus rigorously proved for a wave of arbitrary shape.

4. Run-up of a Monochromatic Wave

As an illustration, we shall first consider the classical problem of monochromatic wave run-up on a beach. The solution of the linear problem is well known (in dimensional variables):

$$\eta(x, t) = RJ_0\left(\sqrt{\frac{4\omega^2|x|}{g\alpha}}\right)\cos\omega t. \tag{24}$$

Far from the shore, using asymptotic expressions for the Bessel function J_0, we find that formula (24) corresponds to a standing wave

$$\eta = H(x)\left\{\sin\left[\omega\left(t - \int\frac{dx}{\sqrt{gh}}\right) + \frac{\pi}{4}\right] + \sin\left[\omega\left(t + \int\frac{dx}{\sqrt{gh}}\right) - \frac{\pi}{4}\right]\right\}, \tag{25}$$

where

$$H(x) = R\left(\frac{\alpha}{\pi\omega}\sqrt{\frac{g}{\alpha|x|}}\right)^{1/2}. \tag{26}$$

Far from the shore, the wave amplitude changes in accordance with Green's law $H \sim |x|^{-1/4} \sim h^{-1/4}$, and near the shore, this formula is not valid because of the reflection so that the wave amplitude remains bounded. From Equation (26), we get an important formula for the wave amplification coefficient on the slope, or the fluid elevation height on the shore:

$$\frac{R}{H_0} = \left(\frac{\pi\omega}{\alpha}\sqrt{\frac{h_0}{g}}\right)^{1/2} = 2\pi\sqrt{\frac{2L}{\lambda_0}}, \tag{27}$$

where H_0 is the amplitude of a wave of length λ_0 at a distance L from the shoreline and h_0 is depth at the distance L from the shoreline. We emphasize that formula (27), which determines the extreme run-up characteristics, is also exact in the

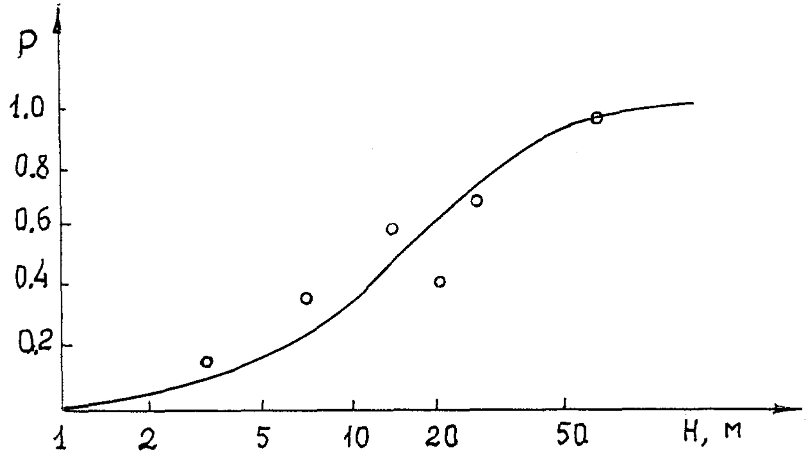

Fig. 2. The dependence of the wave breaking probability P on run-up height H; ○ – cloud statistics data.

nonlinear theory. Using (24), we can easily find the critical value of the breaking parameter $Br_* = 1$. Assuming, for example, that the tsunami wave period of 10 min, $\alpha \sim 10^{-2}$ and $Br_* = 1$, we find the run-up height at which the breaking is started (this height is approximately 10 m). Such a conclusion is confirmed by the tsunami data analysis in the Pacific Ocean (Mazova *et al.*, 1983). Figure 2 represents the dependence of the tsunami wave-breaking probability of run-up height, averaged over 114 tsunamis. It is seen that waves with heights of more than 10 m generally break. Totally, 75% tsunami waves do not break; this confirms that the theory proposed is applicable to the tsunami problem. Combining (27) with the critical value $Br_* = 1$, we obtain the maximum amplification coefficient

$$\mathrm{Max}\,\frac{R}{H_0} = \frac{2\pi\sqrt{2}}{(8\pi^3\sqrt{2})^{1/5}}\left(\frac{H_0}{h_0}\right)^{-1,5} \simeq 2.75\left(\frac{H_0}{h_0}\right)^{-1,5}, \tag{28}$$

(at $Br > Br_*$, the wave will break and the run-up height will not increase). We emphasize that the maximum run-up height is rather weakly dependent on the incident wave nonlinearity parameter H_0/h_0. Thus, for tsunami waves in the open ocean, we have: $H_0 \sim 1$ m, $h_0 \sim 1$ km, $\mathrm{Max}\,R/H_0 \sim 11$. Usually, the hydromodeling yields large values of $H_0/h_0 \sim 10^{-1}$ and $\mathrm{Max}\,R/H_0 \sim 4.3$. This fact must be taken into account when the hydromodelling data are applied to full-scale experiments.

