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Mellin Representations of Pseudo-Differential Operators on Manifolds with Corners

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1. The Hierarchy of Operator Structures

1.1. Preliminary Remarks

This paper studies pseudo-differential operators (ψDO's) on manifolds with corners. The analysis of operators on non-smooth spaces is necessary for many reasons. Concrete problems in partial differential equations, differential geometry or applied sciences lead to questions on the solvability in the sense of representing parametrices in suitable algebras of operators and the regularity of solutions close to the singularities. It is convenient to look at the singularities in terms of an iterative geometric procedure, starting with a closed compact C^∞ manifold X. Then we can form a cone $C = X \times \bar{\mathbf{R}}_+/X \times \{0\}$ with base X, and further a wedge $W = C \times \mathbf{R}^q$ with cone C and edge $\mathbf{R}^q$. These are the local models of spaces with conical singularities and edges, respectively (with C^∞ structures outside the singularities). Below we shall recall the precise definitions.

Now if B is a space with conical singularities, we can pass to a corner $K = B \times \bar{\mathbf{R}}_+/B \times \{0\}$, further to a cone based on a manifold with edges, or to a wedge of the form $K \times \mathbf{R}^q$. In this way we can successively generate higher singularities.

This concept covers in particular situations that are interesting for applications such as screen (or crack) problems, mixed elliptic problems, or of quarter plane type. Also the case of standard boundary value problems is included, since the local model of a domain Ω with C^∞ boundary is a wedge with cone $\bar{\mathbf{R}}_+$ and edge $\mathbf{R}^q$, $\dim \Omega = q + 1$.

The program for the analysis is now, parallel to this geometric picture, to establish a hierarchy of operator structures, beginning with the standard ψDO calculus over X. In other words on $C, W, K, \ldots$ is required a ψDO calculus, aimed at solving elliptic problems for differential operators within the corresponding operator classes.

Operators on manifolds with conical singularities and edges have been studied by many authors, cf., for instance, Kondrat'ev [K1], Plamenevskij [P1], Roßmann [R2], Dauge [D1], Rempel/Schulze [R1], Schulze [S2], [S5], [S6], [S7], and the references given there. Conical singularities can be treated in the context of totally characteristic operators (or of Fuchs type) written as Mellin operators.

Edges require constructions analogous to boundary value problems with extra trace and potential conditions along the edges, similarly to Boutet de Monvel [B1].

The case of corners $K = B \times \bar{\mathbf{R}}_+/B \times \{0\}$ has to unify both aspects, the totally characteristic one in direction to the corner axis $\mathbf{R}_+$ and that of trace and potential conditions along the one-dimensional edges emanating from the corners.

The present paper develops the Mellin operator conventions for ψDO's near corners. In [S3] we shall add further elements on parameter-dependent cone operator families. This will finally be used in [S4] for an algebra of corner ψDO's including the concept of ellipticity, parametrix constructions and the asymptotics of solutions.

The main idea is to employ a Mellin ψDO calculus on $\mathbf{R}_+$ with meromorphic operator-valued symbols acting on the base B of the corner. They will reflect a totally characteristic structure close to the conical points of B as well as an edge degeneracy in the Mellin covariable. In addition the complete corner algebra will contain operators with meromorphic smoothing Mellin symbols and Green operators, induced by the parametrix construction for simpler operators, e.g. differential operators.

The non-smoothing Mellin symbols are linked to an interior symbolic structure by the Mellin operator convention. Analogous Mellin conventions exist on manifolds with conical singularities and edges, and the constructions will show how to proceed for higher singularities.

In this sense the present paper wants to emphasize aspects of a more axiomatic approach in dealing with ψDO's on manifolds with higher singularities.

The operator conventions will lead to natural analogues of Sobolev spaces, defined on the manifold with singularities, where the operators induce continuous actions, similarly as in the classical case of closed compact C^∞ manifolds (cf. [S1], [S3], [S4]).

Another aspect will be a symbolic structure consisting in a system of leading symbol components, a part of them being operator-valued, where the bijectivity of every component is by definition the ellipicity of a given operator. The interaction of the various symbolic levels can be illustrated already for boundary value problems. In the comparatively simple special case of Boutet de Monvel's algebra we have the interior and the boundary symbol, where the ellipticity of the latter one corresponds just to the Shapiro-Lopatinskij condition, whereas the ellipticity of the interior symbol concerns as usual the given pseudo-differential operator.

For conical singularities we have an interior and a Mellin symbol of leading interior (usual) and conormal order, respectively. Edge singularities require an extra edge symbolic level, and so on.

The present paper is organized as follows. In 1.2. we establish Mellin operator conventions on manifolds with conical singularities. 1.3. deals with manifolds with edges. The constructions are close to the corresponding chapters of [S2]. The main difference here is that we allow arbitrary cone bases and more general interior symbols. This causes a larger class of ψDO's and extra contributions to the edge symbols from interior lower order terms (restricted to the edge).

In 1.4. we shall briefly discuss polar coordinates in ψDO's that lead to totally characteristic symbols. In 2.1. we will introduce the Mellin operator conventions for corner singularities and obtain a representation of operators in terms of the Mellin transform along the corner axis with cone operator-valued symbols. These are families of cone operators with edge degeneracy at the parameter varying on a weight line. Such families will be the starting point of [S3]. Finally in 2.2. we shall describe the system of symbolic levels for corners with the compatibility conditions.

The operator conventions will often be formulated in terms of mappings between spaces of amplitude functions. They are "non-canonical" but canonical up to elements of order $-\infty$. The remainders that are systematically neglected will have no influence to the final operator algebra of [S4], because of the additional smoothing Mellin and Green operators. The latter ones will be briefly defined for the case of cones and wedges, here based on the spaces with continuous conormal asymptotics (cf. [S5], [S6], [S2]). A subclass is associated with the discrete asymptotics, which is defined along $\mathbf{R}_+ \ni r$, for $r \to 0$ as

$$u(r) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{m_j} \zeta_{jk} r^{-p_j} \log^k r\,,$$

with complex p_j, $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$, and integers m_j. The analogous objects for corners will be introduced in [S4].

