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Hypercomplex Structures on Stiefel Manifolds

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Abstract: This paper describes a family of hypercomplex structures $\{\mathcal{I}^a(\mathbf{p})\}_{a=1,2,3}$ depending on n real non-zero parameters $\mathbf{p} = (p_1, \ldots, p_n)$ on the Stiefel manifold of complex 2-planes in $\mathbb{C}^n$ for all $n > 2$. Generally, these hypercomplex structures are inhomogeneous with the exception of the case when all the p_i's are equal. We also determine the Lie algebra of infinitesimal hypercomplex automorphisms for each structure. Furthermore, we solve the equivalence problem for the hypercomplex structures in the case that the components of $\mathbf{p}$ are pairwise commensurable. Finally, some of these examples admit discrete hypercomplex quotients whose topology we also analyze.

Key words: *Hypercomplex, Stiefel manifolds, foliations, automorphisms*
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Introduction

A hypercomplex structure on a smooth manifold M is a G-structure where $G = GL(n, \mathbb{H})$ that admits a necessarily unique torsion free connection, the Obata connection [Bon, Ob]. In particular, every such M has three complex structures I, J, and K which satisfy the relations of the algebra of imaginary quaternions and thus generate an entire two-sphere's worth of complex structures on M. Until recently, there were few known examples of compact, irreducible, hypercomplex manifolds in dimension 8 and higher. The first class of such examples are the hyperkähler twisted products of $K3$ surfaces constructed by Beauville [Bea]. Examples of hypercomplex manifolds that are not hyperkähler were very scarce, the simplest ones being the Hopf manifolds $S^{4n+3} \times S^1$ which are locally conformally hyperkähler. Recently the authors [BGM2] gave a class of new compact locally conformally hyperkähler manifolds by replacing S^{4n+3} with any 3-Sasakian manifold. Similar examples involving the quaternionic Heisenberg group were found by Hernandez [Her]. None of these examples, however, are simply connected.

In contrast to the 4-dimensional case, where all hypercomplex structures on compact 4-manifolds are locally conformally hyperkähler [Boy], in higher dimensions this is no longer true. A class of hypercomplex manifolds that are not locally conformally hyperkähler was studied by physicists interested in supersymmetric σ-models. In this regard, Spindel et. al. [SSTP] classified compact Lie groups which admit

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hypercomplex structures. This is the generalization to the hypercomplex category of the classic work of Samelson [Sam] and Wang [W] on the classification of compact Lie groups [Sam] and homogeneous spaces [W] admitting complex structures, respectively. Using different methods, Joyce [Joy1] later recovered this [SSTP] classification and developed a theory of homogeneous hypercomplex manifolds which generalizes Wang's [W] result.

In this paper, guided by our previous work on 3-Sasakian manifolds [BGM1, BGM2], we prove the existence of uncountably many distinct hypercomplex structures on Stiefel manifolds of complex 2-planes in complex n-space. An announcement of the main results of this paper has appeared in [BGM3], and a general theory of hypercomplex structures on circle bundles over 3-Sasakian manifolds is currently being developed [BGM4].

Our main results are as follows.

Theorem A. *Let $n > 2$ and $\mathbf{p} = (p_1, \ldots, p_n) \in (\mathbb{R}^*)^n$ be an n-tuple of non-zero real numbers. For each such $\mathbf{p}$ there is a compact hypercomplex manifold $(\mathcal{N}(\mathbf{p}), \mathcal{I}^a(\mathbf{p}))$, where $\mathcal{N}(\mathbf{p})$ is diffeomorphic to $V^{\mathbb{C}}_{n,2}$, the Stiefel manifold of 2-frames in $\mathbb{C}^n$.*

Since $V^{\mathbb{C}}_{n,2}$ is $(2n-4)$-connected, it cannot admit a locally conformally hyperkähler structure. A priori, all the hypercomplex structures $\{\mathcal{I}^a(\mathbf{p})\}$ on $V^{\mathbb{C}}_{n,2}$ could be equivalent. In particular, permuting the coordinates of $\mathbf{p}$ or changing their signs does not change the hypercomplex structure. Thus, we can assume that $\mathbf{p}$ is an n-tuple of positive, non-decreasing, real numbers; i.e., $\mathbf{p}$ is an element in the positive cone $C_n = \{\mathbf{p} \in \mathbb{R}^n \mid 0 < p_1 \le p_2 \le \ldots \le p_n\}$. However, we prove

Theorem B. *If $\mathbf{p}$ and $\mathbf{q}$ are both commensurable sequences in the positive cone C_n then the hypercomplex manifolds $\mathcal{N}(\mathbf{p})$ and $\mathcal{N}(\mathbf{q})$ are hypercomplex equivalent if and only if $\mathbf{p} = \mathbf{q}$. Here $\mathbf{p}$ is said to be commensurable if each of the ratios $\frac{p_i}{p_j}$ is a rational number. Furthermore, the manifold $\mathcal{N}(\mathbf{p})$ is hypercomplex homogeneous if and only if $\mathbf{p} = \lambda(1, \ldots, 1)$ for some $\lambda \in \mathbb{R}^*$.*

The one parameter family of distinct $U(n)$-homogeneous hypercomplex structures on $V^{\mathbb{C}}_{n,2}$ given in Theorem B was found implicitly by Joyce [Joy1] (see also [Bat]); however the remaining inhomogeneous hypercomplex structures on $V^{\mathbb{C}}_{n,2}$ are new. They are analogous to the inhomogeneous complex structures found by Griffiths [Gr] in the versal deformation space of homogeneous complex structures. The equivalence problem in the case of incommensurable sequences $\mathbf{p}$ appears to be beyond the scope of the techniques used in this paper and is relegated to a future work.

Our next main result determines the connected component of the group of hypercomplex automorphisms of $\mathcal{N}(\mathbf{p})$. In particular, it is shown that this group depends only on the multiplicities m_i of the components p_i of $\mathbf{p}$. More precisely,

Theorem C. *For all $\mathbf{p} \in (\mathbb{R}^*)^n$ the Lie algebra $\mathfrak{h}(\mathbf{p})$ of infinitesimal hypercomplex automorphisms of $\mathcal{N}(\mathbf{p})$ is isomorphic to $\bigoplus_{i=1}^{k} \mathfrak{u}(m_i)$. Hence, the connected component of the group of hypercomplex automorphisms of $\mathcal{N}(\mathbf{p})$ is $\prod_{i=1}^{k} U(m_i)$. In particular,* *there exists a natural hyperhermitian metric $h(\mathbf{p})$ on $\mathcal{N}(\mathbf{p})$ such that every infinitesimal automorphism is an infinitesimal isometry with respect to $h(\mathbf{p})$.*

Notice that Theorem C implies that the set of multiplicities in $\mathbf{p}$ is an invariant of the hypercomplex structure even when $\mathbf{p}$ is not a commensurable sequence.

Many of our examples admit discrete quotients which are also hypercomplex. To state this result we need some additional notation. A commensurable sequence $\mathbf{p}$ is called basic if all the coordinates are integers and the greatest common divisor of all the coordinates is one. A basic sequence is said to be coprime if the coordinates are pairwise relatively prime. If $\mathbf{p}$ is an integer multiple of a basic sequence and if the triples (p_i, p_j, k) have no common factor for all $1 \leq i < j \leq n$ then $\mathbf{p}$ is called k-coprime.

Theorem D. *Let $\mathbf{p}$ be k-coprime. Then there is a compact hypercomplex manifold $\mathcal{H}(\mathbf{p}, k)$ with universal cover $\rho_k : \mathcal{N}(\mathbf{p}) \rightarrow \mathcal{H}(\mathbf{p}, k)$ such that $\pi_1(\mathcal{H}(\mathbf{p}, k)) \cong \mathbb{Z}_k$ and ρ_k is a hypercomplex map. Moreover, $\mathcal{H}(\mathbf{p}, k)$ is never locally conformally hyperkähler and is hypercomplex homogeneous if and only if $\mathbf{p} = (p, p, \dots, p)$.*

Theorems B and C immediately extend to these non-simply connected examples so $\mathcal{H}(\mathbf{p}, k)$ is hypercomplex equivalent to $\mathcal{H}(\mathbf{q}, l)$ if and only if $\mathbf{p} = \mathbf{q}$ and $k = l$. Furthermore, the Lie algebra of infinitesimal hypercomplex automorphisms of $\mathcal{H}(\mathbf{p}, k)$ is the Lie algebra $\mathfrak{h}(\mathbf{p})$ given in Theorem C. Moreover, if $\mathbf{p}$ is commensurable but not k-coprime then $\mathcal{H}(\mathbf{p}, k)$ is a hypercomplex orbifold.

This paper is organized as follows: In Section 1 we construct by the method of symmetry reduction two equivalent models for the n-parameter family of hypercomplex structures on the Stiefel manifold $V^{\mathbb{C}}_{n,2}$. Both of these models are then used in Section 2 together with a crucial scaling argument in the parameter space to completely describe the connected component to the identity of the group of hypercomplex automorphisms for each hypercomplex structure. The precise statement is Theorem C above. Next we turn our attention to the question of when the hypercomplex structures on $V^{\mathbb{C}}_{n,2}$ defined by different $\mathbf{p}$ are inequivalent and prove Theorem B. This is done in the case of commensurable $\mathbf{p}$ by analyzing the holonomy groups associated to certain foliations canonically attached to the hypercomplex structures. We briefly address a few questions in the more general case when some of the $\frac{p_i}{p_j}$ ratios are irrational. This is done in Sections 3 and 4. This leads to the study of the moduli space of hypercomplex structures on the complex Stiefel manifold $V^{\mathbb{C}}_{n,2}$. Finally, in the last section we construct the hypercomplex structures on certain discrete quotients $\mathcal{H}(\mathbf{p}, k)$ and analyze their topology.

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1. The Hypercomplex Geometry of Stiefel Manifolds

Recall that the Stiefel manifold $V^{\mathbb{C}}_{n,2}$ of complex 2-planes in $\mathbb{C}^n$ is the homogeneous manifold $U(n)/U(n-2)$. Alternatively, $V^{\mathbb{C}}_{n,2}$ can be described as an embedded submanifold $\mathbf{V}^{\mathbb{C}}_{n,2} \subset S^{4n-1} \subset \mathbb{H}^n$. It is the zero set of a 3-Sasakian moment map $\mu : S^{4n-1} \to \mathbb{R}^3$ associated with the central $U(1)$ subgroup of $U(n)$ [BGM2].

We shall induce the hypercomplex structures on $\mathbf{V}^{\mathbb{C}}_{n,2}$ from the flat hypercomplex geometry of $\mathbb{H}^n$. This is the flat G-structure with $G = GL(n, \mathbb{H})$. Since the first prolongation $\mathfrak{gl}(n,\mathbb{H})^{(1)}$ vanishes [Sal] the automorphism group of this flat G-structure is the semidirect product $GL(n,\mathbb{H}) \rtimes Q$ where Q is the quaternionic translation group generated by $\mathbf{u} \mapsto \mathbf{u}+\mathbf{q}$. Owing to the noncommutativity of the quaternions there are two (equivalent) flat hypercomplex structures on $\mathbb{H}^n$. With respect to the standard quaternionic coordinates $(u_1, \ldots, u_n)$ they are given by

$$I^a_{\pm} = \sum \left(\frac{\partial}{\partial u^a_j} \otimes du^0_j - \frac{\partial}{\partial u^0_j} \otimes du^a_j \pm \epsilon^{abc} \frac{\partial}{\partial u^b_j} \otimes du^c_j \right). \tag{1.1}$$

The two hypercomplex structures are distinguished as follows: On $\mathbb{H}^n$ there is a natural action of $Sp(1) \cdot Sp(1) \simeq Sp(1) \times Sp(1)/\mathbb{Z}_2$ acting diagonally on each coordinate u_i by $u_i \mapsto l u_i r$, where $l, r \in Sp(1)$. This action induces representations of $Sp(1) \cdot Sp(1)$ on the trivial vector bundles $V_{\pm}$ spanned by $I^a_{\pm}$, respectively. These representations are $\mathrm{id} \otimes \mathrm{Ad}$ on V_+ and $\mathrm{Ad} \otimes \mathrm{id}$ on V_-. We shall mainly confine ourselves to the hypercomplex structure defined by I^a_+. Let ξ^a_+ denote the infinitesimal generators of the right $Sp(1)$ action on $\mathbb{H}^n$, and let Ψ denote the Euler field that generates dilatations on $\mathbb{H}^n$. Together this gives an action of $\mathbb{H}^*$ on $\mathbb{H}^n$ that acts freely on the open submanifold $\mathbb{H}^n \setminus \{0\}$. The hypercomplex structure I^a_+ on $\mathbb{H}^n$ restricts to give a hypercomplex structure, also denoted by I^a_+, on $\mathbb{H}^n \setminus \{0\}$ with automorphism group $GL(n,\mathbb{H})$. The Lie algebra $\mathfrak{gl}(n,\mathbb{H})$ of $GL(n,\mathbb{H})$ has the Levi decomposition

$$\mathfrak{gl}(n,\mathbb{H}) \simeq \mathfrak{sl}(2n,\mathbb{H}) \oplus \mathfrak{r}, \tag{1.2}$$

where the radical $\mathfrak{r}$ is the central one-dimensional Lie algebra generated by the Euler vector field Ψ. One can easily check that the followings compatibility relations hold

$$I^a_+ \xi^b_+ = -\epsilon^{abc} \xi^c_+ + \delta^{ab} \Psi. \tag{1.3}$$

This leads to

Definition 1.1. An almost hypercomplex structure $\{I^a\}^3_{a=1}$ on a smooth manifold M is said to be $Sp(1)$-*compatible* if there are a smooth action of $Sp(1)$ and a vector field Ξ on M such that

(i) Ξ is an infinitesimal automorphism of I^a for each $a = 1, 2, 3$.
(ii) The vector space V spanned by $\{I^a\}^3_{a=1}$ is the adjoint representation of $Sp(1)$.
(iii) For all $a, b = 1, \ldots, 3$ we have $I^a \xi^b = -\epsilon^{abc} \xi^c + \delta^{ab} \Xi$, where ξ^a are the infinitesimal generators of the $Sp(1)$ action.