The time-dependence of water level is calculated using Equations (19) and (20) and is presented in Figure 3. As Br increases, the velocity profile becomes asymmetric, the leading front steepens, so that the wave becomes a shock one at $Br = 1$, as was to be expected. The water-level profile remains symmetric with respect to the vertical axis; at $Br = 1$ (at the time of breaking), the curve $R(t)$ contains a kink (Figure 3). That is why it is difficult to distinguish the wave-

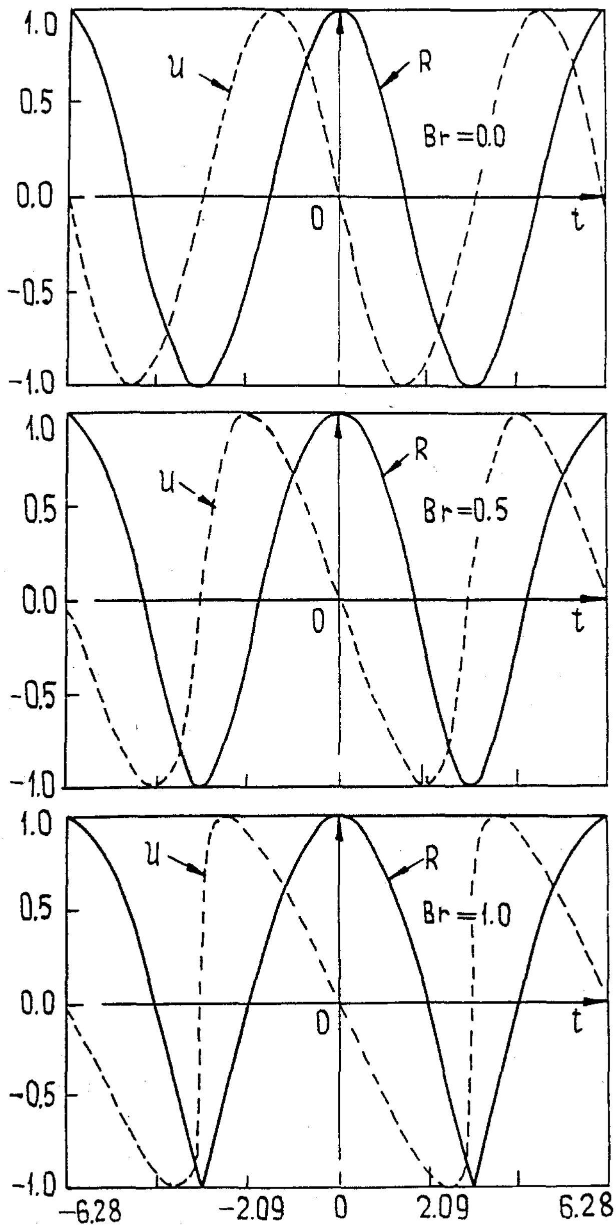

Fig. 3. The maximum run-up R and moving shoreline velocity u for a monochromatic wave.

breaking moment in numerical calculations using the fixed shoreline records; it is easier to use the moving shoreline velocity records. Note that the run-up height profile became asymmetric with respect to the horizontal axis; this corresponds to water elevation on the average. The same conclusion follows from (20) after some transformations:

$$\bar{\eta}/R = \tfrac{1}{4}\,\mathrm{Br} > 0. \tag{29}$$

As a result, the run-up time will exceed the run-down time. We omit cumbersome expressions and give a simple approximation formula which is valid in the region $\mathrm{Br} < 1$:

$$\omega t_{\text{run-up}} = \pi + \mathrm{Br}, \qquad \omega t_{\text{run-down}} = \pi - \mathrm{Br}. \tag{30}$$

Such behaviour of a moving shoreline in the case of monochromatic wave run-up, confirms the results of numerical experiments by Synolakis (1986), Gogodre *et al.* (1985).

5. Run-up of Pulse Perturbations

In practice, tsunamis represent a finite train of waves or even a single perturbation. As was mentioned above, the run-up of such waves can be investigated in two stages. At the first stage, the linear problem is solved, and at the second, the nonlinearity is taken into account through the Riemann transformation of the solution. We begin by considering the first stage. The general solution of the linear system (16), as is well known, can be written in two equivalent forms: the Poisson formula:

$$\eta(0,t) = \begin{cases} \dfrac{\partial}{\partial t}\displaystyle\int_0^t \frac{x\eta_0(x)\,\mathrm{d}x}{\sqrt{t^2 - x^2}} + \int_0^t \frac{xu_0(x)\,\mathrm{d}x}{\sqrt{t^2 - x^2}}, & t > x, \\ 0, & t < x, \end{cases} \tag{31}$$

and the Fourier integral:

$$\eta(x,t) = \int A(\omega) J_0\left(\sqrt{\frac{4\omega^2|x|}{g\alpha}}\right)\cos(\omega t - \psi(\omega))\,\mathrm{d}\omega, \tag{32}$$

where the choice of $A(\omega)$ and $\psi(\omega)$ depends on the initial conditions for tsunami waves $\eta(x,0) = h_0(x)$, $u(x,0) = u_0(x)$ (the piston model of tsunami wave-generation is often considered with the assumption $u_0(x) = 0$). These solutions are equivalent, at the corresponding choice of A and φ, to those obtained from the initial conditions. Then it is necessary to put $x = 0$ in (32); such a solution describes the shoreline oscillations in the linear problem. It should be borne in mind that although the initial conditions can be formally arbitrary within the framework of the linear system (16), the initial assumption that the wave is linear in the source permits one not to consider the initial problem and to solve the boundary-value problem assuming that a tsunami wave, with assigned properties, comes from the open ocean. Indeed, at large distances from the source, the ray approximation is valid so that the initial perturbation decays into oppositely directed waves on an even bottom. Thus, the wave propagating to the shore should be considered as the initial one. This fact is also important, since the real geometry of the ocean bottom does not change linearly; such an approximation is convenient only for the coastal zone and it is reasonable to assign the initial condition at the boundary of this zone (a similar situation is usually realized in laboratory and numerical experiments).

In the following, we shall assume as given the parameters of a wave moving to the shore at some (fixed) depth h_0, spaced at a distance L from the shore. We shall suppose that the wave is characterized by only two parameters: the height H_0 and the length λ_0. Then, investigating (32) for extremum at $x = 0$, we find a parametric formula for the run-up height of a wave of arbitrary form:

$$\frac{R}{H_0} = P_0 \sqrt{\frac{L}{\lambda_0}}, \tag{33}$$

where P_0 is the form factor defined as $P_0 = \max P(t)$;

$$P(t) = 2\sqrt{\pi} \int \sqrt{|\Omega|}\varphi(\Omega)\, e^{i(\Omega t - \delta - (\pi/4)\,\mathrm{sgn}\,\Omega)}\, d\Omega, \tag{34}$$

where φ and δ are the dimensionless amplitude and phase spectra of the approaching wave.