1.2. Pseudo-Differential Operators on Manifolds With Conical Singularities

This section will formulate a Mellin operator convention on manifolds with conical singularities. Let X be a closed compact C^∞ manifold, $n = \dim X$, and set

$$X^\wedge = X \times \mathbf{R}_+\,. \tag{1}$$

(1) will be interpreted as the (open stretched) cone with base X. A compact C^∞ manifold $\mathbf{B}$ with boundary $\partial\mathbf{B} = X$ is called the stretched manifold belonging to a space B with conical singularity if for a given tubular neighbourhood U of $\partial\mathbf{B}$ there is fixed a diffeomorphism $\delta\colon X^\wedge \to \operatorname{int} U$. The $\mathbf{R}_+$ action on $X^\wedge$, given by $(x, r) \to (x, \lambda r)$, $\lambda \in \mathbf{R}_+$, corresponds to an $\mathbf{R}_+$ action on U. On U it is needed only in the local form for points close to $\partial\mathbf{B}$ and small λ. Then $\mathbf{B}/\partial\mathbf{B} = B$ is the corresponding space with conical singularity, where the local $\mathbf{R}_+$ action is canonically defined close to the vertex. Note that X may have several connection components. We might distinguish between several conical points belonging to the connection components. For simplicity we shall neglect the aspect of different conical points. Clearly B is no C manifold unless X is not a sphere.

We adopt here the standard notations of the ψD (pseudo-differential) calculus. If $\Omega \subset \mathbf{R}^n$ is open, we denote by $S^\mu(\Omega \times \mathbf{R}^m)$ the space of all $a(x, \xi) \in C^\infty(\Omega \times \mathbf{R}^m)$ satisfying

$$|D_x^\alpha D_\xi^\beta a(x, \xi)| \leqq c(1 + |\xi|)^{\mu - |\beta|} \tag{2}$$

for all $(x, \xi) \in K \times \mathbf{R}^m$ and arbitrary $K \Subset \Omega, \alpha \in \mathbf{N}^n, \beta \in \mathbf{N}^m$, with constants $c = c(\alpha, \beta, K)$. Further $S^\mu_{\mathrm{cl}}(\Omega \times \mathbf{R}^m)$ denotes the subclass of classical amplitude functions, i.e. with asymptotic expansions

$$a(x, \xi) \sim \sum_{j=0}^{\infty} a_{\mu - j}(x, \xi),$$

$a_{\mu-j} \in C^\infty(\Omega \times \mathbf{R}^m)$, $a_{\mu-j}(x, \lambda\xi) = \lambda^{\mu-j} a_{\mu-j}(x, \xi)$ for all $\lambda \geqq 1$, $|\xi| \geqq$ const, $x \in \Omega$.

The best constants in (2) form a semi-norm system on $S^\mu(\Omega \times \mathbf{R}^m)$ under which this space is Fréchet. $S^\mu_{\mathrm{cl}}(\Omega \times \mathbf{R}^m)$ will be equipped with the topology of the projective limit with respect to the canonical mappings

$$h_j \colon S^\mu_{\mathrm{cl}}(\Omega \times \mathbf{R}^m) \to C^\infty(\Omega \times (\mathbf{R}^m \setminus \{0\})),$$

$$r_k \colon S^\mu_{\mathrm{cl}}(\Omega \times \mathbf{R}^m) \to S^{\mu - (k+1)}(\Omega \times \mathbf{R}^m)$$

for $j, k \in \mathbf{N}$, where $h_j(a) = a_{(\mu - j)}$ with $a_{(\mu - j)}$ being the unique homogeneous function that equals $a_{\mu - j}$ for large $|\xi|$, and $r_k(a) = \sum_{i=0}^{k} \chi(\xi)\, a_{(\mu - i)}(x, \xi)$ for an excision function χ in $\mathbf{R}^m$ (i.e. $\chi \in C^\infty(\mathbf{R}^m)$, $\chi = 0$ in a neighbourhood of $\xi = 0$, $\chi \equiv 1$ for $|\xi| >$ const). Then $S^\mu_{\mathrm{cl}}(\Omega \times \mathbf{R}^m)$ is a Fréchet space and the topology is independent of the choice of χ.

$L^\mu(\Omega)$ is defined as the class of all ψDO's A in Ω, i.e. $A = A_1 + A_0$, with

$$A_1 u(x) = \mathrm{op}_{\psi, x}(a)\, u := \iint e^{i(x - x')\xi}\, a(x, \xi)\, u(x')\, dx'\, đ\xi, \tag{3}$$

$đ\xi = (2\pi)^{-n}\, d\xi$, $a(x, \xi) \in S^\mu(\Omega \times \mathbf{R}^n)$, and A_0 being an operator with kernel in $C^\infty(\Omega \times \Omega)$. The space $L^\mu_{\mathrm{cl}}(\Omega)$ of classical ψDO's is defined by $a(x, \xi) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbf{R}^n)$.

Analogous notations will be used for ψDO's on C^∞ manifolds. In particular we can talk about $L^\mu(\mathrm{int}\, \mathbf{B})$, $L^\mu_{\mathrm{cl}}(\mathrm{int}\, \mathbf{B})$.

The spaces $L^\mu(\ldots)$, $L^\mu_{\mathrm{cl}}(\ldots)$ are Fréchet in a natural way (cf. [S2], Section 2.1.4.).

Now if $\mathbf{B}$ is interpreted as the (stretched) manifold with conical singularities we look at special subclasses of $L^\mu(\mathrm{int}\, \mathbf{B})$, called ψDO's on the manifold with conical singularities. In the set-up of operator algebras filtered by orders it is convenient to deal with classical operators. They will mainly be discussed from now on.