We denote an $Sp(1)$ compatible hypercomplex structure by the triple (I^a, ξ^a, Ξ). One easily checks that the above conditions imply the following facts:

(i) The vector fields Ξ and ξ^a commute, that is, $[\Xi, \xi^a] = 0$.
(ii) Ξ is nowhere vanishing on M if and only if the $Sp(1)$ action is locally free.
(iii) All complex structures in the two sphere of complex structures of the hypercomplex structure I^a are equivalent.

Assume that the vector field Ξ is nowhere vanishing on M. Then ξ^1, ξ^2, ξ^3, Ξ span a trivial subbundle $\mathcal{V}_4$ of the tangent bundle TM. Furthermore, since these vector fields give a basis for the Lie algebra $\mathfrak{sp}(1) \oplus \mathbb{R}$, for each $a = 1,2,3$ there is a nested sequence of foliations $\mathcal{F}_1 \subset \mathcal{F}_2^a \subset \mathcal{F}_4$ of M generated by the subbundles $\mathcal{V}_1 = \mathrm{span}\{\Xi\}$, $\mathcal{V}_2^a = \mathrm{span}\{\Xi, \xi^a\}$ and $\mathcal{V}_4$, respectively. We refer to this nested sequence of foliations on M as the *multifoliate structure* associated with the $Sp(1)$ compatible hypercomplex structure. Clearly, we have

Proposition 1.2. *The triple (I_+^a, ξ_+^a, Ψ) defines an $Sp(1)$ compatible hypercomplex structure on both $\mathbb{H}^n$ and $\mathbb{H}^n \setminus \{0\}$.*

Remark 1.3. The space $\mathbb{H}^n \setminus \{0\}$ is the Swann $\mathbb{H}^*$ bundle over quaternionic projective space $\mathbb{HP}^{n-1}$, and the spaces of leaves of the foliations $\mathcal{F}_1$ and $\mathcal{F}_2^a$, are the 3-Sasakian manifold S^{4n-1} and the twistor space $\mathbb{CP}^{2n-1}$, respectively. One might thus expect that under suitable conditions, in the general case of a given $Sp(1)$ compatible hypercomplex manifold, the geometry of these foliations persist. That is, that $\mathcal{F}_1$ describes 3-Sasakian geometry, $\mathcal{F}_2$ describes twistor geometry with a complex contact structure, and $\mathcal{F}_4$ describes quaternionic Kähler geometry.

A particularly efficient way of constructing explicit new examples of a given geometry is by the method of symmetry reduction. In the case at hand there are two ways of performing this reduction. The first is related to 3-Sasakian reduction [BGM2] and is tied to a Riemannian metric. The second is Joyce's [Joy1] hypercomplex reduction and is ostensibly independent of any metric. In our case it will be shown that these two methods give equivalent hypercomplex structures on the Stiefel manifold $\mathbb{V}_{n,2}^{\mathbb{C}}$. We now consider symmetry reduction of the flat hypercomplex structure on $\mathbb{H}^n \setminus \{0\}$ by a one-parameter subgroup of its group of automorphisms $GL(n, \mathbb{H})$. The maximal compact subgroup of $GL(n, \mathbb{H})$ is $Sp(n)$ with maximal torus T^n. This torus lies in a $U(n)$ subgroup of $Sp(n)$, and thus fixes a complex structure, say I_-^1, of the "opposite" hypercomplex structure I_-^a. The action of this torus on $\mathbb{H}^n$ is linear and given by the diagonal representation

$$\tilde{\theta}(\mathbf{t}, \mathbf{u}) = (e^{2\pi i t_1} u_1, \ldots, e^{2\pi i t_n} u_n). \tag{1.4}$$

Thus, the corresponding representation of the Lie algebra $\mathfrak{t}_n$ of T_n is given by the multiplication by diagonal matrices $\mathrm{diag}(p_1, \ldots, p_n)$ on the quaternionic vector $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{H}^n$. Thus, we identify $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{R}^n$ as an element of the Lie algebra $\mathfrak{t}_n$. Let $\Xi : \mathfrak{t}_n \to \Gamma(T(\mathbb{H}^n \setminus \{0\}))$ be the map associating to each element in $\mathfrak{t}_n$ the corresponding vector field on $\mathbb{H}^n \setminus \{0\}$, where $\Gamma(TM)$ denotes the smooth vector fields on a smooth manifold M. Let e_i denote the standard basis of $\mathbb{R}^n \simeq \mathfrak{t}_n$, then we have

$$\Xi(e_j) = H_j = u_j^0 \frac{\partial}{\partial u_j^1} - u_j^1 \frac{\partial}{\partial u_j^0} + u_j^2 \frac{\partial}{\partial u_j^3} - u_j^3 \frac{\partial}{\partial u_j^2}. \tag{1.5}$$

We shall be interested in the vector field $\Xi(\mathbf{p})$ on $\mathbb{H}^n \setminus \{0\}$ corresponding to an element $\mathbf{p} \in \mathfrak{t}_n$. If $p_i \neq 0$ for all $i = 1, \ldots, n$ the vector field $\Xi(\mathbf{p})$ is nowhere vanishing on $\mathbb{H}^n \setminus \{0\}$. So we define the subset

$$\hat{\mathfrak{t}}_n = \{\mathbf{p} \in \mathfrak{t}_n : p_i \neq 0 \ \text{ for all } \ i = 1, \ldots, n\} \simeq (\mathbb{R}^*)^n. \tag{1.6}$$

In [BGM2] we constructed a 3-Sasakian moment map from the hyperkähler moment map $\mu(\mathbf{p})$ on $\mathbb{H}^n$. This moment map is given by

$$\mu(\mathbf{p})(\mathbf{u}^*, \mathbf{u}) = \frac{1}{2}\eta_+(\Xi(\mathbf{p})) = \mathbf{u}^* \cdot i\mathbf{p}\mathbf{u}, \tag{1.7}$$

where $\eta_+ = i\eta_+^1 + j\eta_+^2 + k\eta_+^3$, the 1-forms $\{\eta_+^a\}_{a=1}^3$ are dual to the vector fields ξ_+^a, $*$ denotes the quaternionic adjoint, $\cdot$ is the flat Hermitian inner product on $\mathbb{H}^n$, and $\mathbf{p}$ is identified with the diagonal matrix diag $(p_1, \ldots, p_n)$.

Definition 1.4.

$$\mathcal{N}(\mathbf{p}) = \mu(\mathbf{p})^{-1}(0) \cap S^{4n-1} \subset \mathbb{H}^n \setminus \{0\}.$$

Our first result is a slight generalization of Proposition 7.5 of [BGM2] whose proof is identical.

Theorem 1.5. *For every $\mathbf{p} \in \hat{\mathfrak{t}}_n$ the subspace $\mathcal{N}(\mathbf{p})$ is a smooth manifold of real dimension $4n-4$ diffeomorphic to the complex Stiefel manifold $\mathbf{V}^{\mathbb{C}}_{n,2}$.*

From their definitions one easily sees that the restrictions of the vector fields H_i and ξ_+^a to $\mathcal{N}(\mathbf{p})$ are tangent to $\mathcal{N}(\mathbf{p})$. These restrictions we denote by H_i and ξ^a, respectively. The vector fields $\{H_i\}_{i=1}^n$ generate a commutative Lie algebra sheaf $\mathcal{SF}$ that is a subsheaf of the tangent sheaf $\mathcal{TN}(\mathbf{p})$ for all $\mathbf{p} \in \hat{\mathfrak{t}}_n$. This sheaf defines a singular foliation [Mol] on $\mathcal{N}(\mathbf{p})$ which we also denote by $\mathcal{SF}$. One easily sees that when $\mathbf{p} \in \hat{\mathfrak{t}}_n$ (which we shall hereafter always assume), the vector fields $\Xi(\mathbf{p})$ and ξ^a are everywhere linearly independent on $\mathcal{N}(\mathbf{p})$. These vector fields give rise to locally free actions of certain Lie group on $\mathcal{N}(\mathbf{p})$, and associated foliations. We now give an easy generalization of the theory of connections in principal fiber bundles to the case of foliations described by locally free actions.

Let $\mathcal{F}$ be the foliation determined by a locally free action of a Lie group G on a smooth manifold M, and let $\mathcal{V}$ denote the vertical subbundle of TM generated by the fundamental vector fields determined by the action of G. We have the exact sequence of G-modules

$$0 \longrightarrow \mathcal{V} \longrightarrow TM \longrightarrow TM/\mathcal{V} \longrightarrow 0. \tag{1.8}$$

Definition 1.6. A *principal connection in the foliation $\mathcal{F}$* is a splitting $TM \simeq \mathcal{V} \oplus \mathcal{H}$ as G-modules. Given a chain of Lie subgroups $G_{k_n} < \cdots < G_{k_1} < G_{k_0} = G$ and an associated multi-foliate structure $\mathcal{F}_\kappa = \mathcal{F}_{k_n} \subset \cdots \subset \mathcal{F}_{k_1} \subset \mathcal{F}_{k_0}$ with $k_n < \cdots < k_1 < k_0$, we call the sequence of splittings $TM \simeq \mathcal{V}_{k_i} \oplus \mathcal{H}_{k_i}$ satisfying $\mathcal{H}_{k_n} \supset \cdots \supset \mathcal{H}_{k_1} \supset \mathcal{H}_{k_0}$ a *principal connection in the multi-foliate structure $\mathcal{F}_\kappa$* or simply a *multi-foliate principal connection.*

We construct almost hypercomplex structures $\mathcal{I}^a(\mathbf{p})$ on the Stiefel manifold $\mathbf{V}^{\mathbb{C}}_{n,2}$ for each $\mathbf{p} \in \hat{\mathfrak{t}}_n$. In order to minimize notational baggage we use the same notation for objects on the submanifold $\mathcal{N}(\mathbf{p})$ and on the Stiefel manifold $\mathbf{V}^{\mathbb{C}}_{n,2}$. There are two cases to consider, namely, $\Xi(\mathbf{p})$ generates an S^1 action on $\mathbf{V}^{\mathbb{C}}_{n,2}$, or $\Xi(\mathbf{p})$ generates an $\mathbb{R}$ action on $\mathbf{V}^{\mathbb{C}}_{n,2}$. We denote the respective groups together with their actions by $S^1(\mathbf{p})$ and $\mathbb{R}(\mathbf{p})$. We have the chain of subgroups $G_1(\mathbf{p}) < G_2^a(\mathbf{p}) < G_4(\mathbf{p})$ given by $G_1(\mathbf{p}) = S^1(\mathbf{p})$, $G_2(\mathbf{p}) = S^1(\mathbf{p}) \times S^1_a$, $G_4(\mathbf{p}) = S^1(\mathbf{p}) \times SU(2) \simeq U(2)(\mathbf{p})$ or $G_1(\mathbf{p}) = \mathbb{R}(\mathbf{p})$, $G_2(\mathbf{p}) = \mathbb{R}(\mathbf{p}) \times S^1_a$, $G_4(\mathbf{p}) = \mathbb{R}(\mathbf{p}) \times SU(2)$, where S^1_a denotes the circle action generated by ξ^a. For each $a = 1,2,3$ these subgroup chains give associated multi-foliate structures

$$\mathcal{F}_1(\mathbf{p}) \subset \mathcal{F}_2^a(\mathbf{p}) \subset \mathcal{F}_4(\mathbf{p}) \tag{1.9}$$

on $\mathcal{N}(\mathbf{p})$. Let $g_0(\mathbf{p})$ denote the pushforward to $\mathbf{V}_{n,2}^{\mathbb{C}}$ of the restriction to $\mathcal{N}(\mathbf{p})$ of the flat metric on $\mathbb{H}^n$. The vector fields ξ^a are mutually orthogonal and of unit norm on $\mathbf{V}_{n,2}^{\mathbb{C}}$ and $\Xi(\mathbf{p})$ is orthogonal to these vector fields. Thus, we get splittings of $T\mathbf{V}_{n,2}^{\mathbb{C}}$ and horizontal subbundles $\mathcal{H}_1(\mathbf{p}) \supset \mathcal{H}_2^a(\mathbf{p}) \supset \mathcal{H}_4(\mathbf{p})$ corresponding to the multifoliate structure (1.9). This allows us to construct almost hypercomplex structures on $\mathbf{V}_{n,2}^{\mathbb{C}}$ as follows:

Definition 1.7. Let $\mathcal{I}^a(\mathbf{p})$ denote the sections of $\operatorname{End}\mathcal{V}_4(\mathbf{p}) \oplus \operatorname{End}\mathcal{H}_4(\mathbf{p})$ defined by

(i) On $\mathcal{V}_4(\mathbf{p})$, $\mathcal{I}^a(\mathbf{p})\xi^a = -\epsilon^{abc}\xi^c + \delta^{ab}\Xi$ and $\mathcal{I}^a(\mathbf{p})\Xi(\mathbf{p}) = -\xi^a$.
(ii) On $\mathcal{H}_4(\mathbf{p})$, $\mathcal{I}^a(\mathbf{p}) = I_+^a$.

One easily checks that

Proposition 1.8. *For each $\mathbf{p} \in \hat{\mathfrak{t}}_n$ the endomorphisms $\mathcal{I}^a(\mathbf{p})$ define an $Sp(1)$-compatible almost hypercomplex structure on $\mathbf{V}_{n,2}^{\mathbb{C}}$.*

We now investigate the integrability of $\mathcal{I}^a(\mathbf{p})$. First, we have

Proposition 1.9. *The chain of horizontal subbundles $\mathcal{H}_1(\mathbf{p}) \supset \mathcal{H}_2^a(\mathbf{p}) \supset \mathcal{H}_4(\mathbf{p})$ defines a principal connection in the multifoliate structure* (1.9).