Thus, functionally, the run-up height of an arbitrary pulse is described by the same formula as that of a monochromatic wave; therefore, the dependences on H_0, λ_0, and L are universal. Meanwhile, formula (33) also includes a form factor. Since almost nothing can be said about the form of a wave spaced well enough from the shore because of the poor measurement data, the problem arises to determine the possible dispersion of P_0 among the most probable wave forms, at least.

As an example, we shall consider the initial perturbation of a Lorenz form:

$$\eta = \frac{H_0}{1 + \left(\frac{2t}{T_0}\right)^2} \left[\cos\left(\theta - \frac{\pi}{4}\right) - \frac{2t}{T_0} \sin\left(\theta - \frac{\pi}{4}\right)\right], \tag{35}$$

where the parameter θ is arbitrary. T_0 is the period of the pulse. The chosen class of perturbations has a wide variety of forms (Figure 4) with equal energies thus being, in a sense, standard. A substitution of (35) into (34) permits one to find the extrema of the function P (Figure 5), in particular P_+ determining the run-up

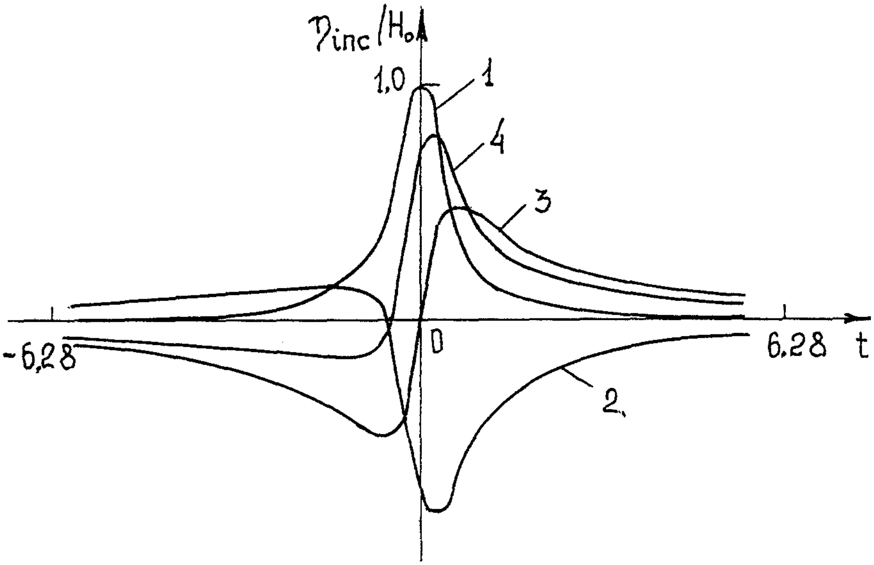

Fig. 4. Forms of the incident wave for different θ: 1 – 45°; 2 – 180°; 3 – 315°; 4 – 360°.

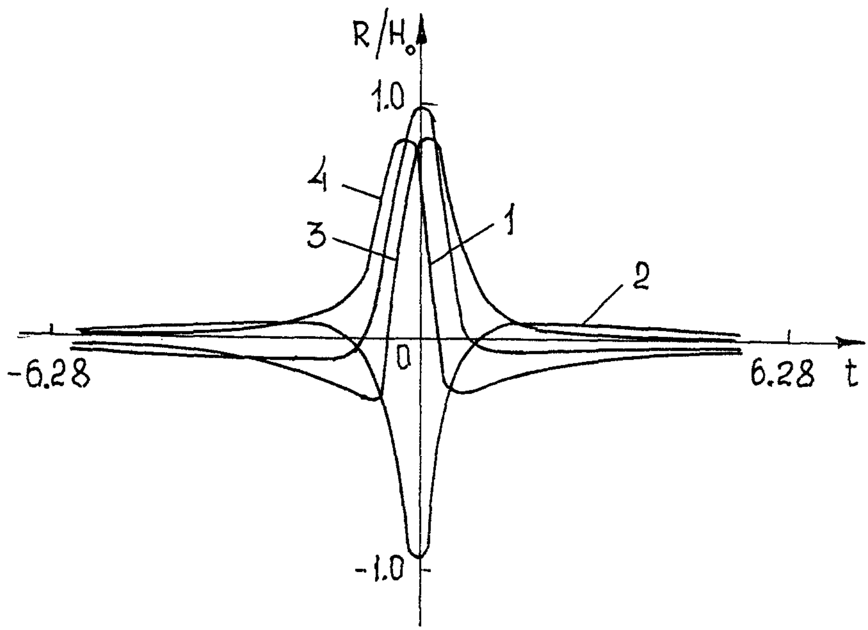

Fig. 5. Lorentz pulse run-up on a beach: R/H_0 is the relative run-up value θ: 1 – 45°; 2 – 180°; 3 – 315°; 4 – 360°.

height and P_- determining the run-down depth:

$$P_+ = \pi\sqrt{2}\cos^{5/2}\frac{2\theta}{5},$$
$$P_- = -\pi\sqrt{2}\cos^{5/2}\frac{2}{5}(\pi-\theta). \tag{36}$$