Let V be a coordinate neighbourhood on X (= the base of the cone) with local coordinates $x \in \mathbf{R}^n$. Set $V^\wedge = V \times \mathbf{R}_+ \ni (x, r)$. Then $S^\mu_{\mathrm{cl}}(cV^\wedge \times \mathbf{R}^{n+1})$ denotes the subspace of all $a(x, r, \xi, \varrho) \in S^\mu_{\mathrm{cl}}(V^\wedge \times \mathbf{R}^{n+1})$ with $a = a_1|_{V^\wedge \times \mathbf{R}^{n+1}}$ for some $a_1 \in S^\mu_{\mathrm{cl}}(V \times \mathbf{R} \times \mathbf{R}^{n+1})$. Moreover we define

$$\tilde{S}^\mu_{\mathrm{cl}}(cV^\wedge \times \mathbf{R}^{n+1}) = \{a(x, r, \xi, r\varrho) \colon a(x, r, \xi, \varrho) \in S^\mu_{\mathrm{cl}}(cV^\wedge \times \mathbf{R}^{n+1})\}. \tag{4}$$

The elements of (4) are called totally characteristic.

We say that $a(x, r, \xi, \varrho) \in S^{\mu}_{\mathrm{cl}}(\mathrm{c}V^{\wedge} \times \mathbf{R}^{n+1})$ satisfies the *exit condition* if there exists a sequence $e_j(x, \xi, \varrho) \in S^{\mu}_{\mathrm{cl}}(\mathrm{c}V^{\wedge} \times \mathbf{R}^{n+1})$ such that

$$\chi(r)\, a(x, r, \xi, \varrho) \sim \chi(r) \sum_{j=0}^{\infty} r^{-j} e_j(x, \xi, \varrho) \tag{5}$$

for any excision function χ (i.e. $\chi \in C^{\infty}$, $\chi \equiv 0$ close to $r = 0$, $\chi = 1$ for $r >$ const). The asymptotics (5) mean that

$$\left| D_r^k D_x^{\alpha} D_{\xi,\varrho}^{\beta} \chi(r) \left\{ a - \sum_{j=0}^{N} r^{-j} e_j \right\} \right| \leqq c(1 + |\xi, \varrho|)^{\mu - |\beta|} (1 + r)^{-(N+1)-k}$$

for all $k \in \mathbf{N}$, $\alpha \in \mathbf{N}^n$, $\beta \in \mathbf{N}^{n+1}$, with a constant $c = c(k, \alpha, \beta, K)$ for all $x \in K \Subset V$, $r \in \mathbf{R}_+$, $(\xi, \varrho) \in \mathbf{R}^{n+1}$.

Remember that more general "exit conditions" have been studied by Parenti [P3], [P4], and Cordes [C1], Schrohe [S8]. $S^{\mu}_{\mathrm{cl}}(\mathrm{c}V^{\wedge} \times \mathbf{R}^{n+1})_{\mathrm{e}}$ will denote the subspace of all $a(x, r, \xi, \varrho) \in S^{\mu}_{\mathrm{cl}}(\mathrm{c}V^{\wedge} \times \mathbf{R}^{n+1})$ satisfying the exit condition, and we set

$$\tilde{S}^{\mu}_{\mathrm{cl}}(\mathrm{c}V^{\wedge} \times \mathbf{R}^{n+1})_{\mathrm{e}} = \{a(x, r, \xi, r\varrho) : a(x, r, \xi, \varrho) \in S^{\mu}_{\mathrm{cl}}(\mathrm{c}V^{\wedge} \times \mathbf{R}^{n+1})_{\mathrm{e}}\} . \tag{6}$$

For $\tilde{a} \in \tilde{S}^{\mu}_{\mathrm{cl}}(\mathrm{c}V^{\wedge} \times \mathbf{R}^{n+1})_{\mathrm{e}}$ we shall also say that it satisfies the exit condition.

The ψDO's on int $\mathbf{B}$ that refer to the conical singularities will have (complete) symbols in (4) close to $r = 0$ (possibly up to a weight factor). In order to get a precise control near $r = 0$ we shall employ a Mellin operator convention along $\mathbf{R}_+ \ni r$.

Remember that the Mellin transform M is defined by

$$Mu(w) = \int_0^{\infty} r^{w-1} u(r)\, \mathrm{d}r ,$$

where first $u \in C_0^{\infty}(\mathbf{R}_+)$, $w \in \mathbf{C}$. Then

$$(M^{-1} g)(r) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\beta}} r^{-w} g(w)\, \mathrm{d}w$$

with

$$\Gamma_{\beta} = \{w \in \mathbf{C} : \mathrm{Re}\, w = \beta\} ,$$

$\beta \in \mathbf{R}$ arbitrary. If $G \subseteq \mathbf{C}$ is open, we denote by $\mathscr{A}(G)$ the space of holomorphic functions in G in the topology of uniform convergence on compact subsets. Analogously $\mathscr{A}(G, E)$ denotes the space of holomorphic E-valued functions in G, with E being a Fréchet space. Moreover for any open $\Omega \subseteq \mathbf{R}^n$ we use notations such as $C^{\infty}(\Omega, E)$ ($C_0^{\infty}(\Omega, E)$) for the spaces of C^{∞} (C_0^{∞}) functions on Ω with values in E. The Mellin transform can be applied also to $u \in C_0^{\infty}(\mathbf{R}_+, E)$ with respect to r. The result is then in $\mathscr{A}(\mathbf{C}, E)$. It is well known that M extends by continuity to natural classes of spaces, for instance,

$$M : r^{\beta} L^2(\mathbf{R}_+) \underset{\cong}{\to} L^2(\Gamma_{1/2-\beta}) ,$$

$\beta \in \mathbf{R}$, cf. [S2], Section 1.1.1.