Proof. The vector field $\Xi(\mathbf{p})$ has unit norm with respect to $g_0(\mathbf{p})$ only if $\mathbf{p} = \mathbf{1}$. Thus, we set $\rho^2 = g_0(\mathbf{p})(\Xi(\mathbf{p}), \Xi(\mathbf{p}))$. One checks directly that ρ^2 is basic with respect to the foliation $\mathcal{F}_4(\mathbf{p})$. Letting $\eta^0(\mathbf{p}), \eta^1, \eta^2, \eta^3$ denote the 1-forms on $\mathbf{V}_{n,2}^{\mathbb{C}}$ dual to the vector fields $\Xi(\mathbf{p}), \xi^1, \xi^2, \xi^3$, respectively, we define another metric $h(\mathbf{p})$ on $\mathbf{V}_{n,2}^{\mathbb{C}}$ by rescaling $g_0(\mathbf{p})$ in the leaves of the foliation $\mathcal{F}_1(\mathbf{p})$ by the factor ρ^{-2}. That is,

$$h(\mathbf{p}) = \pi_h g(\mathbf{p}) + \rho^{-2}\pi_v g(\mathbf{p}), \tag{1.10}$$

where π_h and π_v denote the projections onto $\mathcal{H}_1(\mathbf{p})$ and $\mathcal{V}_1(\mathbf{p})$, respectively. One now sees that the result follows from the fact that $\Xi(\mathbf{p})$ and ξ^a are all Killing vector fields with respect to $h(\mathbf{p})$ for $a = 1,2,3$. □

The 1-forms $\{\eta^0(\mathbf{p}), \eta^a\}$ are the connection 1-forms of the connection $\mathcal{H}_4(\mathbf{p})$ with respect to the standard basis $\{e^0, e^a\}$ of the Lie algebra $\mathfrak{g}_4(\mathbf{p})$ of $G_4(\mathbf{p})$.

Theorem 1.10. *For each $\mathbf{p} \in \hat{\mathfrak{t}}_n$ the almost hypercomplex structures $\mathcal{I}^a(\mathbf{p})$ are integrable.*

Proof. Let $\{\omega^0, \omega^a\}$ denote the components with respect to the standard basis of the curvature of the connection $\eta = \eta^0(\mathbf{p})e^0 + \sum_a \eta^a e^a$. A straightforward but tedious computation shows that the vanishing of the Nijenhuis tensor of $\mathcal{I}^a(\mathbf{p})$ is equivalent to the two facts which are direct to verify:

(i) The curvature components ω^a are, up to a factor of 2, the restriction of the fundamental hyperhermitian forms to the horizontal subspaces $\mathcal{H}_4(\mathbf{p})$.

(ii) The curvature ω^0 is type $(1,1)$ with respect to all of the complex structures in the hypercomplex structure $\mathcal{I}^a(\mathbf{p})$.

□

It is straightforward to verify the following

Proposition 1.11. *The metric $h(\mathbf{p})$ is hyperhermitian with respect to the hypercomplex structure $\mathcal{I}^a(\mathbf{p})$ on $\mathbb{V}^{\mathbb{C}}_{n,2}$. Furthermore, the leaves of the foliations $\mathcal{F}_1(\mathbf{p})$, $\mathcal{F}^a_2(\mathbf{p})$, $\mathcal{F}_4(\mathbf{p})$ are totally geodesic with respect to $h(\mathbf{p})$. The vector field $\Xi(\mathbf{p})+\xi^a$ is holomorphic with respect to the complex structure $\mathcal{I}^a(\mathbf{p})$ for fixed a; hence, the foliation $\mathcal{F}^a_2(\mathbf{p})$ is holomorphic.*

As in [BGM2] the Weyl group $\mathcal{W}$ of $\mathfrak{sp}(n)$ acts on $\hat{\mathfrak{t}}_n \times \mathbb{V}^{\mathbb{C}}_{n,2}$ sending the hypercomplex structure $\mathcal{I}^a(\mathbf{p})$ to the hypercomplex structure $\mathcal{I}^a(w\mathbf{p})$ for $w\in\mathcal{W}$, and a representative for a Weyl group orbit is obtained by restricting $\mathbf{p}$ to the positive Weyl chamber $C_n \subset \hat{\mathfrak{t}}_n$, defined by $0<p_1\leq\cdots\leq p_n$. Thus, we have proved a more precise version of Theorem A of the introduction.

Theorem 1.12. *On $\mathbb{V}^{\mathbb{C}}_{n,2}$ two hypercomplex structures $\mathcal{I}^a(\mathbf{p})$ and $\mathcal{I}^a(\mathbf{p}')$ are equivalent if they lie on the same $\mathcal{W}$ orbit.*

Henceforth, we restrict $\mathbf{p}$ to lie in the positive Weyl chamber C_n. Now if the components of $\mathbf{p}$ are commensurable each leaf of the foliation $\mathcal{F}_4(\mathbf{p})$ is a hypercomplex Hopf surface of the form $S^1\times SU(2)/\Gamma$, where Γ is one of the finite subgroups of $SU(2)$, and each leaf of the foliation $\mathcal{F}^a_2(\mathbf{p})$ is an elliptic curve. If some pair of components of $\mathbf{p}$ are incommensurable the leaves of both foliations are noncompact, and the spaces of leaves are not Hausdorff. We say that $\mathbf{p}\in C_n$ is *commensurable* if all the components of $\mathbf{p}$ are commensurable. Otherwise $\mathbf{p}$ is called *incommensurable.* When $\mathbf{p}$ is commensurable the space of leaves $\mathcal{N}(\mathbf{p})/\mathcal{F}_1(\mathbf{p})$ is a 3-Sasakian orbifold $\mathcal{S}(\mathbf{p})$, and the space of leaves $\mathcal{Z}(\mathbf{p})=\mathcal{N}(\mathbf{p})/\mathcal{F}^a_2(\mathbf{p})$ is a complex orbifold. In fact, in a future work, we shall prove that the twistor space $\mathcal{Z}$ associated to a 3-Sasakian orbifold is always a projective algebraic variety. Notice that the orbifolds $\mathcal{S}(\mathbf{p})$ and $\mathcal{Z}(\mathbf{p})$ do not depend on the scale of $\mathbf{p}$; hence, if $\mathbf{p}'=\lambda\mathbf{p}$ for some $\lambda\in\mathbb{R}^+$, the respective hypercomplex structures $\mathcal{I}^a(\mathbf{p}')$ and $\mathcal{I}^a(\mathbf{p})$ can only differ along the leaves of the elliptic foliation $\mathcal{F}^a_2(\mathbf{p})$. We shall prove in Section 4 that this is indeed the case when $\mathbf{p}$ is commensurable (see Proposition 4.9).

We now discuss another isomorphic model of $\mathcal{N}(\mathbf{p})$ that will be useful in the next section. This model is related to Joyce's [Joy2] hypercomplex reduction. We begin again with the flat hypercomplex structure I^a_+ on $\mathbb{H}^n\setminus\{0\}$. We can easily check that the Euler vector field Ψ is annihilated by η^a_+. The consequence of this is that the moment map (1.7) is also a moment map with respect to the vector field $\Psi+\Xi(\mathbf{p})$ on $\mathbb{H}^n\setminus\{0\}$. For each $\mathbf{p}\in C_n$ it will be convenient to consider the one-parameter family of vector fields defined by $\Psi(\lambda)=\Psi+\Xi(\lambda\mathbf{p})$. Each vector field $\Psi(\lambda)$ generates an $\mathbb{R}$-action on $\mathbb{H}^n\setminus\{0\}$ which we denote by $\mathbb{R}(\lambda)$, and one can use the corresponding flow to show that the sphere S^{4n-1} is a totally transverse submanifold to this action. (Notice that if $\lambda=0$ this $\mathbb{R}$-action is just the radial flow on $\mathbb{H}^n\setminus\{0\}$.) Since the vector fields $\Xi(\mathbf{p})$ and Ψ are tangent to the zero set $\mu(\mathbf{p})^{-1}(0)=\mu(\lambda\mathbf{p})^{-1}(0)$, the action $\mathbb{R}(\lambda)$ restricts to an $\mathbb{R}$-action on $\mu(\mathbf{p})^{-1}(0)$. Thus, we have

Proposition 1.13. *For each $\lambda \in \mathbb{R}$ there is a natural identification $\mu(\mathbf{p})^{-1}(0)/\mathbb{R}(\lambda) \simeq \mathcal{N}(\mathbf{p})$ given by sending each $\mathbb{R}(\lambda)$ orbit in $\mu(\mathbf{p})^{-1}(0)$ to its point of intersection with the unit sphere S^{4n-1}. The inverse map sends $x \in \mathcal{N}(\mathbf{p})$ to the corresponding $\mathbb{R}(\lambda)$ orbit in $\mu(\mathbf{p})^{-1}(0)$.*

We shall use this quotient model in the next section to compute the Lie algebra $\mathfrak{h}(\mathbf{p})$ of infinitesimal automorphisms of $\mathcal{N}(\mathbf{p})$. From this point on we assume that $\lambda > 0$. First we show that $\mu(\mathbf{p})^{-1}(0)/\mathbb{R}(\lambda)$ has an induced hypercomplex structure given essentially by Joyce's hypercomplex reduction [Joy2]. To see this we define some vector bundles on $\mu(\mathbf{p})^{-1}(0)$. Let $\mathcal{L}_{\lambda\mathbf{p}}$ denote the subbundle of $T\mu(\mathbf{p})^{-1}(0)$ generated by $\Xi(\lambda\mathbf{p})+\Psi$, and $\mathcal{Q}$ its quotient bundle. Let $\iota : \mu(\mathbf{p})^{-1}(0) \to \mathbb{H}^n\setminus\{0\}$ be the natural inclusion, and let $N(\mathbf{p})$ denote the normal bundle. Note that $N(\mathbf{p})$ is spanned by the gradient fields $i\mathrm{grad}\ \mu^a(\mathbf{p})$. We also denote by $N_\lambda(\mathbf{p})$ the subbundle of $\iota^*T(\mathbb{H}^n\setminus\{0\})$ spanned by the vector fields $\xi^a_+ - i\mathrm{grad}\ \mu^a(\lambda\mathbf{p})$, and define $\mathcal{M}_{\lambda\mathbf{p}} = \mathcal{L}_{\lambda\mathbf{p}} \oplus N_\lambda(\mathbf{p})$. We have following commutative diagram of exact sequences

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{L}_{\lambda\mathbf{p}} \arrow[r] \arrow[d] & T\mu^{-1}(0) \arrow[r] \arrow[d] & \mathcal{Q}_{\lambda\mathbf{p}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{M}_{\lambda\mathbf{p}} \arrow[r] \arrow[d] & \iota^*T(\mathbb{H}^n\setminus\{0\}) \arrow[r] \arrow[d] & \mathcal{R}_{\lambda\mathbf{p}} \arrow[r] \arrow[d] & 0. \\
0 \arrow[r] & N_\lambda(\mathbf{p}) \arrow[r, "\simeq"] \arrow[d] & N(\mathbf{p}) \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd} \tag{1.11}$$

The hypercomplex structure I^a_+ restricts to give automorphisms of the bundle $\iota^*T(\mathbb{H}^n\setminus\{0\})$. Furthermore, the computations

$$I^a_+\Xi(\lambda\mathbf{p}) = -i\mathrm{grad}\ \mu^a(\lambda\mathbf{p}) \quad \text{and} \quad I^a_+\Psi = \xi^a_+ \tag{1.12}$$

show that the subbundle $\mathcal{M}_{\lambda\mathbf{p}}$ is invariant under I^a_+. Thus we have an induced hypercomplex structure on the quotient $\mathcal{R}_{\lambda\mathbf{p}}$ and hence, on $\mathcal{Q}_{\lambda\mathbf{p}}$ which we denote by $\bar{I}^a_+(\lambda\mathbf{p})$. Notice that on $\mathcal{Q}_{\lambda\mathbf{p}}$ the tensor field $\bar{I}^a_+$ depends on λ and $\mathbf{p}$, but only through the combination $\lambda\mathbf{p}$. Using the natural isomorphism between the fibers of $\mathcal{Q}_{\lambda\mathbf{p}}$ and the tangent spaces of $\mu(\mathbf{p})^{-1}(0)/\mathbb{R}(\lambda)$, this provides $\mu(\mathbf{p})^{-1}(0)/\mathbb{R}(\lambda)$ with a hypercomplex structure, denoted by $\bar{\bar{I}}^a_+(\lambda\mathbf{p})$.

Theorem 1.14. *Under the identifications $\mathcal{Q}_{\lambda\mathbf{p}} \simeq T\mu(\mathbf{p})^{-1}(0)/\mathbb{R}(\lambda) \simeq T\mathcal{N}(\mathbf{p})$ described above, the hypercomplex structures $\bar{I}^a_+(\lambda\mathbf{p})$, $\bar{\bar{I}}^a_+(\lambda\mathbf{p})$, and $\mathcal{I}^a(\lambda\mathbf{p})$ are equivalent.*

Proof. First Proposition 1.13 gives a diffeomorphism $\mu(\mathbf{p})^{-1}(0)/\mathbb{R}(\lambda) \simeq \mathcal{N}(\mathbf{p})$. We give the equivalence between the first and third hypercomplex structures. The hypercomplex structure $\{\bar{I}^a_+\}^3_{a=1}$ on $\mathcal{Q}_{\lambda\mathbf{p}}$ is that obtained by restriction and computing modulo the ideal generated by the vector field $\Xi(\lambda\mathbf{p}) + \Psi$. Definition 1.7 and the fact that the vector bundle $\tilde{\mathcal{H}}$ is independent of the scale of $\mathbf{p}$ show that the two hypercomplex structures coincide on $\tilde{\mathcal{H}}$. Then for a fixed a one easily checks that the endomorphisms $\bar{I}^a_+(\mathbf{p})$ and $\mathcal{I}^a(\mathbf{p})$ coincide as well on the subbundle spanned by ξ^b_+, ξ^c_+ for a, b, c all different. Using the definitions, (1.12), and computing modulo the $\mathcal{L}_{\lambda\mathbf{p}}$, we have

$$\bar{I}^a_+\Xi(\lambda\mathbf{p}) = -\bar{I}^a_+\Psi = -\xi^a_+ = \mathcal{I}^a(\lambda\mathbf{p})\Xi(\lambda\mathbf{p}). \tag{1.13}$$

A similar computation on ξ^a_+ concludes the proof. □

2. The Infinitesimal Automorphisms of $\mathcal{N}(\mathbf{p})$

In this section we compute the Lie algebra $\mathfrak{h}(\mathbf{p})$ of infinitesimal automorphisms of the hypercomplex Stiefel manifolds $\mathcal{N}(\mathbf{p})$ and thereby prove Theorem C of the introduction. Our proof crucially uses the scaling behavior in the parameter space C_n. The Lie algebra $\mathfrak{h}(\mathbf{p})$ will depend on the number of equalities among the components of $\mathbf{p} \in C_n$. Accordingly, we write $\mathbf{p} = (p_1^{m_1}, \ldots, p_k^{m_k})$ for some positive integer $k = k(\mathbf{p})$ where the component p_i occurs m_i times, and m_i is called the *multiplicity* of p_i. We define the *multiplicity* of $\mathbf{p}$ by $\mathbf{m} = (m_1, \ldots, m_k)$. Notice that $\sum_{i=1}^k m_i = n$.