The coefficients P_+ and P_- change rather weakly. We are dealing, however, with the ambiguous interpretation of H_0. Specifically, in this example H_0 is the run-up height (i.e., a sum of the crest height and the trough depth). If in (33) by P_0, only the crest height is meant, then the coefficients must be divided by $\cos^2(\theta/2)$ so that the dependence on θ becomes strong. Thus, to interpret experimental data, it is important to know the particular form of the running-up wave. It is also interesting to compare these coefficients in the case where only a crest of a Lorenzian or sinusoidal form comes to the shore. Calculations show that in the first case, we have $P_+ = 4.4$; $P_- = -0.23$, while in the second case, $P_+ = 3.9$; $P_- = -1.4$. Bearing in mind that the real form of the tsunami wave is indefinite but assuming it as a single crest, we propose that for rough preliminary estimates, we can conservatively adopt $P_0 = 5$. Such a formula was used to plot the tsunami zonation scheme of the U.S.S.R. (Pelinovsky, 1988; Go *et al.*, 1982).

The existence of the run-up and run-down phases, even when a unipolar pulse comes to a beach, permits one to formulate the problem of 'dangerous' forms of tsunami waves. So, if the crest motion is accompanied by a precursor seen as a trough, then a run-down will first occur on the shore, followed by a greater run-up than without a precursor. Calculations show that most dangerous is the wave with a short deep or very long negative phase, as well as with a short high hump behind the trough (Figure 6). (A more detailed analysis of this problem will appear separately.)

Let us note another important feature of the run-up of single perturbations. Since short spectral components are appreciably amplified on a beach, discontinuities can arise in the wave, thus indicating the wave-breaking. Generally speaking, a localized pulse whose derivatives have discontinuities at the pulse edge, does not satisfy the shallow-water approximation: the vertical acceleration becomes unlimited. It should be borne in mind, however, that the vertical acceleration reaches high values in a very narrow zone near the front; therefore, it is difficult to ascertain whether or not the wave-breaking really occurs, since it is necessary to take into account the dispersion that smooths out such a feature on the front. Meanwhile, the wave crest remains smooth and satisfies the unbroken wave approximation under the condition $\mathrm{Br} < \mathrm{Br}_*$.

Using the linear solutions (31) or (32) for a wave field on the shoreline, we can

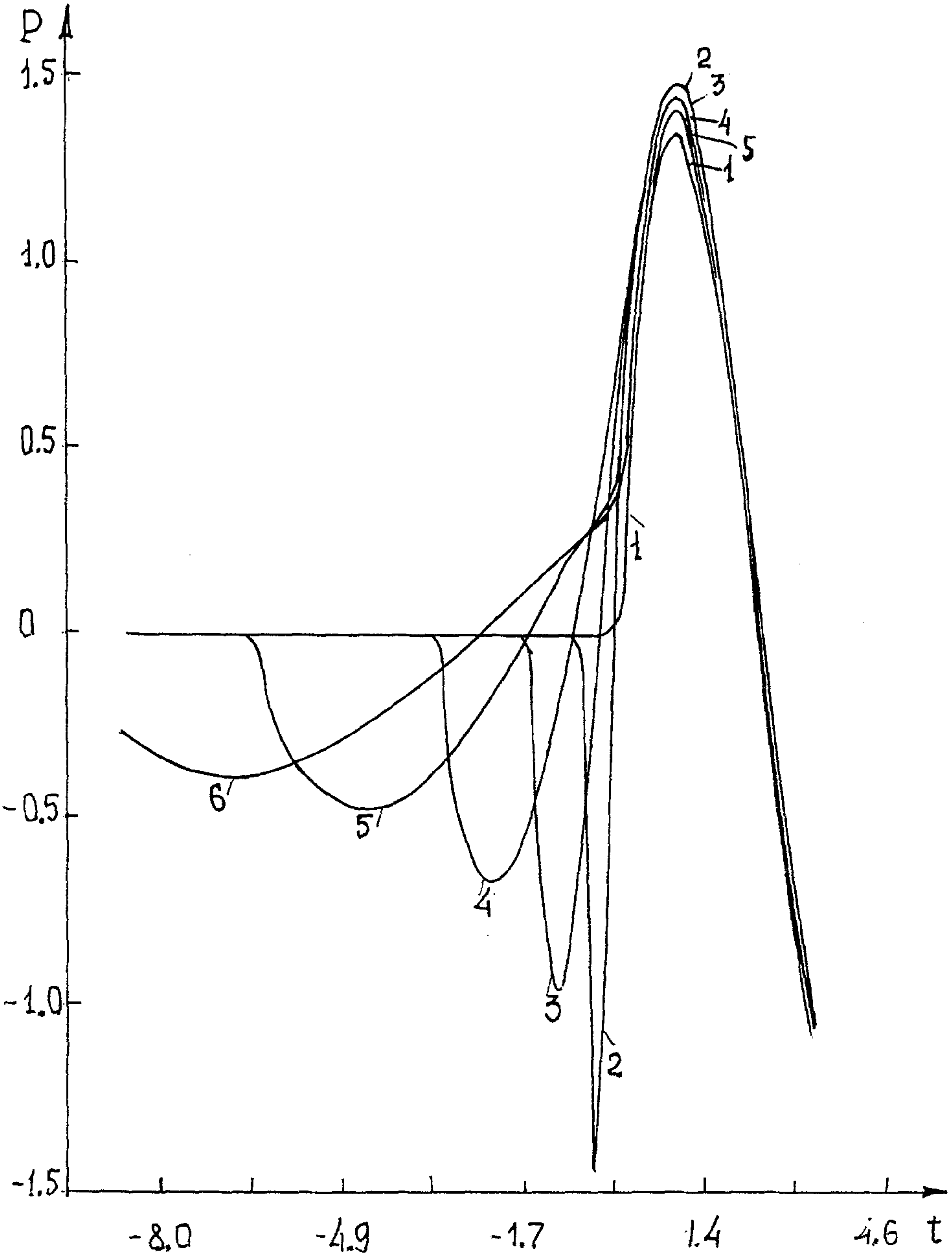

Fig. 6. The form coefficients' dependence on the initial parameters for an alternating pulse; $H_1 = 0.5$; $H_2 = 1$; $\lambda_2 = 1$; λ_1: 1 – 0; 2 – 0.2; 3 – 0.5; 4 – 2; 5 – 3; 6 – 5.