For $h(r, w) = h(r, 1/2 + \mathrm{i}\,\varrho) \in S^{\mu}(\mathbf{R}_+ \times \mathbf{R})$, $(r, \varrho) \in \mathbf{R}_+ \times \mathbf{R}$, we can form the Mellin ψDO

$$\mathrm{op}_M\,(h)\,u(r) = M^{-1}_{w \to r} h(r, w)\, M_{r' \to w} u(r'),$$

$u \in C_0^{\infty}(\mathbf{R}_+)$. Set

$$\mathrm{op}_M^{\beta}\,(h) = r^{\beta}\,\mathrm{op}_M\,(T^{-\beta} h)\, r^{-\beta}, \tag{7}$$

$(T^{-\beta} h)\,(r, w) := h(r, w - \beta)$. If we impose smoothness of h in r up to $r = 0$ and, for instance, bounded support in r, then (7) extends by continuity to an action between weighted (totally characteristic) Sobolev spaces

$$\mathrm{op}_M^{\beta}\,(h) \colon \mathscr{H}^{s,\beta}(\mathbf{R}_+) \to \mathscr{H}^{s-\mu,\beta}(\mathbf{R}_+)$$

(cf. [S2], 1.2.3. Proposition 16). Analogous relations hold in the vector-valued case.

Let

$$S^{\mu}_{\mathrm{cl}}(\mathrm{c}V^{\wedge} \times \mathbf{R}^n \times \Gamma_{1/2-\beta}) \tag{8}$$

denote the space of all $h(x, r, \xi, w)$, defined for $w \in \Gamma_{1/2-\beta}$, such that

$$h(x, r, \xi, 1/2 - \beta + \mathrm{i}\varrho) \in S^{\mu}_{\mathrm{cl}}(\mathrm{c}V^{\wedge} \times \mathbf{R}^{n+1}_{\xi,\varrho}). \tag{9}$$

Moreover,

$$S^{\mu}_{\mathrm{cl}}(\mathrm{c}V^{\wedge} \times \mathbf{R}^n \times \mathbf{C})_{\mathrm{hol}} \tag{10}$$

denotes the space of all $h(x, r, \xi, w) \in \mathscr{A}(\mathbf{C}_w, C^{\infty}(V \times \bar{\mathbf{R}}_+ \times \mathbf{R}^n))$ such that (9) holds uniformly in $c_1 \leqq \beta \leqq c_2$ for all $c_1, c_2 \in \mathbf{R}$. Here and in the sequel a relation for elements in a Fréchet space uniformly in a parameter interval means that the semi-norms on the corresponding parameter dependent elements are uniformly bounded in the interval.

The following lemma is quite elementary. We shall give a proof, since assertions of this sort will be employed below in many variants, where we then drop the analogous kernel cut-off arguments.

1. Lemma. *For every* $h \in S^{\mu}_{\mathrm{cl}}(\mathrm{c}V^{\wedge} \times \mathbf{R}^n \times \Gamma_{1/2-\beta})$ *there exists an* $h_{-\infty} \in S^{-\infty}(\mathrm{c}V^{\wedge} \times \mathbf{R}^n \times \Gamma_{1/2-\beta})$ *such that* $h - h_{-\infty}$ *extends to an element in* (10).

Proof: Let us show the assertion in the version for $S^{\mu}_{\mathrm{cl}}(\mathbf{R}^n_{\xi} \times \mathbf{R}_{\varrho})$ where $\mathrm{c}V^{\wedge}$ disappears and $\Gamma_{1/2-\beta}$ is replaced by $\mathbf{R}_{\varrho}$. Then the result in general follows by an obvious modification. For $a(\xi, \varrho) \in S^{\mu}_{\mathrm{cl}}(\mathbf{R}^n \times \mathbf{R})$ we write

$$K(a)\,(\zeta, \varkappa) = \int \mathrm{e}^{\mathrm{i}\zeta\xi + \mathrm{i}\varkappa\varrho}\, a(\xi, \varrho)\, đ\xi\, đ\varrho\,.$$

Let $\omega(\varkappa)$ be a cut-off function, i.e., $\omega \in C^{\infty}(\mathbf{R})$, $\omega \equiv 1$ close to 0, $\omega \equiv 0$ for $|\varkappa| > \text{const.}$ Then we form

$$b(\xi, \varrho) = \int \mathrm{e}^{-\mathrm{i}\zeta\xi - \mathrm{i}\varkappa\varrho}\, \omega(\varkappa)\, K(a)\,(\zeta, \varkappa)\, \mathrm{d}\zeta\, \mathrm{d}\varkappa\,. \tag{11}$$

The symbol estimates for a imply that $(1 - \omega(\varkappa))\, K(a)\,(\zeta, \varkappa) \in \mathscr{S}(\mathbf{R}^{n+1}_{\zeta,\varkappa})$ ($\mathscr{S}(\ldots)$ denotes the Schwartz space). Thus $a(\xi, \varrho) - b(\xi, \varrho) \in S^{-\infty}(\mathbf{R}^{n+1}_{\xi,\varrho})$.

If we replace (11) by

$$\tilde{b}(\xi, \varrho) = \int e^{-i\zeta\xi - i\varkappa\varrho}\, \psi(\varkappa)\, K(a)\,(\zeta, \varkappa)\, d\zeta\, d\varkappa$$

for any $\psi(\varkappa) \in C_0^\infty(\mathbf{R})$, then we get $\tilde{b}(\xi, \varrho) \in S^\mu_{\mathrm{cl}}(\mathbf{R}^n \times \mathbf{R})$. If $\psi(\varkappa)$ is flat at $\varkappa = 0$ of order N, then it follows even $\tilde{b} \in S^{\mu-N}_{\mathrm{cl}}(\mathbf{R}^n \times \mathbf{R})$. All this holds uniformly in β, $c_1 \leqq \beta \leqq c_2$, if we replace ψ by a function $\psi(\varkappa, \beta) \in C^\infty(\mathbf{R}^2)$, $\psi \equiv 0$ for $|\varkappa| >$ const. The latter properties follow by simple calculations in terms of the symbol spaces and are left to the reader.