Definition 2.1. Let $\mathfrak{h}(\mathbf{p})$, $\bar{\mathfrak{h}}(\mathbf{p})$, and $\bar{\bar{\mathfrak{h}}}(\mathbf{p})$ denote the Lie algebras of infinitesimal automorphisms of the hypercomplex structures $\mathcal{I}^a(\mathbf{p})$, $\bar{I}^a_+(\mathbf{p})$, and $\bar{\bar{I}}^a_+(\mathbf{p})$, respectively.

Notice that elements in $\bar{\bar{\mathfrak{h}}}(\mathbf{p})$ are vector fields on $\mu(\mathbf{p})^{-1}(0)/\mathbb{R}(\lambda)$, whereas elements in $\bar{\mathfrak{h}}(\mathbf{p})$ are equivalence classes of vector fields on $\mu(\mathbf{p})^{-1}(0)$ modulo sections of $\mathcal{L}_{\lambda\mathbf{p}}$. Then Theorem 1.14 gives the identifications

$$\mathfrak{h}(\mathbf{p}) \simeq \bar{\mathfrak{h}}(\mathbf{p}) \simeq \bar{\bar{\mathfrak{h}}}(\mathbf{p}). \tag{2.1}$$

In particular, every vector field X representing an element $\bar{X} \in \bar{\mathfrak{h}}(\mathbf{p})$ is projectible to a vector field $\bar{\bar{X}} \in \bar{\bar{\mathfrak{h}}}(\mathbf{p})$. It follows from the vanishing of the first prolongation $\mathfrak{gl}(n, \mathbb{H})^{(1)}$ that the Lie algebra $\mathfrak{h}(\mathbf{p})$ is finite dimensional.

Definition 2.2. Let $\mathfrak{h}_1(\mathbf{p})$ denote the Lie algebra of vector fields on $\mu(\mathbf{p})^{-1}(0)$ that satisfy $\mathcal{L}_X I^a_+(\mathbf{p}) = 0$.

Recall the vector fields H_j of equation (1.5). It is easy to check that $\Psi, H_1, \cdots, H_n$ generate an $(n+1)$-dimensional abelian Lie algebra which we denote by $\mathfrak{t}_{n+1}$. Notice that for all $\mathbf{p}$ the vector field $\Xi(\mathbf{p})$ lies in the subalgebra $\mathfrak{t}_n$ of $\mathfrak{t}_{n+1}$ spanned by the H_j.

Lemma 2.3. *For each $\mathbf{p} \in C_n$, $\mathfrak{t}_{n+1}$ is a Lie subalgebra of $\mathfrak{h}_1(\mathbf{p})$. Moreover, for each $\lambda \in \mathbb{R}^+$ we have $\mathfrak{h}_1(\lambda\mathbf{p}) = \mathfrak{h}_1(\mathbf{p})$.*

Proof. The first statement follows by a straightforward computation. The second statement follows from the equalities $\mu(\lambda\mathbf{p})^{-1}(0) = \mu(\mathbf{p})^{-1}(0)$ and $I^a_+(\lambda\mathbf{p}) = I^a_+(\mathbf{p})$. □

Lemma 2.3 allows us to define linear maps $\pi_\lambda : \mathfrak{h}_1(\mathbf{p}) \to \bar{\mathfrak{h}}(\lambda\mathbf{p})$ for each $\lambda \in \mathbb{R}^+$ by sending each $X \in \mathfrak{h}_1(\mathbf{p})$ to its equivalence class modulo $\Psi + \Xi(\lambda\mathbf{p})$.

We have

Proposition 2.4. *For every* $\mathbf{p} \in C_n$ *and each* $\lambda \in \mathbb{R}^+$ *there is an exact sequence of Lie algebras*

$$0 \longrightarrow \{\Psi + \Xi(\lambda\mathbf{p})\} \longrightarrow \mathfrak{h}_1(\mathbf{p}) \xrightarrow{\pi_\lambda} \bar{\mathfrak{h}}(\lambda\mathbf{p}) \longrightarrow 0.$$

This exact sequence splits so there are Lie algebra isomorphisms:

$$\mathfrak{h}_1(\mathbf{p}) \simeq \bar{\mathfrak{h}}(\mathbf{p}) \oplus \{\Psi + \Xi(\mathbf{p})\} \simeq \mathfrak{h}(\mathbf{p}) \oplus \{\Psi\}.$$

Moreover, we have

(i) *The two-dimensional abelian algebra spanned by* Ψ *and* $\Xi(\mathbf{p})$ *is central in* $\mathfrak{h}_1(\mathbf{p})$.

(ii) $\mathfrak{h}(\lambda\mathbf{p}) = \mathfrak{h}(\mathbf{p})$ *for all* $\lambda \in \mathbb{R}^+$.

Proof. The one-dimensional vector space spanned by $\Psi + \Xi(\lambda\mathbf{p})$ is clearly the kernel of π_λ, and since all vector fields in $\mathfrak{h}_1(\mathbf{p})$ are projectible this vector space is a Lie algebra ideal. So the first statement will follow from the surjectivity of the map π_λ. We prove the surjectivity together with (ii) by considering the equivalence of the two models $(\mathcal{N}(\mathbf{p}), \mathcal{I}^a(\mathbf{p}))$ and $(\mathcal{Q}_\mathbf{p}, I^a_+(\mathbf{p}))$. Let $\phi_{\lambda\mathbf{p}}(t)$ denote the flow generated by the vector field $\Psi + \Xi(\lambda\mathbf{p})$. We see that

$$\mu(\lambda\mathbf{p})^{-1}(0) = \mu(\mathbf{p})^{-1}(0) = \{\phi_{\lambda\mathbf{p}}(t)(\mathcal{N}(\lambda\mathbf{p})) \mid t \in \mathbb{R}\}.$$

This gives an isomorphism $\pi_\lambda \circ \phi_{\lambda\mathbf{p}}(t)_* : T\mathcal{N}(\lambda\mathbf{p}) \to \mathcal{Q}_{\lambda\mathbf{p}}$, and induces the equivalence between $\mathcal{I}^a(\lambda\mathbf{p})$ and $\bar{I}^a_+(\lambda\mathbf{p})$. Then we have for any vector field X on $\mathcal{N}$

$$\mathcal{L}_{\pi_\lambda \circ \phi_{\lambda\mathbf{p}}(t)_* X} \bar{I}^a_+(\lambda\mathbf{p}) = \pi_\lambda \phi_{\lambda\mathbf{p}}(t)_* \mathcal{L}_X \mathcal{I}^a(\lambda\mathbf{p}) \circ (\pi_\lambda \phi_{\lambda\mathbf{p}}(t)_*)^{-1}. \tag{2.2}$$

Now choose any splitting of the top horizontal exact sequence in diagram (1.11), and accordingly consider $\mathcal{Q}_{\lambda\mathbf{p}}$ as a subspace of $T\mu(\mathbf{p})^{-1}(0)$. Let X be any vector field on $\mu(\mathbf{p})^{-1}(0)$ that commutes with $\Xi(\mathbf{p})$. Then $\phi_{\lambda\mathbf{p}}(t)_* X = \psi(t)_* X$, where $\psi(t)$ is the flow of the Euler field Ψ. Notice that this vector field is independent of both λ and $\mathbf{p}$. Moreover, $I^a_+(\lambda\mathbf{p}) = I^a_+(\mathbf{p})$ for all λ, so on the subspace $\mathcal{Q}_{\lambda\mathbf{p}}$ equation (2.2) lifts to

$$\mathcal{L}_{\psi(t)_* X} I^a_+(\mathbf{p}) = (\Psi + \Xi(\lambda\mathbf{p})) \otimes b_{\psi(t)_* X} + \phi_{\lambda\mathbf{p}}(t)_* \mathcal{L}_X \mathcal{I}^a(\lambda\mathbf{p}) \circ (\phi_{\lambda\mathbf{p}}(t)_*)^{-1}, \tag{2.3}$$

where $b_{\psi(t)_* X}$ is some 1-form on $\mu(\mathbf{p})^{-1}(0)$. Now the left hand side of (2.3) is independent of λ, and the Euler field Ψ is not in the image of $\phi_{\lambda\mathbf{p}}(t)_*$. It follows that the two terms in the right hand side of equation (2.3) must both be separately independent of λ. In particular, if $X \in \mathfrak{h}(\lambda\mathbf{p})$ for a fixed λ it must be in $\mathfrak{h}(\lambda\mathbf{p})$ for all λ. This proves (ii).

Proving the surjectivity of π_λ amounts to proving that the isomorphism

$$\pi_\lambda \circ \phi_{\lambda\mathbf{p}}(t)_* : \mathfrak{h}(\lambda\mathbf{p}) \to \bar{\mathfrak{h}}(\lambda\mathbf{p})$$

factors through $\mathfrak{h}_1(\mathbf{p})$. From (2.3) we have for $X \in \mathfrak{h}(\lambda\mathbf{p})$

$$\mathcal{L}_{\psi(t)_* X} I^a_+(\mathbf{p}) = (\Psi + \Xi(\lambda\mathbf{p})) \otimes b_{\psi(t)_* X} \tag{2.4}$$

on $\mathcal{Q}_{\lambda\mathbf{p}}$. The independence of λ once again implies that $b_{\psi(t)_* X}$ vanishes on $\mathcal{Q}_{\lambda\mathbf{p}}$. We claim that it vanishes on all of $T\mu(\mathbf{p})^{-1}(0)$. A simple calculation using (2.4) shows that

$$[\psi(t)_* X, \Psi] = b_{\psi(t)_* X}(\Psi) I^a_+(\mathbf{p})(\Psi + \Xi(\lambda\mathbf{p})).$$

Again the left hand side is independent of λ so that $b_{\psi(t)_*X}(\Psi) = 0$. But since the vector space generated by Ψ is complementary to $\mathcal{Q}_{\lambda\mathbf{p}}$ we must have $b_{\psi(t)_*X} = 0$, implying that $\psi(t)_*X \in \mathfrak{h}_1(\mathbf{p})$. Notice that this argument also implies that Ψ is central in $\mathfrak{h}_1(\mathbf{p})$. Similarly it is not difficult to show that $\Xi(\mathbf{p})$ is central in $\mathfrak{h}_1(\mathbf{p})$ proving (i).

It is now easy to see that the map $\psi(t)_* \circ (\pi_\lambda \circ \phi_{\lambda\mathbf{p}}(t)_*)^{-1} : \bar{\mathfrak{h}}(\lambda\mathbf{p}) \to \mathfrak{h}_1(\mathbf{p})$ splits the exact sequence, and the string of isomorphisms follows easily. □

Proposition 2.5. *Every infinitesimal hypercomplex automorphism $X \in \mathfrak{h}(\mathbf{p})$ commutes with the vector fields ξ^a. In particular, every such X is foliate with respect to the foliation $\mathcal{F}_4(\mathbf{p})$. Furthermore, every infinitesimal hypercomplex automorphism is an infinitesimal isometry with respect to the metric $h(\mathbf{p})$.*

Proof. Consider the endomorphism

$$\mathcal{I}^a(\lambda\mathbf{p}) - \mathcal{I}^a(\mathbf{p}) = (\lambda - 1)\Xi(\mathbf{p}) \otimes \hat{\eta}^a - (\lambda^{-1} - 1)\hat{\xi}^a \otimes \hat{\eta}^0(\mathbf{p}). \tag{2.5}$$

Now for $X \in \mathfrak{h}(\mathbf{p})$ (ii) of Proposition 2.4 implies that the Lie derivative of the left hand side of this expression with respect to X vanishes. This gives

$$(\lambda - 1)\Xi(\mathbf{p}) \otimes \mathcal{L}_X\eta^a - (\lambda^{-1} - 1)([X, \xi^a] \otimes \eta^0 + \xi^a \otimes \mathcal{L}_X\eta^0) = 0. \tag{2.6}$$

Now one easily checks that $(\mathcal{L}_X\eta^0)(\Xi(\mathbf{p})) = (\mathcal{L}_X\eta^a)(\Xi(\mathbf{p})) = 0$. Thus, evaluating (2.6) on $\Xi(\mathbf{p})$ proves the first statement, and then post-composing (2.6) with η^0 and η^a gives

$$\mathcal{L}_X\eta^a = \mathcal{L}_X\eta^0 = 0. \tag{2.7}$$

This in turn implies that the tensor field $\mathcal{L}_X h(\mathbf{p})$ is horizontal with respect to the foliation $\mathcal{F}_4(\mathbf{p})$. But for any sections Y, Z of $\mathcal{H}_4(\mathbf{p})$ we have $2h(\mathbf{p})(\mathcal{I}^a(\mathbf{p})Y, Z) = d\eta^a(Y, Z)$. The last statement now follows from (2.7). □

Corollary 2.6. *The group* $\operatorname{Aut}\mathcal{N}(\mathbf{p})$ *of hypercomplex automorphisms of* $\mathcal{N}(\mathbf{p})$ *is compact.*

Proof. It follows from Proposition 2.5 $\operatorname{Aut}\mathcal{N}(\mathbf{p})$ is a subgroup of the isometry group $I(\mathcal{N}(\mathbf{p}), h(\mathbf{p}))$ which is compact by a well-known result of Myers and Steenrod. But it is immediate that any subgroup of a Lie transformation group that is defined by an invariance condition on a tensor field is a closed subgroup. □

Lemma 2.7. *If $X \in \mathfrak{h}(\mathbf{p})$ then $X\rho^2 = 0$; hence, X is a Killing vector field with respect to $g(\mathbf{p})$.*