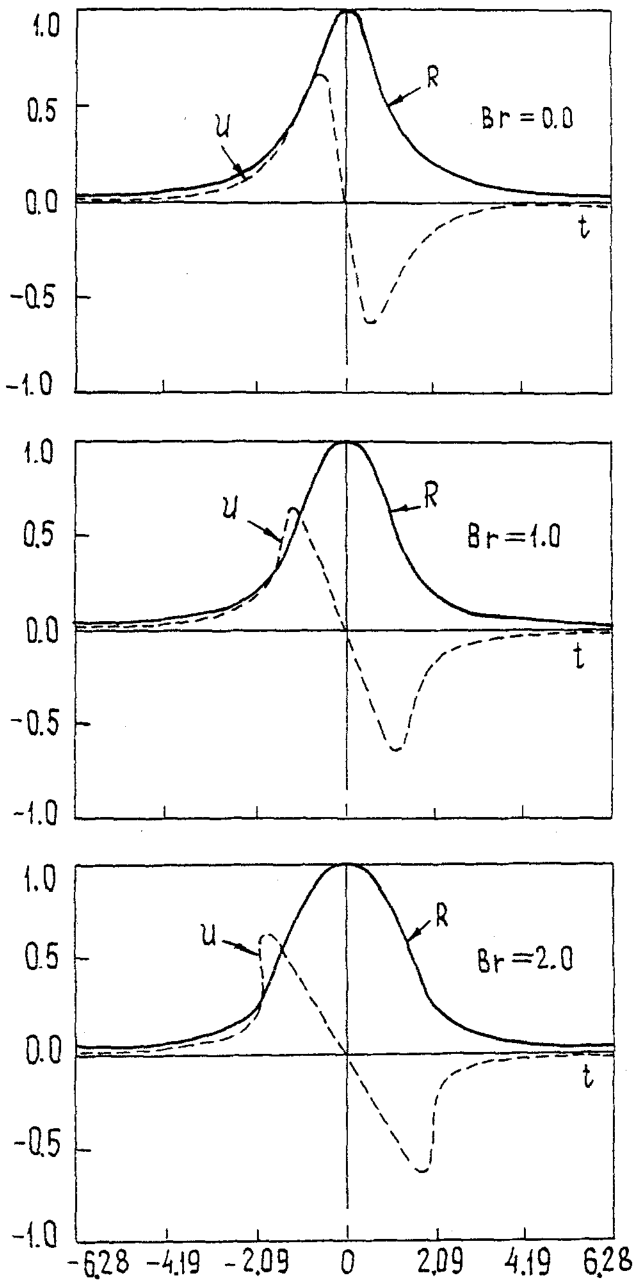

Fig. 7. A typical calculation of pulsed perturbation using formulae (19) and (20).

restore the nonlinear dynamics of a moving shoreline by (19) and (20). A typical calculation is given in Figure 7. As Br increases, the shoreline velocity profile takes on a characteristic shock form and the curve $R(t)$ remains symmetric to the vertical axis. We emphasize that the critical value of Br is 2 in this problem.

6. Wave Reflection from a Beach

In the nonbreaking case, besides the run-up characteristics, it is easy to obtain simple formulae for a reflected wave. If we neglect, as before, the dissipation and breaking, then the whole energy of the wave will transform to the reflected wave energy. The reflection characteristics are investigated in the region where waves are linear, while the wave field is described by (25). From this formula, it is seen that the reflected-wave amplitude remains equal to the incident-wave amplitude and the phase shift becomes $\pi/2$. Consequently, in the pulse perturbation case, the spectral amplitudes do not change and all phases undergo a $\pi/2$ shift. Such a situation is characteristic for total internal reflection and the relationship between the incident and the reflected wave in the pulse perturbation case can be written through the Hilbert transformation:

$$\eta_{\text{refl}}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta_{\text{inc}}(\tau)}{t-\tau}\,\mathrm{d}\tau. \tag{37}$$

Figure 8 is a marigram of incident and reflected waves for the Lorentz pulse run-up (35). It is seen that the form of the reflected wave changes appreciably as compared to the incident wave.

7. Wave Run-up on a Beach with a Kink

Let us now consider the case of wave run-up on a beach conjugate with an even bottom. Such a situation is most important from the viewpoint of laboratory modeling (Figure 9). An exact solution of the nonlinear problem for this case has not yet been obtained. Nevertheless, if the kink point is relatively far from the shoreline, it seems reasonable to use the linear approximation to find the run-up height. Omitting elementary calculations, we shall write down the final expression for shoreline oscillations:

$$R(t) = \int_{-\infty}^{\infty} \frac{2H(\omega)}{\sqrt{J_0^2(z) + J_1^2(z)}} \times \exp\left\{i\left[\omega\left(t + \frac{L}{\sqrt{gh}}\right) - \operatorname{arc}tg\frac{J_1(\omega)}{J_0(\omega)}\operatorname{sgn}(\omega)\right]\right\}\mathrm{d}\omega, \tag{38}$$

where $H(\omega)$ is the spectrum of the approaching wave, $z = 4\pi L/\lambda_0$. If $z \gg 1$, then using the asymptotic representation of the Bessel function, we can find a more simple expression:

$$R = 2\sqrt{\frac{\pi_4}{\alpha}}\sqrt{\frac{h}{g}}\int \sqrt{|\omega|}\,H(\omega)\exp i\left[\omega\left(t + \frac{L}{\sqrt{gh}}\right) + \frac{\pi}{4}\operatorname{sgn}\omega\right]\mathrm{d}\omega. \tag{39}$$