Since supp ω is compact, $b(\xi, \varrho)$ extends to an element in $\mathscr{A}(\mathbf{C}, C^\infty(\mathbf{R}^n))$ in the complex variable $\varrho + i\beta$. Let us show that for every β this belongs again to $S^\mu_{\mathrm{cl}}(\mathbf{R}^n \times \mathbf{R})$.

We have

$$\begin{aligned} b(\xi, \varrho + i\beta) &= \int e^{-i\zeta\xi - i\varkappa\varrho}\, e^{\varkappa\beta}\, \omega(\varkappa)\, K(a)\,(\zeta, \varkappa)\, d\zeta\, d\varkappa \\ &= \int e^{-i\zeta\xi - i\varkappa\varrho} \left\{ \sum_{j=0}^{N} \frac{(\varkappa\beta)^j}{j!} + R_N(\varkappa, \beta) \right\} \omega(\varkappa)\, K(a)\,(\zeta, \varkappa)\, d\zeta\, d\varkappa \end{aligned}$$

with $\varkappa^{-j} R_N(\varkappa, \beta) \in C^\infty(\mathbf{R}^2)$ for $0 \leqq j \leqq N$. Now standard manipulations with Fourier integrals show that $\varkappa^j K(a)\,(\zeta, \varkappa) = (-1)^j K(D^j_\varrho a)\,(\zeta, \varkappa)$, i.e.

$$\begin{aligned} &\int e^{-i\zeta\xi - i\varkappa\varrho} \sum_{j=0}^{N} \frac{(\varkappa\beta)^j}{j!}\, \omega(\varkappa)\, K(a)\,(\zeta, \varkappa)\, d\zeta\, d\varkappa \\ &= \int e^{-i\zeta\xi - i\varkappa\varrho}\, \omega(\varkappa) \sum_{j=0}^{N} \frac{(-\beta)^j}{j!}\, K(D^j_\varrho a)\,(\zeta, \varkappa)\, d\zeta\, d\varkappa\,. \end{aligned}$$

This belongs certainly to $S^\mu_{\mathrm{cl}}(\mathbf{R}^n \times \mathbf{R})$ for every β, since $D^j_\varrho a \in S^{\mu-j}_{\mathrm{cl}}$ and the summands have the same structure as the right-hand side of (11). This holds obviously uniformly in $c_1 \leqq \beta \leqq c_2$.

Finally we can apply the above remarks to $\psi_N(\varkappa, \beta) = R_N(\varkappa, \beta)\, \omega(\varkappa)$. Thus

$$\int e^{-i\zeta\xi - i\varkappa\varrho}\, \psi_N(\varkappa, \beta)\, K(a)\,(\zeta, \varkappa)\, d\zeta\, d\varkappa \in S^{\mu-N}_{\mathrm{cl}}(\mathbf{R}^n \times \mathbf{R})\,. \tag{12}$$

It was also mentioned that (12) is true uniformly in $c_1 \leqq \beta \leqq c_2$ (it would be needed here only in the sense of $S^{\mu-N}(\mathbf{R}^n \times \mathbf{R})$). □

The amplitude functions in (8) give rise to Mellin-Fourier ψDO's

$$\mathrm{op}^\beta_M\, \mathrm{op}_{\psi,x}\,(h)\, u(x, r) := r^\beta M^{-1}_{w \to r} F^{-1}_{\xi \to x} (T^{-\beta} h)\,(x, r, \xi, w)\, F_{x' \to \xi} M_{r' \to w} (r')^{-\beta}\, u(x', r')\,, \tag{13}$$

with F being the Fourier transform in the x-space. A standard consideration on equivalence of phase functions yields

$$\mathrm{op}^\beta_M\, \mathrm{op}_{\psi,x}\,(h) \in L^\mu_{\mathrm{cl}}(V^\wedge)\,.$$

With h we can also associate the family of ψDO's on V

$$\mathrm{op}_{\psi,x}\,(h)\,(r, w)\, v(x) = F^{-1}_{\xi \to x} h(x, r, \xi, w)\, F_{x' \to \xi} v(x')\,,$$

dependent on the parameters $r \in \bar{\mathbf{R}}_+$, $w \in \Gamma_{1/2-\beta}$. Then

$$\mathrm{op}_{\psi,x}(h)(r,w) \in C^\infty(\bar{\mathbf{R}}_+, L^\mu_{\mathrm{cl}}(V;\Gamma_{1/2-\beta}))\,. \tag{14}$$

Here $\Gamma_{1/2-\beta}$ is identified with $\mathbf{R} \ni \varrho$, and $L^\mu_{\mathrm{cl}}(V;\mathbf{R})$ is the space of parameter-dependent ψDO's $A(\varrho) = A_1(\varrho) + A_0(\varrho)$, where $A_1(\varrho)$ is of the form (3), where we have to insert $a(x,\xi,\varrho) \in S^\mu_{\mathrm{cl}}(V_x \times \mathbf{R}^n_\xi \times \mathbf{R}_\varrho)$, and $A_0(\varrho) \in \mathscr{S}(\mathbf{R}_\varrho, C^\infty(V \times V))$ with the Schwartz space of $C^\infty(V\times V)$-valued functions. In (14) we have employed a natural Fréchet topology of $L^\mu_{\mathrm{cl}}(V;\mathbf{R})$, cf. [S2], Section 2.3.1. (Below in 1.3. we shall employ analogously parameter-dependent ψDO's of the classes $L^\mu_{\mathrm{cl}}(V;\mathbf{R}^q)$ in the corresponding Fréchet topology.)