Proof. The second statement follows from the first since $g(\mathbf{p}) = h(\mathbf{p}) + (\rho^2 - 1)(\hat{\eta}^0)^2$, where ρ^2 is the basic function defined in the previous section. To prove the first statement we consider the curvature form $\omega^0 = d\eta^0$. From (2.7) this is an invariant of the infinitesimal hypercomplex automorphisms. From this fact a direct computation shows that the 1-form

$$\phi = -\rho^4 \frac{d(\rho^{-2}(X\rho^2))}{2X\rho^2}$$

is closed. But it is easy to see that this implies the existence of a smooth function F depending only on ρ^2 such that $X\rho^2 = F(\rho^2)$. Now by (ii) of Proposition 2.4 we can take X to be independent of λ and ρ^2 is a polynomial of degree 2 in the components of $\mathbf{p}$. This implies the scaling behavior $F(\rho^2(\lambda\mathbf{p})) = \lambda^2 F(\rho^2(\mathbf{p}))$, and this implies that $F(\rho^2) = \alpha\rho^2$ for some real constant α. Then the 1-parameter group generated by X is a dilatation of ρ^2. But on $\mathcal{N}(\mathbf{p})$ we have the bound $\rho^2 \leq |\mathbf{p}|^2$. This gives a contradiction unless $\alpha = 0$. □

This lemma allows us to use the restriction of the flat metric to the surface $\iota : \mathcal{N}(\mathbf{p}) \to \mathbb{H}^n \setminus \{0\}$. On $\mathcal{N}(\mathbf{p})$ the normal bundle has a global orthonormal frame (and is thus trivial) given by $(n^0 = \Psi, n^a = \frac{-1}{\rho} i\mathrm{grad}\ \mu^a(\mathbf{p}))$. The dual coframe is $\tau^0 = dr$, $\tau^a = \frac{i}{\rho} d\mu^a(\mathbf{p})$, where $r^2 = \sum_i |\mathbf{u}_i|^2$. Thus the metric in the vector bundle $\iota^* T\mathbb{H}^n$ on $\mathcal{N}(\mathbf{p})$ obtained by restricting the flat metric to $\mathcal{N}(\mathbf{p})$ is $g_0 = g(\mathbf{p}) + \sum_a (\tau^a)^2 + (\tau^0)^2$. We now have

Proposition 2.8. *Every $X \in \mathfrak{h}(\mathbf{p})$ is the restriction to $\mathcal{N}(\mathbf{p})$ of a Killing vector field $\hat{X}$ on S^{4n-1} with respect to the canonical metric.*

Proof. By Lemma 2.7 we know that any $X \in \mathfrak{h}(\mathbf{p})$ is a Killing vector field with respect to the metric $g(\mathbf{p})$ on $\mathcal{N}(\mathbf{p})$. Moreover, according to either Theorem 18 or 19 of volume IV of Spivak [Sp], if the metric g_0, the second fundamental forms, and the normal fundamental forms are invariant under every $X \in \mathfrak{h}(\mathbf{p})$, then X is the restriction of an infinitesimal Euclidean motion. But since X itself is already tangent to the sphere, any extension to an infinitesimal Euclidean motion will be in $\mathfrak{so}(4n)$. To show that the fundamental forms are invariant we prove

Lemma 2.9. *If $X \in \mathfrak{h}(\mathbf{p})$ then $\mathcal{L}_X n^0 = \mathcal{L}_X n^a = \mathcal{L}_X \tau^0 = \mathcal{L}_X \tau^a = 0$.*

Proof. The invariance of n^0 and τ^0 is immediate. To see the other conditions we compute $[X, -i\mathrm{grad}\ \mu^a(\mathbf{p})] = [X, I^a_+(\mathbf{p})\Xi(\mathbf{p})] = (\mathcal{L}_X I^a_+)(\Xi(\mathbf{p})) + [X, \Xi(\mathbf{p})] = 0$. Then the result follows from Lemma 2.7 □

Continuing with the proof of Proposition 2.8 we notice that Lemmas 2.7 and 2.9 imply that $\mathcal{L}_X g_0 = 0$. Thus, if we denote by ∇^0 the connection in $\iota^* T\mathbb{H}^n$ induced by the flat Levi-Civita connection on $\mathbb{H}^n$, then also $\mathcal{L}_X \nabla^0 = 0$. The second fundamental form is given by the projection $P_N \nabla^0$ where $P_N = \sum_{\nu=0}^3 n^\nu \otimes n^\nu \lrcorner g_0$. Thus, Lemma 2.9 implies the invariance of the second fundamental form. Similarly, the normal fundamental forms are given by $\beta^b_a = g_0(\nabla^0 n^a, n^b)$, so again $\mathcal{L}_X \beta^b_a = 0$ by Lemma 2.9. □

To conclude the proof of Theorem C we see that any $X \in \mathfrak{h}(\mathbf{p})$ generates a unique linear map on $\mathbb{H}^n \setminus \{0\}$ that also satisfies $\mathcal{L}_X I^a_+(\mathbf{p}) = 0$. It follows that X is the restriction of an element $\hat{X} \in \mathfrak{gl}(n, \mathbb{H})$. But we must also satisfy $[X, \Xi(\mathbf{p})] = 0$. The proof of the following lemma is standard and left to the reader

Lemma 2.10. *The centralizer of $\Xi(\mathbf{p})$ in $\mathfrak{gl}(n, \mathbb{H})$ is $\bigoplus_{i=1}^{k} \mathfrak{gl}(m_i, \mathbb{C})$.*

Now Theorem C follows from $\mathfrak{so}(4n) \cap \bigoplus_{i=1}^{k} \mathfrak{gl}(m_i, \mathbb{C}) = \bigoplus_{i=1}^{k} \mathfrak{u}(m_i)$. □

Finally we remark that $\mathfrak{h}(\mathbf{p})$ is clearly an invariant of the hypercomplex structure $\mathcal{I}^a(\mathbf{p})$. However, since $\mathfrak{h}(\mathbf{p})$ depends on $\mathbf{p}$ only through the multiplicities m_i as an unordered set of natural numbers, it is a very crude invariant. We shall prove in Section 4 that, for commensurable $\mathbf{p}$, $\mathcal{N}(\mathbf{p})$ and $\mathcal{N}(\mathbf{p}')$ are equivalent only if $\mathbf{p} = \mathbf{p}'$, whereas Theorem C implies

Corollary 2.11. *The Lie algebras $\mathfrak{h}(\mathbf{p})$ and $\mathfrak{h}(\mathbf{p}')$ are abstractly isomorphic if and only if $k(\mathbf{p}) = k(\mathbf{p}')$ and there is a permutation σ in the symmetric group Σ_k on k letters such that $\mathbf{m}' = \sigma\mathbf{m}$.*

3. Singularities and Leaf Holonomy

In this section we analyze the holonomy of the singular leaves of the $\mathcal{F}_1$ and $\mathcal{F}_4$ foliations on $\mathcal{N}(\mathbf{p})$. In the special case when $\mathbf{p} = (q, p, \ldots, p)$ such an analysis can be found in [GL]. Here we generalize this study to arbitrary $\mathbf{p}$ in order to solve the hypercomplex equivalence problem in the next section.

Definition 3.1. Let $\mathbf{q} = (q_1, \ldots, q_n) \in C_n$. If $\mathbf{q}$ is commensurable then there is a positive number λ so that $\mathbf{q} = \lambda\mathbf{p} = \lambda(p_1, \ldots, p_n)$ where $\mathbf{p} \in C_n$ with each $p_i \in \mathbb{Z}$ and the greatest common divisor $\gcd(p_1, \ldots, p_n) = 1$. We shall call this $\mathbf{p}$ the *basic commensurable sequence* associated to $\mathbf{q}$. A basic commensurable sequence is called *coprime* if the p_i coordinates are pairwise relatively prime.

Given a basic commensurable sequence $\mathbf{p} = (p_1, \ldots, p_n) \in C_n$ we consider ordered subsequences $\mathbf{p}^{(l)} = (p_{i_1}, \ldots, p_{i_l})$ (which are not necessarily basic), and denote the length of the subsequence by l. To each such subsequence $\mathbf{p}^{(l)}$ with $l \geq 2$ we can associate an embedded submanifold $\mathcal{N}(\mathbf{p}^{(l)}) \subset \mathcal{N}(\mathbf{p})$ defined by setting the quaternionic coordinates $u_j = 0$ for all $j \neq i_1, \ldots, i_l$. We only consider subsequences with length $l \geq 2$. As the notation indicates one can easily verify

Proposition 3.2. *The submanifolds $\mathcal{N}(\mathbf{p}^{(l)})$ are hypercomplex Stiefel manifolds of real dimension $4l - 4$ with hypercomplex structure $\mathcal{I}^a(\mathbf{p}^{(l)})$.*

The submanifolds $\mathcal{N}(\mathbf{p}^{(2)})$ of length 2 will play a special role in what follows. For $\mathbf{p}^{(2)} = (p_i, p_j)$ we denote these submanifolds by H_{ij} and refer to them as *vertices.* Our next task is to compute the holonomy groups of the singular leaves of the foliations $\mathcal{F}_4(\mathbf{p})$ and $\mathcal{F}_1(\mathbf{p})$ when $\mathbf{p}$ is a basic commensurable sequence. Recall that these foliations were defined in Section 1. First we analyze the foliation $\mathcal{F}_1$. For a leaf L of $\mathcal{F}_1(\mathbf{p})$ we denote its holonomy group by $G_1(L)$. The circle action on $\mathbb{H}^n$ generated by the vector field $\Xi(\mathbf{p})$ is given by

$$(u_1, \ldots, u_n) \mapsto (e^{2\pi i p_1 t} u_1, \ldots, e^{2\pi i p_n t} u_n). \tag{3.1}$$

Notice that if a leaf L of $\mathcal{F}_1$ intersects one of the submanifold $\mathcal{N}(\mathbf{p}^{(l)})$ then L lies entirely in $\mathcal{N}(\mathbf{p}^{(l)})$. Thus, we have

Lemma 3.3. *Let L be a leaf of $\mathcal{F}_1$ lying on the submanifold $\mathcal{N}(\mathbf{p}^{(l)})$ corresponding to the subsequence $\mathbf{p}^{(l)} = (p_{i_1}, \ldots, p_{i_l})$ of length l. Then the holonomy group of L is $G_1(L) = \mathbb{Z}_{r_{i_1 \ldots i_l}}$ where $r_{i_1 \ldots i_l} = \gcd(p_{i_1}, \cdots, p_{i_l})$.*

Next we consider the holonomy groups of the foliation $\mathcal{F}_4$. If L is a leaf of $\mathcal{F}_4$ we denote its holonomy group by $G_4(L)$. Since the action of $SU(2)$ on the quaternionic coordinates $\mathbf{u}$ is the diagonal action on it components, a leaf of $\mathcal{F}_4$ is either disjoint from the submanifold $\mathcal{N}(\mathbf{p}^{(l)})$ or it lies entirely in it.

Lemma 3.4. *The manifolds H_{ij} are Hopf surfaces diffeomorphic to $S^1 \times S^3$ and there is a covering map $H_{ij} \to L_{ij}$ where L_{ij} is a leaf of the foliation $\mathcal{F}_4(\mathbf{p})$. Moreover, the holonomy group $G_4(L_{ij})$ is given by*

$$G_4(L_{ij}) = \begin{cases} \mathbb{Z}_{\frac{p_i+p_j}{2}} & \text{if } p_i + p_j \text{ is even for all } i,j; \\ \mathbb{Z}_{p_i+p_j} & \text{otherwise.} \end{cases}$$

The leaf L_{ij} is singular unless $p_i = p_j = 1$ and all other sums $p_k + p_m$ are even. Furthermore, the holonomy group of any singular leaf L of $\mathcal{F}_4$ that lies in a submanifold $\mathcal{N}(\mathbf{p}^{(l)})$ but not in a submanifold of smaller length has the form

$$G_4(L) = \begin{cases} \mathbb{Z}_{\frac{g}{2}} & \text{if } p_i + p_j \text{ is even for all } i,j; \\ \mathbb{Z}_g & \text{otherwise.} \end{cases}$$

Here g is the gcd *of the sequences of length at least $l-1$ whose elements are of the form $p_{i_l} \pm p_j$ where $j = i_1, \ldots, i_{l-1}$. In particular, if the sequence $\mathbf{p}^{(l)}$ contains both i and j and U_{ij} is a neighborhood of the vertex H_{ij} in $\mathcal{N}(\mathbf{p}^{(l)})$, then $g \le p_i + p_j$ for all leaves intersecting U_{ij}.*

Proof. Clearly H_{ij} is equal to $\mathcal{N}(\mathbf{p}^{(2)})$ where $\mathbf{p}^{(2)} = (p_i, p_j)$ which is, in turn, diffeomorphic to $\mathcal{N}(1,1) \simeq U(2)$. Thus, H_{ij} is a Hopf surface. This last diffeomorphism is given by writing $u_i = z_i^1 + z_i^2 j$ and forming the $U(2)$ matrix

$$Z = \begin{pmatrix} z_i^1 & z_i^2 \\ z_j^1 & z_j^2 \end{pmatrix}.$$

Now the action of $U(2)$ on $\mathcal{N}(\mathbf{p}^{(2)})$ restricts to a $U(2)$ action on H_{ij}. Furthermore, the diffeomorphism above intertwines the action of $U(2)$ on $\mathcal{N}(\mathbf{p}^{(2)})$ with its action on $\mathcal{N}(\mathbf{p})$. This later action is given by sending the $Z \in U(2)$ to the matrix

$$\begin{pmatrix} e^{2\pi i p_i t} & 0 \\ 0 & e^{2\pi i p_j t} \end{pmatrix} Z\sigma, \tag{3.2}$$

where $t \in [0,1]$ and $\sigma \in SU(2)$. By Lemma 3.3 the isotropy subgroup of the circle group $U(1)_{\mathbf{p}}$ acting on H_{ij} is $\mathbb{Z}_{r_{ij}}$. Thus, the quotient of H_{ij} with respect to this circle action is a 3-Sasakian manifold S^3/Γ. We need to determine the group Γ. Since S^3/Γ is a homogeneous manifold with respect to the $SU(2)$ action we can use $\sigma \in SU(2)$ to set $z_j^2 = 0$. The moment map equations for H_{ij} then imply that $z_i^1 = z_j^2 = 0$. The isotropy subgroup of this set of points is the torus subgroup T^2 obtained by setting $\sigma = \begin{pmatrix} e^{2\pi i s} & 0 \\ 0 & e^{-2\pi i s} \end{pmatrix}$ in (3.2). Then the isotropy subgroup $\Gamma \subset T^2$ is determined by the conditions

$$s - p_j t = 0 \quad \text{and} \quad s + p_i t = 1.$$

This implies that $\Gamma = \mathbb{Z}_{p_i+p_j}$. This is the holonomy group if $SU(2)$ acts effectively on $\mathcal{N}(\mathbf{p})$, and then, as in the proof of Proposition 7.16 of [BGM2], this occurs only if $p_k + p_l$ is odd for some pair (p_k, p_l) of components of $\mathbf{p}$. If all of the $\frac{n(n-1)}{2}$ sums of pairs of components of $\mathbf{p}$ are even, the action of $SU(2)$ is not effective and one has an effective action of the factor group $SO(3)$. In this case $\Gamma = \mathbb{Z}_{\frac{p_i+p_j}{2}}$.