As a result, the formulae for run-up height and run-down can be parametrized

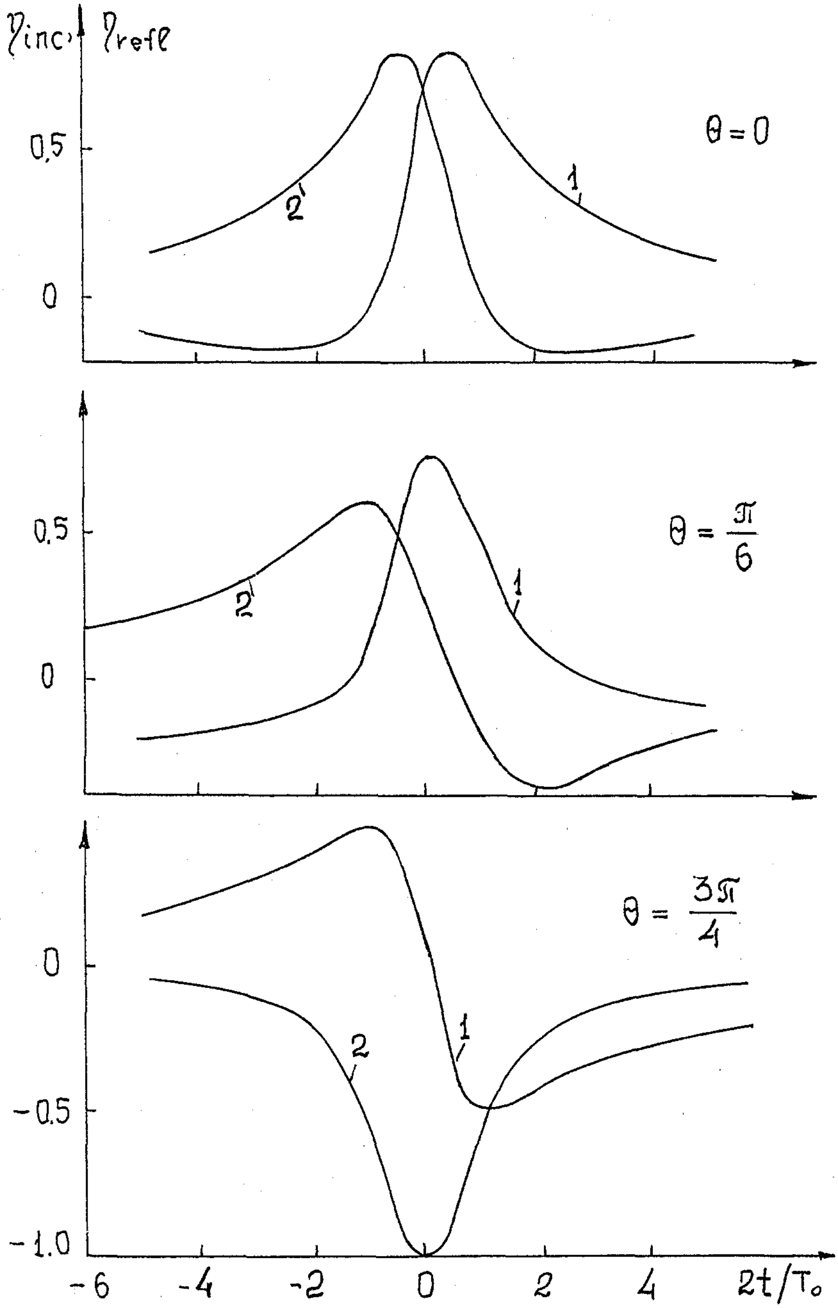

Fig. 8. Marigram of the incident η_{inc}/H_0 and the reflected η_{refl}/H_0 wave for a Lorentz pulse: 1 – η_{inc}/H_0; 2 – η_{refl}/H_0.

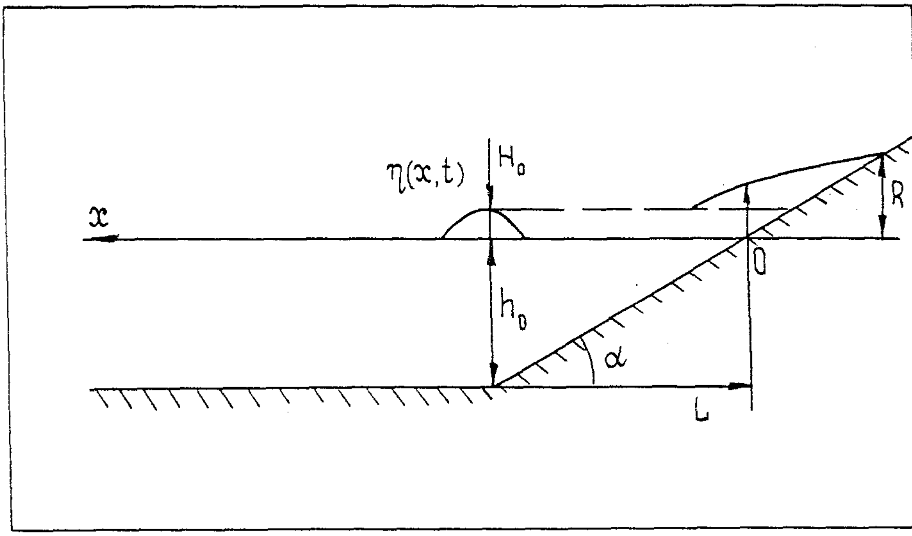

Fig. 9. Scheme of wave run-up on a slope conjugate to an even bottom.