On the other hand let

$$\tilde{a}(x,r,\xi,\varrho) = a(x,r,\xi,r\varrho) \in \tilde{S}^\mu_{\mathrm{cl}}(\mathrm{c}V^\wedge \times \mathbf{R}^{n+1})\,. \tag{15}$$

Then we can form $\mathrm{op}_{\psi,(x,r)}(\tilde{a}) \in L^\mu_{\mathrm{cl}}(V^\wedge)$, where $\mathrm{op}_{\psi,(x,r)}$ refers to the Fourier transform in $\mathbf{R}^n_x \times \mathbf{R}^1_r$.

2. Theorem. *For every $\beta \in \mathbf{R}$ there exists a non-canonical mapping*

$$m^\beta\colon S^\mu_{\mathrm{cl}}(\mathrm{c}V^\wedge \times \mathbf{R}^{n+1}_{\xi,\varrho}) \to S^\mu_{\mathrm{cl}}(\mathrm{c}V^\wedge \times \mathbf{R}^n_\xi \times \mathbf{C}_w)_{\mathrm{hol}} \tag{16}$$

such that for $h = m^\beta(a)$ and $\tilde{a}$ being defined by (15)

$$\mathrm{op}_{\psi,(x,r)}(\tilde{a}) \sim \mathrm{op}^\beta_M\, \mathrm{op}_{\psi,x}(h)\,, \tag{17}$$

where $\sim$ means equivalence mod operators with C^∞ kernels.

Here and in the sequel a mapping between spaces of amplitude functions is called *non-canonical* if it is linear and unique mod elements of order $-\infty$. In the present case it means that (16) induces a linear mapping

$$S^\mu_{\mathrm{cl}}(\mathrm{c}V^\wedge \times \mathbf{R}^{n+1})/S^{-\infty}(\mathrm{c}V^\wedge \times \mathbf{R}^{n+1}) \to S^\mu_{\mathrm{cl}}(\mathrm{c}V^\wedge \times \mathbf{R}^n \times \mathbf{C})_{\mathrm{hol}}/S^{-\infty}(\mathrm{c}V^\wedge \times \mathbf{R}^n \times \mathbf{C})_{\mathrm{hol}}$$

Clearly instead of (16) we might also talk about

$$\tilde{m}^\beta\colon \tilde{S}^\mu_{\mathrm{cl}}(\mathrm{c}V^\wedge \times \mathbf{R}^{n+1}_{\xi,\varrho}) \to S^\mu_{\mathrm{cl}}(\mathrm{c}V^\wedge \times \mathbf{R}^n_\xi \times \mathbf{C}_w)_{\mathrm{hol}} \tag{18}$$

with $\tilde{m}^\beta(\tilde{a}) := m^\beta(a)$.

Proof of Theorem 2: The dependence of symbols on x, ξ is not the specific point. So for simplicity we will consider a in the form $a(r,\varrho)$. Then $\tilde{a}(r,\varrho) = a(r,r\varrho)$. Let us start with $\beta = 1/2$. Choose an $h(r,w) \in S^\mu_{\mathrm{cl}}(\bar{\mathbf{R}}_+ \times \Gamma_0)$. Then the associated Mellin ψDO is of the form

$$\begin{aligned}\mathrm{op}^{1/2}_M(h)\,u(r) &= \int_{\mathbf{R}} r^{-i\varrho} h(r, i\varrho) \int_{\mathbf{R}_+} (r')^{i\varrho} u(r')\,dr'/r'\,đ\varrho \\ &= \iint e^{-i\varrho(\log r - \log r')}\, b(r,\varrho)\, u(r')\,dr'/r'\,đ\varrho\,,\end{aligned} \tag{19}$$

$b(r,\varrho) := h(r,-i\varrho)$. Now we set $y = \log r$, $y' = \log r'$. Then

$$\mathrm{op}^{1/2}_M(h)\,u(e^y) = \iint e^{i(y-y')\varrho}\, b(e^y,\varrho)\, u(e^{y'})\,dy'\,đ\varrho\,.$$

For the diffeomorphism $\varkappa\colon \mathbf{R} \to \mathbf{R}_+$, $\varkappa(y) = e^y$, we have $(\varkappa^* v)(y) = v(e^y)$. Thus

$$\mathrm{op}_M^{1/2}(h)\, u = (\varkappa^*)^{-1}\, \mathrm{op}_{\psi,y}(f)\, \varkappa^* u \quad \text{for} \quad f(y,\eta) = b(e^y, \eta).$$

In other words, $\mathrm{op}_M^{1/2}(h)$ is the push-forward of a ψDO on $\mathbf{R}$ with the amplitude function $f(y,\eta)$ under $\varkappa$. If in general $\varkappa\colon \Omega_y \to \tilde{\Omega}_{\tilde{y}}$ is a diffeomorphism, then the push-forward $\varkappa_*\colon L^\mu(\Omega) \to L^\mu(\tilde{\Omega})$ is given on the level of corresponding complete symbols $f(y,\eta)$, $\tilde{f}(\tilde{y},\tilde{\eta})$ (i.e. in the sense $\varkappa_*\, \mathrm{op}_{\psi,y}(f) \sim \mathrm{op}_{\psi,\tilde{y}}(\tilde{f})$) by the asymptotic formula

$$\tilde{f}(\tilde{y},\tilde{\eta})|_{\tilde{y}=\varkappa(y)} \sim \sum_\alpha \frac{1}{\alpha!} f^{(\alpha)}(y, ({}^t\mathrm{d}\varkappa(y))\,\tilde{\eta}) \cdot \varphi_\alpha(y,\tilde{\eta}) \tag{20}$$

where $f^{(\alpha)}(y,\eta) = (\partial_\eta^\alpha f)(y,\eta)$,

$$\varphi_\alpha(y,\tilde{\eta}) = D_z^\alpha\, e^{i\delta(y,z)\cdot\tilde{\eta}}|_{y=z} \tag{21}$$

for $\delta(y,z) = \varkappa(z) - \varkappa(y) - \mathrm{d}\varkappa(y)(z-y)$. In the present case we have $\tilde{y} = r$, $\tilde{\eta} = \varrho$, and $\varphi_\alpha(y,\tilde{\eta}) = P_\alpha(r\varrho)$, where P_α is a polynomial in $r\varrho$ of degree $\alpha/2$, $\alpha \in \mathbf{N}$, and $P_0 = 1$. Moreover $\mathrm{d}\varkappa$ is in this case the multiplication by r. Thus $\varkappa_*\, \mathrm{op}_{\psi,y}(f) \sim \mathrm{op}_{\psi,r}(\tilde{f})$ with

$$\tilde{f}(r,\varrho) \sim \sum_{\alpha=0}^{\infty} \frac{1}{\alpha!} f^{(\alpha)}(y, r\varrho)\, P_\alpha(r\varrho), \qquad y = \log r.$$