To determine the holonomy group of singular leaves L that are not covered by a vertex we argue as above and, by using a choice of σ, set $z_{i_l}^2 = 0$ and then choose $s = p_{i_l}t$. The conditions for fixed points are

$$z_j^1 = e^{2\pi i(p_{i_l}-p_j)t} z_j^1 \quad \text{and} \quad z_j^2 = e^{2\pi i(p_{i_l}+p_j)t} z_j^2 \tag{3.3}$$

for all $j = i_1, \ldots, i_{l-1}$. Since $\mathbf{u}$ does not lie on a sub-Stiefel of smaller length than l, both z_j^1 and z_j^2 cannot vanish. Thus, at most $l-1$ of the z_j^1's and z_j^2's can vanish while the moment map constraints imply that not all the z_j^2's can vanish. It follows that the order of $G_4(L)$ is as stated in Lemma 3.4 where g is the gcd of a sequence of elements of the form $p_{i_l} \pm p_j$ of length at least $l-1$. To prove the last statement we notice that in a neighborhood of a vertex H_{ij} in $\mathcal{N}(\mathbf{p}^{(l)})$ we must have $u_i, u_j \neq 0$. As before we can set $z_j^2 = 0$ and choose $s = p_j t$. Then since $u_i \neq 0$, g must divide either $|p_j - p_i|$ or $p_j + p_i$. In either case $g \leq p_i + p_j$. □

The proof of Lemma 3.4 implies that there are two types of Stiefel manifolds $\mathcal{N}(\mathbf{p})$, namely, those for which $SU(2)$ acts effectively, and those where its $\mathbb{Z}_2$ factor group $SO(3)$ acts effectively. We refer to these two type as *type* 1 and *type* 2, respectively. The type also gives the generic fibre of the orbifold fibration $\mathcal{N}(\mathbf{p}) \to \mathcal{O}(\mathbf{p})$. For type 1 the generic fibre is $S^1 \times SU(2)$, whereas for type 2 it is $S^1 \times SO(3)$. As illustrated in the proof above type 2 occurs if and only if the sums $p_i + p_j$ are even for all i, j.

Given a basic commensurable sequence $\mathbf{p}$, consider the case that a component p_i has multiplicity $m_i > 1$. Then we let $\mathbf{p}_i^{(m_i)}$ denote the subsequence $(p_i, \ldots, p_i)$ with m_i repetitions. We have

Proposition 3.5. *If $m_i \geq 2$ then $\mathcal{N}(\mathbf{p}_i^{(m_i)})$ is a homogeneous hypercomplex Stiefel manifold of real dimension $4m_i - 4$. In this case the holonomy group $G_1(L)$ of any leaf L of the foliation $\mathcal{F}_1(\mathbf{p})$ lying in $\mathcal{N}(\mathbf{p}_i^{(m_i)})$ is $\mathbb{Z}_{p_i}$ and the holonomy group $G_4(L)$ of any leaf of the foliation $\mathcal{F}_4(\mathbf{p})$ lying in $\mathcal{N}(\mathbf{p}_i^{(m_i)})$ is given by the equation*

$$G_4(L) = \begin{cases} \mathbb{Z}_{p_i} & \text{if } p_k + p_j \text{ is even for all } k, j; \\ \mathbb{Z}_{2p_i} & \text{otherwise.} \end{cases}$$

Proof. It suffices to prove the homogeneity statement as the remainder of the proposition follows easily from Lemmas 3.3 and 3.4. The group Aut $\mathcal{N}(\mathbf{p})$ of hypercomplex automorphisms of $\mathcal{N}(\mathbf{p})$ is given explicitly in Theorem C, and since $\mathbf{p}$ contains the subsequence $\mathbf{p}_i^{(m_i)}$, Aut $\mathcal{N}(\mathbf{p})$ must contain a subgroup isomorphic to $U(m_i)$. Furthermore, one can check that $U(m_i)$ acts transitively on $\mathcal{N}(\mathbf{p}_i^{(m_i)})$. □

Let $\Sigma(\mathbf{p})$ denote the singular locus of the quaternionic Kähler orbifold $\mathcal{O}(\mathbf{p})$. Since the holonomy group of any leaf of $\mathcal{F}_4(\mathbf{p})$ is a cyclic group, we can stratify $\mathcal{O}(\mathbf{p})$ and thus $\Sigma(\mathbf{p})$ according to the orders of the holonomy groups G_4. For convenience we say that a leaf L of $\mathcal{F}_4(\mathbf{p})$ has *order* p if its holonomy group $G_4(L)$ has order p. Notice that in general a stratum Σ_p can be disconnected. However, the stratum consisting of leaves of maximal order is special. We have

Proposition 3.6. *Let Σ_{p_n} denote the stratum of the singular locus consisting of leaves of maximal order.*

(1) *If* $m_n = 1$ *then* Σ_{p_n} *consists of* m_{n-1} *isolated points and the associated leaves are Hopf surfaces* $S^1 \times S^3/\mathbb{Z}_{r_n}$ *that are covered by vertices. Here* $r_n = p_n + p_{n-1}$ *or* $r_n = \frac{p_n+p_{n-1}}{2}$ *if* $\mathcal{N}(\mathbf{p})$ *is type* 1 *or type* 2, *respectively.*

(2) *If* $m_n \geq 2$ *then* Σ_{p_n} *is a homogeneous Wolf space* $\mathbb{G}_{m_n,2}(\mathbb{C})$. *In particular, if* $m_n = 2$ *then* $\Sigma_{p_n} = \mathbb{G}_{2,2}(\mathbb{C}) = \{pt\}$ *is an isolated point, and the leaf is a homogeneous hypercomplex Hopf surface* $S^1 \times S^3$ *or* $S^1 \times SO(3)$ *if* $\mathcal{N}(\mathbf{p})$ *is type* 1 *or type* 2, *respectively.*

Note that the leaves of maximal order for $m_n \geq 2$ are all Hopf surfaces, just as the generic non-singular leaf. However, they are not isometric to the generic leaf as they have non-trivial $U(1)$ holonomy.

4. Holonomy and the Equivalence Problem

In this section we show that an equivalence between hypercomplex structures on $\mathbb{V}^{\mathbb{C}}_{n,2}$ implies an equivalence between the various foliations, and then use the holonomy of the singular leaves to solve the equivalence problem for the hypercomplex structures $\mathcal{I}^a(\mathbf{p})$ on $\mathbb{V}^{\mathbb{C}}_{n,2}$ for commensurable $\mathbf{p}$. The equivalence problem for the case of incommensurable $\mathbf{p}$ is more involved and is relegated to future work. However, the fact that $\mathcal{I}^a(\mathbf{p})$ and $\mathcal{I}^a(\mathbf{p}')$ are inequivalent when $\mathbf{p}$ is commensurable and $\mathbf{p}'$ is incommensurable will follow immediately from our considerations below.

Definition 4.1. *We say that hypercomplex structures* $\mathcal{I}$ *and* $\acute{\mathcal{I}}$ *are equivalent if there is a choice of generators* $\mathcal{I}^a$ *and* $\acute{\mathcal{I}}^a$ *for* $\mathcal{I}$ *and* $\acute{\mathcal{I}}$, *respectively, and an* $F \in \mathrm{Diff}(\mathbb{V}^{\mathbb{C}}_{n,2})$ *such that*

$$F_*\mathcal{I}^a = \acute{\mathcal{I}}^a F_* \tag{4.1}$$

for all $a = 1, 2, 3$. *Alternatively,* F *is a map of Clifford algebras.*

Recall the definition of $Sp(1)$-compatible hypercomplex structures given by Definition 1.1.

Lemma 4.2. *Let* $\mathcal{I}$ *and* $\acute{\mathcal{I}}$ *be two hypercomplex structures on* $\mathbb{V}^{\mathbb{C}}_{n,2}$ *that are compatible with two given* $Sp(1)$ *actions. Let* A *and* $\acute{A}$, *respectively denote the extension of these actions by the one-parameter groups generated by the vector fields* $\Xi = \mathcal{I}^a\xi^a$, *and* $\acute{\Xi} = \acute{\mathcal{I}}^a\acute{\xi}^a$, *respectively. Furthermore, suppose that* $\mathcal{I}$ *and* $\acute{\mathcal{I}}$ *are equivalent under a diffeomorphism* $F \in \mathrm{Diff}(\mathcal{N})$. *Then* $\acute{A} = F \circ A$.

Proof. Since every element of $U(2)$ lies on a one parameter subgroup, it suffices to work infinitesimally. Let (ξ^a, Ξ) and $(\acute{\xi}^a, \acute{\Xi})$ denote the fundamental vector fields corresponding to the action A and $\acute{A}$, respectively. Let ad_X denote the global endomorphism sending any vector field Y to $[X, Y]$. Then we have

$$\mathcal{L}_{F_*\xi^a}\acute{\mathcal{I}}^b = [ad_{F_*\xi^a}, \acute{\mathcal{I}}^b] = F_*[ad_{\xi^a}, \mathcal{I}^b]F_*^{-1} = 2\epsilon^{abc}\acute{\mathcal{I}}^c. \tag{4.2}$$

This equation implies that for each $a = 1, 2, 3$ the vector fields τ^a defined by $\tau^a = F_*\xi^a - \acute{\xi}^a$ lie in the Lie algebra $\mathfrak{h}'$ of infinitesimal automorphisms of the hypercomplex structure $\acute{\mathcal{I}}$. Thus, by Proposition 2.5, τ^a commutes with $\acute{\xi}^b$ and one easily checks that τ^a span the Lie algebra $\mathfrak{su}(2)$, viz. $[\tau^a, \tau^b] = 2\epsilon^{abc}\tau^c$. Moreover, a similar computation to that above shows that $F_*\Xi \in \mathfrak{h}'$. So τ^0 defined by $\tau^0 = F_*\Xi - \Xi'$ must lie in $\mathfrak{h}'$, and it is direct to check that quadruple (τ^a, τ^0) satisfies the compatibility condition

$$\acute{\mathcal{I}}^a\tau^b = \epsilon^{abc}\tau^c + \delta^{ab}\tau^0. \tag{4.3}$$

But by Proposition 2.5, τ^0 commutes with $\acute{\xi}^a$. So taking the Lie derivative of equation (4.3) with respect to $\acute{\xi}^e$ we find that $\tau^a = \tau^0 = 0$ for all $a = 1, 2, 3$. □

This lemma says that any $F \in \mathrm{Diff}(\mathcal{N})$ which gives an equivalence of $U(2)$-compatible hypercomplex structures also gives an equivalence of the corresponding foliations. In fact F gives an equivalence of the nested sequence (1.9). Now let $\mathcal{F}$ and $\acute{\mathcal{F}}$ denote any of the above foliations on $\mathcal{N}$. Let L be a leaf of $\mathcal{F}$, and let $G(L)$ and $\acute{G}(F(L))$ denote the holonomy groups of L and $F(L)$, respectively. We have the following two immediate consequences of Lemma 4.2:

Corollary 4.3. *Let $F \in \mathrm{Diff}(\mathcal{N})$ be an equivalence of $U(2)$-compatible hypercomplex structures $\mathcal{I}$ and $\acute{\mathcal{I}}$. Then F induces a diffeomorphism of the leaves of $\mathcal{F}$ with the leaves of $\acute{\mathcal{F}}$ and thus, an isomorphism of the holonomy groups $G(L) \simeq \acute{G}(F(L))$. In particular, in accord with the notation of the previous section, if L is a leaf of $\mathcal{F}_1$ (respectively, $\mathcal{F}_4$) then $G_1(L) \simeq \acute{G}_1(F(L))$ (respectively, $G_4(L) \simeq \acute{G}_4(F(L))$.*

Corollary 4.4. *Suppose that $F : \mathcal{N}(\mathrm{p}) \to \mathcal{N}(\mathrm{p}')$ is an equivalence of hypercomplex manifolds. Then F induces orbifold diffeomorphisms F_1, F_2, F_4 such that the following diagram commutes:*

$$\begin{array}{ccc}
\mathcal{N}(\mathrm{p}) & \xrightarrow{F} & \mathcal{N}(\mathrm{p}') \\
\downarrow & & \downarrow \\
\mathcal{S}(\mathrm{p}) & \xrightarrow{F_1} & \mathcal{S}(\mathrm{p}') \\
\downarrow & & \downarrow \\
\mathcal{Z}(\mathrm{p}) & \xrightarrow{F_2} & \mathcal{Z}(\mathrm{p}') \\
\downarrow & & \downarrow \\
\mathcal{O}(\mathrm{p}) & \xrightarrow{F_4} & \mathcal{O}(\mathrm{p}').
\end{array}$$

To prove Theorem B we need the following two propositions.