as (33). In the other limiting case ($z \ll 1$), we deal with a 'wall', where $R_+ = 2H_0$ and $R_- = 0$. Combining these asymptotic forms, we propose a simplified formula to calculate the extreme characteristics of tsunami wave run-up on a beach with a kink:

$$\frac{R_+{}^{\max}}{H_0} = \begin{cases} 2, & L < L_*, \\ \sqrt{L/\lambda_0}\max P^+, & L > L_*, \end{cases}$$
$$\frac{R_-{}^{\max}}{H_0} = \begin{cases} 0, & L < L_*, \\ \sqrt{L/\lambda_0}\max |P^-|, & L > L_*, \end{cases} \tag{40}$$

where P^+, P^- are the maximum and minimum function $P(t)$, $R_{\max}$ is the maximum runup height, $R_{\min}$ the maximum rundown, and L_* is the distance from the shoreline characteristic for concrete pulses.

To find the applicability range of the asymptotic formulae, we used (38) and (39) with corresponding curves given in Figures 10 and 11 (solid and dashed lines, respectivley). It is seen that the asymptotic formula slightly underestimates the run-up height; this can be an important circumstance for the experimental data interpretation.

8. Wave Run-up on a Vertical Obstacle

To estimate the role of nonlinear effects in the problem of wave run-up on a beach with a kink, we shall consider the case of wave reflection from a vertical obstacle (in the other limiting case where the kink is spaced far enough from the shoreline, the nonlinearity does not influence characteristics of the run-up). Using the Riemann invariants

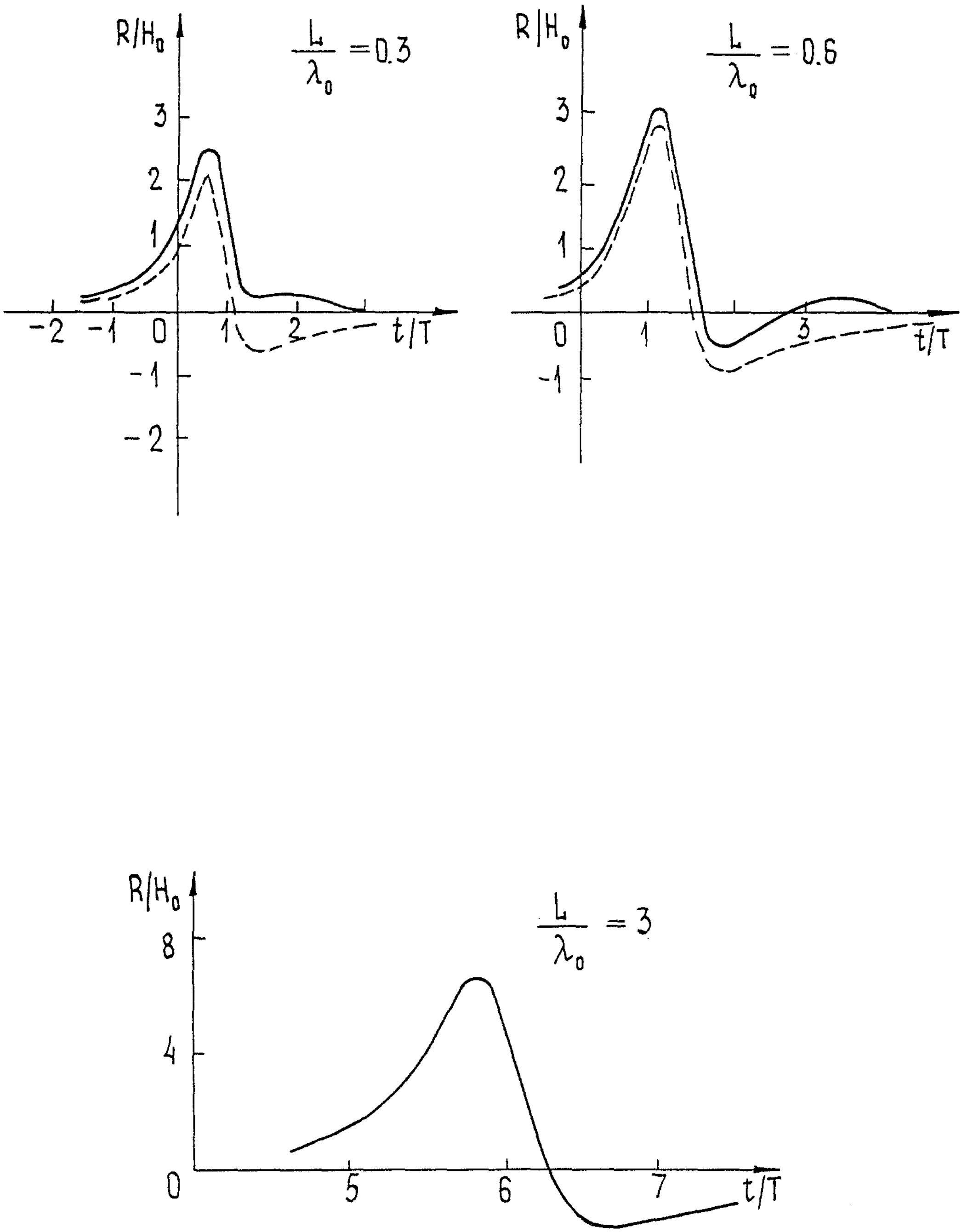

Fig. 10. Calculation by an exact (38) (solid) and an asymptotic (39) formula (dashed line).