Since remainders of order $-\infty$ are accepted, we can first form a convergent sum

$$g(y,\tau) = \sum_{\alpha=0}^{\infty} \frac{1}{\alpha!} f^{(\alpha)}(y,\tau)\, P_\alpha(\tau)\, \chi\left(\frac{\tau}{c_\alpha}\right)$$

with an excision function χ and constants c_α, increasing sufficiently fast as $\alpha \to \infty$, and then define $\tilde{f}(r,\varrho) = g(\log r, r\varrho)$. It is clear that then $\tilde{f}(r,\varrho) \in \tilde{S}^\mu_{\mathrm{cl}}(\bar{\mathbf{R}}_+ \times \mathbf{R}_\varrho)$. Moreover,

$$\tilde{f}(r,\varrho) - h(r, -ir\varrho) \in \tilde{S}^{\mu-1}_{\mathrm{cl}}(\bar{\mathbf{R}}_+ \times \mathbf{R}).$$

In other words, for $h_0(r, i\varrho) := a(r, -\varrho)$ it follows an $\tilde{f}_0(r,\varrho) := \tilde{f}(r,\varrho) \in \tilde{S}^\mu_{\mathrm{cl}}(\bar{\mathbf{R}}_+ \times \mathbf{R})$ such that

$$\mathrm{op}_M^{1/2}(h_0) = \mathrm{op}_{\psi,r}(\tilde{f}_0) = \mathrm{op}_{\psi,r}(\tilde{a}) - \mathrm{op}_{\psi,r}(\tilde{a}_1)$$

with $\tilde{a}_1(r,\varrho) := \tilde{f}_0(r,\varrho) - \tilde{a}(r,\varrho) \in \tilde{S}^{\mu-1}_{\mathrm{cl}}(\bar{\mathbf{R}}_+ \times \mathbf{R})$. By applying the procedure again to $\tilde{a}_1$ we get for $h_1(r, i\varrho) := a_1(r, -\varrho) \in S^{\mu-1}_{\mathrm{cl}}(\bar{\mathbf{R}}_+ \times \Gamma_0)$ an $\tilde{f}_1(r,\varrho) \in \tilde{S}^{\mu-1}_{\mathrm{cl}}(\bar{\mathbf{R}}_+ \times \mathbf{R})$ such that

$$\mathrm{op}_M^{1/2}(h_1) = \mathrm{op}_{\psi,r}(\tilde{f}_1) = \mathrm{op}_{\psi,r}(\tilde{a}_1) - \mathrm{op}_{\psi,r}(\tilde{a}_2)$$

with $\tilde{a}_2 = \tilde{f}_1 - \tilde{a}_1 \in \tilde{S}^{\mu-2}_{\mathrm{cl}}(\bar{\mathbf{R}}_+ \times \mathbf{R})$. This can be continued successively, and we get a sequence $h_k \in S^{\mu-k}_{\mathrm{cl}}(\bar{\mathbf{R}}_+ \times \Gamma_0)$, $k \in \mathbf{N}$. If we form $h' \sim \sum_{k=0}^{\infty} h_k$ in the class $S^\mu_{\mathrm{cl}}(\bar{\mathbf{R}}_+ \times \Gamma_0)$, we obtain

$$\mathrm{op}_M^{1/2}(h') \sim \mathrm{op}_{\psi,r}(a).$$

Applying Lemma 10 we may replace h' by $h \in S^{\mu}_{\mathrm{cl}}(\bar{\mathbf{R}}_+ \times \mathbf{C}_w)_{\mathrm{hol}}$. It is now obvious that an analogous construction applies in the case with (x, ξ) dependence. Thus Theorem 2 is completely proved for $\beta = 1/2$.

For general β we apply the following

3. Lemma. *Let $h(x, r, \xi, w) \in S^{\mu}_{\mathrm{cl}}(cV^{\wedge} \times \mathbf{R}^n_{\xi} \times \Gamma_0)$ be arbitrary and $\beta \in \mathbf{R}$. Then there exists an $h_\beta(x, r, \xi, w) \in S^{\mu}_{\mathrm{cl}}(cV^{\wedge} \times \mathbf{R}^n_{\xi} \times \Gamma_{1/2-\beta})$ such that*

$$\mathrm{op}_M^{1/2}\, \mathrm{op}_{\psi,x}(h) \sim \mathrm{op}_M^{\beta}\, \mathrm{op}_{\psi,x}(h_\beta).$$

Proof: For simplicity let us look again at the (x, ξ)-independent case. By definition we have

$$\mathrm{op}_M^{1/2}(h) = r^{1/2}\, \mathrm{op}_M(f_0)\, (r')^{-1/2}, \qquad \mathrm{op}_M^{\beta}(h_\beta) = r^{\beta}\, \mathrm{op}_M(f_1)\, (r')^{-\beta}$$

with $f_0 = T^{-1/2}h$, $f_1 = T^{-\beta}h_\beta$. Thus it suffices to find $f_1 \in S^{\mu}_{\mathrm{cl}}(\bar{\mathbf{R}}_+ \times \Gamma_{1/2})$ in such a way that

$$r^{1/2-\beta}\, \mathrm{op}_M(f_0)\, (r')^{-1/2+\beta} \sim \mathrm{op}_M(f_1). \tag{22}$$