Proposition 4.5. *Let p, $\mathrm{p}' \in C_n$ be basic commensurable sequences. Then $\mathcal{I}^a(\mathrm{p})$ is equivalent to $\mathcal{I}^a(\mathrm{p}')$ if and only if $\mathrm{p} = \mathrm{p}'$.*

Proof. The if direction is obvious. Let $F : \mathcal{N}(\mathrm{p}) \to \mathcal{N}(\mathrm{p}')$ be a diffeomorphism inducing a hypercomplex equivalence. We assume that $\mathcal{N}(\mathrm{p})$ is type 1 (the proof for type 2 is essentially identical). We first show that $p_n = p'_n$ and then inductively show that this fact forces equality for the remaining p_i, p'_i coordinates. We begin with two cases:

Case 1. $m_n \geq 2$: By Proposition 3.5, $\mathcal{N}(\mathbf{p}_n^{m_n})$ is a homogeneous hypercomplex Stiefel submanifold of $\mathcal{N}(\mathbf{p})$. Moreover, the order of every leaf of the foliation $\mathcal{F}_4(\mathbf{p})$ through this submanifold is maximal, namely $2p_n$. Let L be any such leaf then, by Proposition 4.3, $F(L)$ is a leaf of the foliation $\mathcal{F}_4(\mathbf{p}')$ of order $2p_n$. Thus, $2p_n \leq p'_n + p'_{n-1}$. Applying the same argument to F^{-1} shows that $2p_n = p'_n + p'_{n-1}$. $F(\mathcal{N}(\mathbf{p}_n^{m_n}))$ is a homogeneous submanifold of $\mathcal{N}(\mathbf{p}')$ all of whose leaves of the foliation $\mathcal{F}_4(\mathbf{p}')$ have maximal holonomy. The only way that this can occur is that p'_n have multiplicity m_n, and that $F(\mathcal{N}(\mathbf{p}_n^{m_n})) = (\mathcal{N}(\acute{\mathbf{p}}_n^{m_n}))$. Thus, we have $p'_n = p_n$ and $m_n = m'_n$.

Case 2. $m_n = 1$: By Proposition 3.6 there are m_{n-1} isolated leaves of maximal order $p_n + p_{n-1}$. As above there are precisely m_{n-1} leaves of the foliation $\mathcal{F}_4(\mathbf{p}')$ of order $p_n + p_{n-1}$. So $p_n + p_{n-1} \leq p'_n + p'_{n-1}$. Again applying the same argument to F^{-1} implies that $p_n + p_{n-1} = p'_n + p'_{n-1}$, and that p'_{n-1} must occur with multiplicity $m'_{n-1} = m_{n-1}$. Furthermore, each of the m_{n-1} maximal leaves $F(L_j)$ of the foliation $\mathcal{F}_4(\mathbf{p}')$ where $j = n - m_{n-1}, \ldots, n$ is covered by a vertex H'_{jn}. It follows that the submanifold $\mathcal{N}(\mathbf{p}^{m_{n-1},1})$ of $\mathcal{N}(\mathbf{p})$, where $\mathbf{p}^{m_{n-1},1} = (p_{n-1}, \ldots, p_{n-1}, p_n)$, is mapped diffeomorphically by F to the submanifold $\mathcal{N}(\acute{\mathbf{p}}^{m_{n-1},1})$ of $\mathcal{N}(\mathbf{p}')$. If $m_{n-1} \geq 2$ then each of these submanifolds contains a homogeneous Stiefel submanifold by Proposition 3.5, namely $\mathcal{N}(\mathbf{p}_{n-1}^{m_{n-1}}) \subset \mathcal{N}(\mathbf{p}^{m_{n-1},1})$ and $\mathcal{N}(\acute{\mathbf{p}}_{n-1}^{m_{n-1}}) \subset \mathcal{N}(\acute{\mathbf{p}}^{m_{n-1},1})$. Thus, in this case, $F(\mathcal{N}(\mathbf{p}^{m_{n-1}})) = \mathcal{N}(\acute{\mathbf{p}}_{n-1}^{m_{n-1}})$ and hence that $p_{n-1} = p'_{n-1}$. This implies that $p_n = p'_n$ as well.

Now assume that $m_{n-1} = 1$. We look at leaves of $\mathcal{F}_4(\mathbf{p})$ of order $p_n + p_{n-2}$. All such leaves must lie on the Stiefel submanifold $\mathcal{N}(\mathbf{p}^{m_{n-2},1,1})$ where

$$p^{m_{n-2},1,1} = (p_{m-2}, \ldots, p_{m-2}, p_{n-1}, p_n).$$

There are several types of leaves on $\mathcal{N}(\mathbf{p}^{m_{n-2},1,1})$. There is a unique leaf of maximal order $p_n + p_{n-1}$, and since $p_n + p_{n-2}$ cannot divide $p_n + p_{n-1}$, and $p_n + p_{n-2} \neq p_i + p_j$ for all other i, j, there are precisely m_{n-2} isolated leaves L_j $(j = n-1-m_{n-2}, \ldots, n-2)$ of order $p_n + p_{n-2}$ and each is covered by a vertex H_{jn}. Now since $F(H_{n-1n}) = H'_{n-1n}$ we see that $F(L_j)$ are leaves of $\mathcal{F}_4(\mathbf{p})$ that of maximal order in the set of all leaves excluding the leaf L_{n-1n}. By a maximality argument similar to that used above we see that $p_n + p_{n-2} = p'_n + p'_{n-2}$, and that $F(H_{jn}) = H'_{jn}$ for all $j = n-1-m_{n-2}, \ldots, n-2$. Hence, $F(\mathcal{N}(\mathbf{p}^{m_{n-2},1,1})) = \mathcal{N}(\acute{\mathbf{p}}^{m_{n-2},1,1})$.

If $m_{n-2} \geq 2$ then Proposition 3.5 implies that $p_{n-2} = p'_{n-2}$ and thus that $p_n = p'_n$ and $p_{n-1} = p'_{n-1}$. If $m_{n-2} = 1$ then $\mathcal{N}(p_{n-2}, p_{n-1}, p_n)$ maps diffeomorphically by F to $\mathcal{N}(p'_{n-2}, p'_{n-1}, p'_n)$ and the vertices H_{n-1n} and H_{n-2n} map diffeomorphically to the vertices H'_{n-1n} and H'_{n-2n}, respectively. Furthermore, $\mathcal{N}(p_{n-2}, p_{n-1}, p_n)$ contains at least one leaf of order $p_{n-1} + p_{n-2}$ covered by the vertex H_{n-1n-2} (if $p_{n-1} + p_{n-2}$ divides $p_n + p_{n-1}$ or $p_n + p_{n-2}$ there are many leaves of order $p_{n-1} + p_{n-2}$ that are not covered by vertices but this fact is not relevant to the proof). Applying Proposition 4.4 to the submanifolds $\mathcal{N}(p_{n-2}, p_{n-1}, p_n)$ and $\mathcal{N}(p'_{n-2}, p'_{n-1}, p'_n)$ we obtain a diffeomorphism of spaces of leaves, namely $\mathcal{O}(p_{n-2}, p_{n-1}, p_n) \simeq \mathcal{O}(p'_{n-2}, p'_{n-1}, p'_n)$. Let S and S' denote the suborbifolds of $\mathcal{O}(p_{n-2}, p_{n-1}, p_n)$ and $\mathcal{O}(p'_{n-2}, p'_{n-1}, p'_n)$, respectively, which are obtained by removing the leaves of orders $p_n + p_{n-1}$ and $p_n + p_{n-2}$ in the first case, and those of orders $p'_n + p'_{n-1}$ and $p'_n + p'_{n-2}$, in the second. By applying the maximality argument to $\mathcal{S}$ and $\mathcal{S}'$ we see that $p_{n-1}+p_{n-2}=p'_{n-1}+p'_{n-2}$. This implies that $p_n=p'_n$, $p_{n-1}=p'_{n-1}$, $p_{n-2}=p'_{n-2}$.

Thus, in all cases, $p_n=p'_n$. Now let j be the greatest positive integer less than n such that $p_j\neq p'_j$ but $p_i=p'_i$ for all $i>j$. Let $\mathbf{q}=(p_{j+1},\ldots,p_n)$ then, by the inductive hypothesis, $\mathcal{N}(\mathbf{q})$ is a submanifold of both $\mathcal{N}(\mathbf{p})$ and $\mathcal{N}(\mathbf{p}')$. Moreover, a leaf L of maximal order in $\mathcal{N}(\mathbf{p})$ that is not in $\mathcal{N}(\mathbf{q})$ has order p_n+p_j. Similarly, a leaf of $\mathcal{N}(\mathbf{p}')$ that is not in $\mathcal{N}(\mathbf{q})$ has maximal order $p'_n+p'_j=p_n+p'_j$. Again, by the maximality argument, $p_j=p'_j$. □

Next we relate any commensurable sequence to a basic commensurable sequence.

Proposition 4.6. *Let $\mathbf{p}\in C_n$ be commensurable. For $\lambda\in\mathbb{R}^+$ the hypercomplex structures $\mathcal{I}^a(\mathbf{p})$ and $\mathcal{I}^a(\lambda\mathbf{p})$ are equivalent if and only if $\lambda=1$.*

Proof. Here assume that $\mathcal{N}(\mathbf{p})$ is of type 2 (the proof for type 1 is essentially identical). We identify the complex structure on a generic leaf of the foliation $\mathcal{F}_2$ with a point of the Teichmüller space of a Riemann surface of genus 1. To do this we choose a complex structure in the hypercomplex structure by fixing $a=1$ and notice that the hyperhermitian metric $h(\mathbf{p})$ restricted to a leaf of the foliation $\mathcal{F}_2$ is given by

$$ds^2=(\hat{\eta}^0)^2+(\hat{\eta}^1)^2.$$

The generic leaves of the foliation $\mathcal{F}_4$ are the form $S^1\times SO(3)$, and we can parameterize the elliptic curve E by two angles ϕ, ψ. The angle ψ is the azimuthal angle about the $a=1$ axis in the group $SO(3)$. Its range is $0\leq\psi\leq 2\pi$. To determine the range of the angle ϕ, consider the circle action generated by $\Xi(\mathbf{p})$ on the coordinates u_j of $\mathbb{H}^n$. This action is $u_j\mapsto e^{ip_j\phi}u_j$. Since $\mathbf{p}$ is commensurable there is a $\lambda\in\mathbb{R}^+$ such that $\mathbf{p}=\lambda\mathbf{s}$ where $\mathbf{s}$ is a basic commensurable sequence. Then we see that a complete circle is obtained when ϕ starts at zero and ends at $\frac{2\pi}{\lambda}$. Now the Teichmüller space of a genus 1 Riemann surface is the upper half plane H, and since $\Xi(\mathbf{p})$ is the real part of the holomorphic vector field generating translations on E, we see that the complex structure on E corresponds to the point $i\lambda\in H$. Now two complex structures on a genus one Riemann surface are equivalent if and only if they differ by an element in the subgroup $PSL(2,\mathbb{Z})\subset PSL(2,\mathbb{R})$, where we have the natural action of $PSL(2,\mathbb{R})$ on the upper half plane H. It is easy to see that there is no element other than the identity in $PSL(2,\mathbb{Z})$ that corresponds to a dilatation on $\mathbf{p}$. □

Proof of Theorem B. The first statement is now an immediate consequence of Corollary 4.4, Propositions 4.5 and 4.6, and the fact that $\mathcal{Z}(\lambda\mathbf{p})=\mathcal{Z}(\mathbf{p})$. Next, by Theorem C, the connected component of the hypercomplex automorphism group in the $\mathbf{p}=(1,\ldots,1)$ case is $U(n)$, and the action is transitive; hence, $\mathcal{N}(\lambda\mathbf{1})$ is $U(n)$-homogeneous. For all other $\mathbf{p}$'s the inhomogeneity follows from Propositions 2.4, 2.5, and the fact that the group of 3-Sasakian isometries of $\mathcal{S}(\mathbf{p})$ cannot act transitively unless $\mathbf{p}=\mathbf{1}$ [BGM2]. □

We shall now briefly consider the case of incommensurable $\mathbf{p}$. In this case the vector field $\Xi(\mathbf{p})$ integrates to an action of $\mathbb{R}$ rather than an S^1 action, and the generic leaf of the foliation $\mathcal{F}_1(\mathbf{p})$ is not closed in $\mathcal{N}(\mathbf{p})$. Turning to the foliation $\mathcal{F}_4(\mathbf{p})$, one sees easily from the proof of Lemma 3.4 that the holonomy group $G_4(L)$

of any leaf L contained in a vertex is isomorphic to the integers $\mathbb{Z}$ independent of the value of $p_i + p_j$. Hence, our method of distinguishing hypercomplex structures by the orders of leaf holonomy groups is of little use in the incommensurable case. Nevertheless, if the sequence $\mathbf{p}$ contains a commensurable subsequence, there will be closed $\mathbb{R}$ orbits that are diffeomorphic to S^1, and these can be used to distinguish the hypercomplex structures. More generally we consider

Definition 4.7. Let $\mathbf{p} \in C_n$ be an arbitrary sequence. We define the *integer rank* (or simply *rank*) of $\mathbf{p}$ to be the number of independent constraints over the rational numbers $\mathbb{Q}$ of the form

$$\sum_{i=1}^{n} a_i p_i = 0$$

that are satisfied by the components p_i of $\mathbf{p}$ with $a_i \in \mathbb{Z}$ for $i = 1, \ldots, n$. The integer rank of $\mathbf{p}$ is denoted by $\mathrm{rk}(\mathbf{p})$.