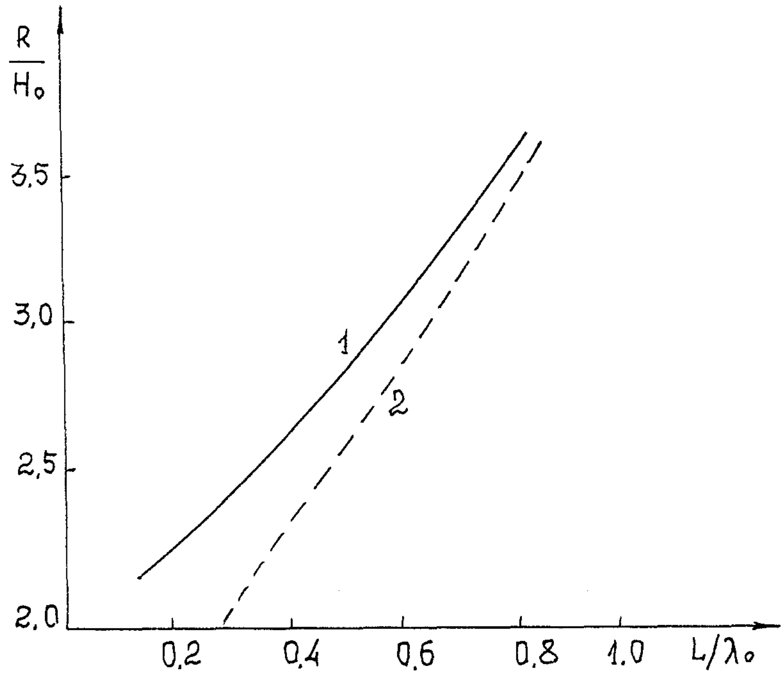

Fig. 11. Exact (1-solid line) nad asymptotic (2-dashed line) dependences for run-up.

$$V_{\pm} = u \pm 2[\sqrt{g(h+\eta)} - \sqrt{gh}], \tag{41}$$

we can write the initial shallow-water equations (in dimensionless variables);

$$\frac{\partial V}{\partial t} + (\pm\sqrt{gh} + \tfrac{3}{4}V_{\pm} + \tfrac{1}{4}V_{\mp})\frac{\partial V_{\mp}}{\partial x} = 0. \tag{42}$$

The wave-wall interaction process is illustrated in Figure 12, where the characteristics $C_{\pm} = \pm\sqrt{gh} + \frac{3}{4}V_{\pm} + \frac{1}{4}V_{\mp}$ are shown. Being straight lines (shown dashed) everywhere in the linear theory, these characteristics are bent in the interaction region between the incident and reflected waves near the wall because of the nonlinearity. From (42), it follows that the quantities $V_{\pm}$ remain unchanged on the characteristics $C_{\pm}$; therefore, the interaction effect reduces to additional time delays of the reflected wave phases. Our primary interest, however, is the wave height near the wall; this value can be calculated exactly. Indeed, from the boundary condition on the wall $u = 0$, we find the relationship between the run-up height and the invariant V_+;

$$V_+ = 2[\sqrt{g(h+R)} - \sqrt{gh}]. \tag{43}$$

Outside the interaction region, we have a common expression for the invariant:

$$V_+ = 4[\sqrt{g(h+H_0)} - \sqrt{gh}. \tag{44}$$

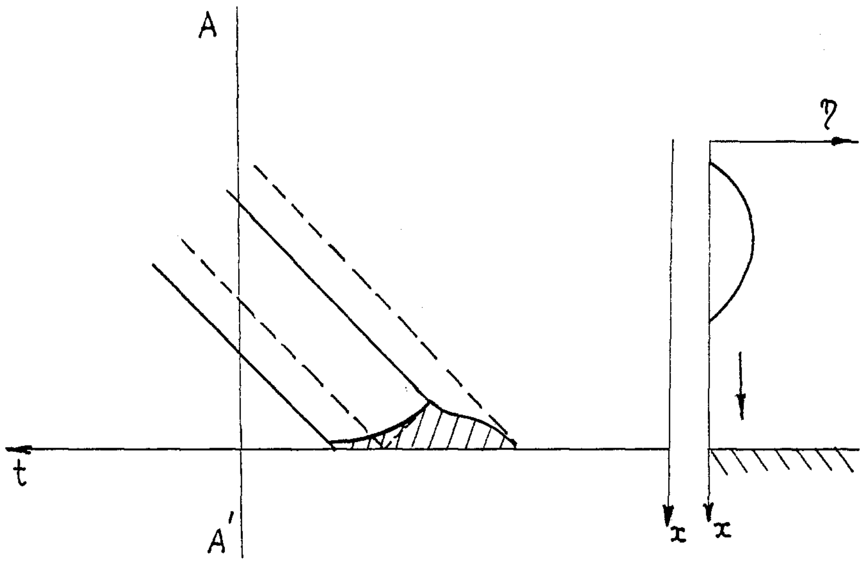

Fig. 12. A quantitative variation of the incident and reflected wave characteristics. Characteristics of the linear problem are indicated by a dashed line; for nonlinear problem – a solid line. The incident and reflected wave interraction region are indicated by hatched lines.

Equating (43) and (44), we find the sought formula:

$$\frac{R}{H} = 4\left[1 + \frac{H_0}{h} - \sqrt{1 - \frac{H_0}{h}}\right]. \tag{45}$$

In the linear approximation, from (45) it follows naturally that $R = 2H_0$. Thus, taking the nonlinearity into account leads to increased water elevation at the wall but such an increase is not so large. Therefore, if $H_0 < h_0$, then the linear theory yields a relatively small error in the run-up height calculation.

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