Note that $0 \le \mathrm{rk}(\mathbf{p}) \le n-1$ and if $\mathbf{p}$ is commensurable then $\mathrm{rk}(\mathbf{p}) = n-1$. We conclude this section by showing that the integer rank $\mathrm{rk}(\mathbf{p})$ is an invariant of the hypercomplex structure $\mathcal{I}^a(\mathbf{p})$. To begin we need to study the closures of the leaves of the foliation $\mathcal{F}_1(\mathbf{p})$. Let $\overline{\mathcal{F}_1}(\mathbf{p})$ denote the partition of $\mathcal{N}(\mathbf{p})$ by the closures of the leaves of $\mathcal{F}_1(\mathbf{p})$. It is known [Mol] that $\overline{\mathcal{F}_1}(\mathbf{p})$ is a singular Riemannian foliation of $\mathcal{N}(\mathbf{p})$. There is another important singular Riemannian foliation [Mol] on $\mathrm{V}^{\mathbb{C}}_{n,2}$, namely that given by the sheaf $\mathcal{SF}$ discussed in the paragraph following Definition 1.4. Now $H_i \in \mathfrak{h}(\mathbf{p})$ for all $\mathbf{p} \in C_n$, so that $\mathcal{SF} \supset \mathcal{F}_1(\mathbf{p})$ for all $\mathbf{p}$. The leaves of $\mathcal{SF}$ are all tori T^k of variable dimension ranging from $k = 2, \ldots, n$. At a generic point of $\mathcal{N}(\mathbf{p})$, that is, a point for which none of the quaternionic coordinates $\mathbf{u}_i$ vanish, any leaf is a maximal torus T^n of Aut $\mathcal{N}(\mathbf{p})$. The leaves of smallest dimension are T^2's lying on the vertices H_{ij}. Notice from the form of the vector field $\Xi(\mathbf{p})$ every leaf $L(\mathbf{p})$ of the foliation $\mathcal{F}_1(\mathbf{p})$ is contained in precisely one leaf T^l of the singular foliation $\mathcal{SF}$, and associated to each such leaf $L(\mathbf{p})$ there is a unique subsequence $\mathbf{p}^{(l)} \subset \mathbf{p}$ of length l such that $T^l \subset \mathcal{N}(\mathbf{p}^{(l)})$. Generally there are two extreme cases: If $\mathbf{p}$ is commensurable then all the leaves of $\mathcal{F}_1(\mathbf{p})$ are closed and $\overline{\mathcal{F}_1}(\mathbf{p}) = \mathcal{F}_1(\mathbf{p})$. On the other extreme if the integer rank $\mathrm{rk}(\mathbf{p}) = 0$, then $\mathbf{p}$ is totally incommensurable and $\overline{\mathcal{F}_1}(\mathbf{p}) = \mathcal{SF}$. More precisely, we have

Lemma 4.8. *Let $L(\mathbf{p})$ be a leaf of $\mathcal{F}_1(\mathbf{p})$ lying on the toral leaf T^l of $\mathcal{SF}$, and let $\mathbf{p}^{(l)}$ be the unique subsequence of $\mathbf{p}$ associated to $L(\mathbf{p})$. Then the closure $\overline{L(\mathbf{p})}$ of the leaf $L(\mathbf{p})$ satisfies the equation*

$$\overline{L(\mathbf{p})} = T^{l-\mathrm{rk}(\mathbf{p}^{(l)})}.$$

Furthermore, every leaf closure $\overline{L(\mathbf{p})}$ is an embedded submanifold of $\mathcal{N}(\mathbf{p})$.

Proof. Suppose that $L(\mathbf{p})$ lies in T^l. If we can show that the closure $\overline{L(\mathbf{p})}$ in $\mathcal{N}(\mathbf{p})$ coincides with the closure of $L(\mathbf{p})$ in T^l, then the result will follow by known results for tori [Mol]. Consider the sequence of inclusions

$$L(\mathbf{p}) \hookrightarrow T^l \hookrightarrow \mathcal{N}(\mathbf{p}^{(l)}) \hookrightarrow \mathcal{N}(\mathbf{p}).$$

Since the right most inclusion is an embedding, it suffices to show that the inclusion $T^n \hookrightarrow \mathcal{N}(\mathbf{p})$ is an embedding, that is, that the maximal tori are closed, embedded submanifolds. But this follows from the form (1.4) of the action of the torus T^n on $\mathbb{H}^n$ and the fact that $\mathcal{N}(\mathbf{p}) \hookrightarrow \mathbb{H}^n$ is an embedding. $\square$

We are now able to prove the following

Proposition 4.9. *Let $F : \mathcal{N}(\mathbf{p}) \to \mathcal{N}(\mathbf{p}')$ be an equivalence of hypercomplex structures, then* $\mathrm{rk}(\mathbf{p}) = \mathrm{rk}(\mathbf{p}')$.

Proof. Lemma 4.8 implies that, for any $\mathbf{p} \in C_n$, the leaves of $\overline{\mathcal{F}_1(\mathbf{p})}$ of maximal dimension are tori of dimension $n - \mathrm{rk}(\mathbf{p})$. Without loss of generality we can assume that $\mathrm{rk}(\mathbf{p}) \leq \mathrm{rk}(\mathbf{p}')$. By Lemma 4.2 F is foliation preserving, so $F(L)$ is a leaf of $\mathcal{F}_1(\mathbf{p}')$. Moreover, F sends closures to closures so that $\overline{F(L)}$ is a leaf of the singular foliation $\overline{\mathcal{F}_1(\mathbf{p}')}$ of dimension $n - \mathrm{rk}(\mathbf{p})$. But the leaves of $\overline{\mathcal{F}_1(\mathbf{p}')}$ of maximal dimension have dimension $n - \mathrm{rk}(\mathbf{p}')$ by Lemma 4.8. So $n - \mathrm{rk}(\mathbf{p}) \leq n - \mathrm{rk}(\mathbf{p}')$ which implies the equality of the ranks. $\square$

So far we have not been able to completely solve the equivalence problem for the incommensurable case, although we do have several invariants including the integer rank $\mathrm{rk}(\mathbf{p})$ and the unordered set $\{m_i\}_{i=1}^{k(\mathbf{p})}$ of multiplicities. We shall return to this classification problem in a future work.

5. Some Hypercomplex Quotients of $\mathbb{V}_{n,2}^{\mathbb{C}}$

In this section we consider certain quotients of the Stiefel manifolds that also carry hypercomplex structures. Our construction here can be seen as a generalization of both the Stiefel manifolds themselves and the hypercomplex structure on the trivial bundle $\mathcal{S}(\mathbf{p}) \times S^1$ noticed in [BGM2]. Let k be a positive integer and $\mathbf{p} \in C_n$ an integer multiple of a basic commensurable sequence. If $\gcd(p_i, p_j, k) = 1$ for all $1 \leq i < j \leq n$ then $\mathbf{p}$ is called *k-coprime*; otherwise such $\mathbf{p}$ is called *k-composite*. Let $\mathbf{p}$ be a commensurable sequence, k a positive integer, and let $\mathcal{H}(\mathbf{p}, k) = \mathcal{N}(\mathbf{p}) \times_{(\tau_{\mathbf{p}}, k)} S^1$, where the action map $(\tau_{\mathbf{p}}, k) : S^1 \times \mathcal{N}(\mathbf{p}) \times S^1 \longrightarrow \mathcal{N}(\mathbf{p}) \times S^1$ is given by $\theta_{\mathbf{p}}$ on the first factor and by the standard map of degree k on the second. More precisely,

$$(\tau_{\mathbf{p}}, k)(\theta, (n, \omega)) = (\theta_{\mathbf{p}}(n), \theta^k \omega). \tag{5.1}$$

We set $\mathcal{H}(\mathbf{p}, 0) = \mathcal{S}(\mathbf{p}) \times S^1$. The following proposition, which is direct to verify, explains why we referred to $\mathcal{N}(\mathbf{p})$ and $\mathcal{S}(\mathbf{p}) \times S^1$ as the two extreme $\mathcal{H}(\mathbf{p}, k)$ subfamilies.

Proposition 5.1. *For all commensurable $\mathbf{p}$ there is a diffeomorphism $\mathcal{N}(\mathbf{p}) \simeq \mathcal{H}(\mathbf{p}, 1)$. Furthermore, if $\mathbf{p}$ is k-coprime then $\mathcal{H}(\mathbf{p}, k)$ is a smooth $(4n-4)$-dimensional manifold with fundamental group $\pi_1(\mathcal{H}(\mathbf{p}, k)) = \mathbb{Z}_k$, and if $k = lm > 0$ then there is a natural projection $\pi(l, k) : \mathcal{H}(\mathbf{p}, l) \longrightarrow \mathcal{H}(\mathbf{p}, k)$ which is an m-fold covering space map. In particular, the universal covering space of $\mathcal{H}(\mathbf{p}, k)$ is $\mathcal{N}(\mathbf{p})$. Here we are using the convention that $\mathbb{Z}_1 = 0$. If $\mathbf{p}$ is k-composite then $\mathcal{H}(\mathbf{p}, k)$ is an orbifold.*

Proof of Theorem D. This follows directly from Theorems 1.10, 1.12 and Proposition 5.1. $\square$

Next we have a direct corollary of Theorems B and C:

Corollary 5.2. *Let $\mathbf{p}$ be k-coprime and $\mathbf{q}$ be l-coprime. Then $\mathcal{H}(\mathbf{p}, k)$ is hypercomplex equivalent to $\mathcal{H}(\mathbf{q}, l)$ if and only if $\mathbf{p} = \mathbf{q}$ and $k = l$. Furthermore, the Lie* *algebra of infinitesimal hypercomplex automorphisms of $\mathcal{H}(\mathbf{p},k)$ is the Lie algebra $\mathfrak{h}(\mathbf{p})$ given in Theorem C.*

It is natural to ask how the homology of $\mathcal{H}(\mathbf{p},k)$ depends on $\mathbf{p}$. For the rest of this section we will restrict our attention to the case when $\mathbf{p}$ is coprime. Here the $\theta_{\mathbf{p}}$ factor in the $(\tau_{\mathbf{p}},k)$ action is free so we have a circle bundle $S^1 \longrightarrow \mathcal{H}(\mathbf{p},k) \longrightarrow \mathcal{S}(\mathbf{p})$ where the base space is one of the 3-Sasakian manifolds $\mathcal{S}(\mathbf{p})$ described in [BGM2]. While the topological structure of $\mathcal{H}(\mathbf{p},1) \simeq \mathcal{N}(\mathbf{p})$ is independent of $\mathbf{p}$, for every $n \geq 3$ the $\mathcal{S}(\mathbf{p})$ run through infinitely many distinct homotopy types. Our next theorem computes the integral cohomology ring $H^*(\mathcal{H}(\mathbf{p},k);\mathbb{Z})$.

Theorem 5.3. *Let $\mathbf{p}$ be coprime and k a positive integer. Then, as rings,*

$$H^*(\mathcal{H}(\mathbf{p},k),\mathbb{Z}) \cong \left(\frac{\mathbb{Z}_k[x_2]}{[x_2^n=0]} \otimes E[y_{2n-3},z_{2n-1}]\right)/\mathcal{R}(\mathcal{H}(\mathbf{p},k)), \tag{5.2}$$

where the subscripts on x_2, y_{2n-3} and z_{2n-1} denote the cohomological dimension of each generator. The relations $\mathcal{R}(\mathcal{H}(\mathbf{p},k))$ are given by

$$d(\mathbf{p},k)x_2^{n-1} = d(\mathbf{p},k)x_2y_{2n-3} = x_2^2y_{2n-3} = x_2^{n-1}z_{2n-1} = x_2y_{2n-3}z_{2n-1} = 0.$$

Here $\sigma_{n-1}(\mathbf{p})$ is the $(n-1)^{st}$ elementary symmetric polynomial in the coordinates of $\mathbf{p}$ and $d(\mathbf{p},k) = \gcd(\sigma_{n-1}(\mathbf{p}),k)$. The conventions here are that $d(\mathbf{p},0) = \sigma_{n-1}(\mathbf{p})$ and $\mathbb{Z}_1 = 0$.

Corollary 5.4. *As abelian groups*

$$H^*(\mathcal{H}(\mathbf{p},k);\mathbb{Z}) = \begin{cases} \mathbb{Z}_k & \textit{if } * = 2,4,\ldots,2n-4, \\ \mathbb{Z} & \textit{if } * = 0,2n-3,4n-4, \\ \mathbb{Z}_{d(\mathbf{p},k)} & \textit{if } * = 2n-2, \\ \mathbb{Z}\oplus\mathbb{Z}_{d(\mathbf{p},k)} & \textit{if } * = 2n-1, \\ \mathbb{Z}_k & \textit{if } * = 2n+1,2n+3,\ldots,4n-5, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof of Theorem 5.3. Given the circle bundle $S^1 \longrightarrow \mathcal{H}(\mathbf{p},k) \longrightarrow \mathcal{S}(\mathbf{p})$ there is an associated complex line bundle with first Chern class c_1. This Chern class determines the key homomorphism in the associated Gysin sequence. Since the cohomology ring of $\mathcal{S}(\mathbf{p})$ is completely known [BGM2: Theorem E], standard techniques using the Gysin sequence and Poincare dualtity gives the result. □

Notice that, with our convention that $\mathbb{Z}_1 = 0$, Theorem 5.3 recovers the classical computation of the torsion free ring $H^*(\mathcal{H}(\mathbf{p},1);\mathbb{Z}) \simeq H^*(\mathcal{N}(\mathbf{p});\mathbb{Z}) \simeq H^*(\mathbf{V}^{\mathbb{C}}_{n,2};\mathbb{Z}) \simeq E(f_{2n-3},f_{2n-1})$. Another immediate corollary of our results is:

Corollary 5.5. *For all coprime $\mathbf{p}$, $n \geq 3$, and $k > 1$ there is one cohomological invariant of $\mathcal{H}(\mathbf{p},k)$ that depends on $\mathbf{p}$; namely, the integer $d(\mathbf{p},k)$ which is the order of the torsion subgroups of the $2n-2$ and $2n-1$ integral cohomology groups of $\mathcal{H}(\mathbf{p},k)$.*

While the fundamental group can be used to distinguish the $\mathcal{H}(\mathbf{p},k)$ for different values of k it is more interesting to look for distinct homotopy types within a fixed choice of dimension and fundamental group. On one extreme, when the fundamental group vanishes, all the examples in any fixed dimension are homeomorphic. On the other extreme, when the fundamental group is $\mathbb{Z}$, there are infinitely many distinct homotopy types for the $\mathcal{H}(\mathbf{p},0)$ in every dimension of the form $4n-4$ for $n \geq 3$. Of course, this cohomological computation does not completely determine the homotopy type of the $\mathcal{H}(\mathbf{p},k)$ but only gives a lower bound on the number of possibly distinct homotopy types that occur. Still, Corollary 5.5 does permit us to detect some interesting distinctions for fixed k. Moreover, the computations when the fundamental group is finite cyclic are also dependent on n. For example, when $n=3$, it is known that the $\sigma_2(p_1,p_2,p_3)$ take on every odd integer value greater than one [BGM2]. This implies the following proposition.

Proposition 5.6. *Let $k=2^r m$ with m odd. Then there are at least $\tau(m)$ distinct homotopy types for the 8-dimensional hypercomplex manifolds $\mathcal{H}(p_1,p_2,p_3;k)$. Here $\tau(m)$ is the number of positive divisors of m. Moreover, all these homotopy distinct examples have isomorphic fundamental groups.*

On the other hand, since it is always the case that $\sigma_2(p_1,p_2,p_3)$ is odd, Theorem 5.4 cannot be used to distinguish homotopy types in dimension 8 for even values of k when $n=3$. However, when $n=4$ and $k=2$ the examples $\mathcal{H}(1,2,3,5;2)$ and $\mathcal{H}(1,3,5,7;2)$ are not homotopy equivalent. There are many other similar examples for other values of k and n.

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