The left-hand side is a Mellin ψDO with r, r'-dependent amplitude function $f(r, r', w) = r^{1/2-\beta} f_0(r, w)\, (r')^{-1/2+\beta}$. Applying the standard Mellin operator calculus (cf. [L1], [S1]) from $f(r, r', w)$ we can pass to an equivalent r'-independent amplitude function $f_1(r, w)$ by

$$f_1(r, w) \sim \sum_{k=0}^{\infty} \frac{1}{k!} \left(-r' \frac{\partial}{\partial r'}\right)^k D^k_{\varrho} f(r, r', w)|_{r=r'},$$

$w = 1/2 + i\varrho$. Every summand belongs to $S^{\mu-k}_{\mathrm{cl}}(\bar{\mathbf{R}}_+ \times \Gamma_{1/2})$ and hence the asymptotic sum can be carried out in $S^{\mu}_{\mathrm{cl}}(\bar{\mathbf{R}}_+ \times \Gamma_{1/2})$. Thus we find f_1 as desired. □

By this we have also finished the proof of Theorem 2. □

4. Remark. *Let $V_1, V_2 \subseteq \mathbf{R}^n$ be open, and $\chi\colon V_1 \to V_2$ be a diffeomorphism, $\chi(x) = y$. Then there exists a non-canonical mapping*

$$\delta\colon \tilde{S}^{\mu}_{\mathrm{cl}}(cV_1^{\wedge} \times \mathbf{R}^{n+1}) \to \tilde{S}^{\mu}_{\mathrm{cl}}(cV_2^{\wedge} \times \mathbf{R}^{n+1})$$

such that for $\tilde{a}_2 = \delta(\tilde{a}_1)$

$$(\chi \times id)_*\, \mathrm{op}_{\psi,(x,r)}(\tilde{a}_1) \sim \mathrm{op}_{\psi,(y,r)}(\tilde{a}_2)$$

with $(\chi \times id)_*$ being the push-forward of ψDO's under the diffeomorphism $\chi \times id\colon V_1 \times \mathbf{R}_+ \to V_2 \times \mathbf{R}_+$.

Note that there is another non-canonical mapping

$$\varkappa\colon S^{\mu}_{\mathrm{cl}}(cV_1^{\wedge} \times \mathbf{R}^n \times \mathbf{C})_{\mathrm{hol}} \to S^{\mu}_{\mathrm{cl}}(cV_2^{\wedge} \times \mathbf{R}^n \times \mathbf{C})_{\mathrm{hol}}$$

such that for $f = \varkappa(h)$

$$(\chi \times id)_*\, \mathrm{op}_M^{\beta}\, \mathrm{op}_{\psi,x}(h) \sim \mathrm{op}_M^{\beta}\, \mathrm{op}_{\psi,y}(f).$$

Moreover, if $\tilde{m}_i^\beta$ are the mappings (18) with respect to V_i, $i = 1, 2$, then the procedure commutes in the sense

$$\varkappa \tilde{m}_1^\beta \sim \tilde{m}_2^\beta \delta ,$$

$\sim$ being equivalence mod elements of order $-\infty$.

Now let $\mathfrak{V} = \{V_j\}_{j=1,\dots,N}$ be an open covering of X by coordinate neighbourhoods. Then a system $\{r^{-\mu} p_j\}_{1 \leqq j \leqq N}$ with

$$p_j \in \tilde{S}^\mu_{\mathrm{cl}}(\mathrm{c}V_j^\wedge \times \mathbf{R}^{n+1}), \qquad j = 1, \dots, N, \tag{23}$$

is called a *complete symbol* on $\bar{X}^\wedge$ (with respect to $\mathfrak{V}^\wedge = \{V_j^\wedge\}_{j=1,\dots,N}$) if

$$\delta_{kj} p_j|_{V_j^\wedge \cap V_k^\wedge} \sim p_k|_{V_j^\wedge \cap V_k^\wedge} \tag{24}$$

for the transition diffeomorphisms $V_j \cap V_k \to V_j \cap V_k$ and the associated mappings δ_{kj} from Remark 4, for all $j, k = 1, \dots, N$. The weight factor $r^{-\mu}$ in front of the symbols is natural for several reasons, as we shall see below. For the moment, of course, it is unessential. For unifying notations we take it into consideration from the very beginning.

Let $\{\varphi_j\}_{j=1,\dots,N}$ be a partition of unity belonging to $\mathfrak{V}$ and $\tilde{\varphi}_j \in C_0^\infty(V_j)$ with $\varphi_j \tilde{\varphi}_j = \varphi_j$ for $j = 1, \dots, N$. Choose a complete symbol (23) and set

$$A_\psi = \sum_{j=1}^N \varphi_j r^{-\mu} \operatorname{op}_{\psi,(x,r)}(p_j)\, \tilde{\varphi}_j , \tag{25}$$

$$A_M = \sum_{j=1}^N \varphi_j \operatorname{op}_M^\beta \operatorname{op}_{\psi,x}(h_j)\, \tilde{\varphi}_j , \tag{26}$$

where $h_j = \tilde{m}^\beta(p_j)$, $j = 1, \dots, N$. For abbreviation the local coordinates on V_j are all denoted by x; clearly they depend on j. By Theorem 2 we have

$$A_\psi - r^{-\mu} A_M \in L^{-\infty}(X^\wedge) .$$

5. Definition. Let $\{r^{-\mu} p_j\}_{1 \leqq j \leqq N}$ be a complete symbol belonging to $\mathfrak{V}^\wedge$. Then the (non-canonical) mapping $\{r^{-\mu} p_j\}_{1 \leqq j \leqq N} \to r^{-\mu} A_M$ (for a fixed choice of φ_j, $\tilde{\varphi}_j$ and of local coordinates on V_j) is called a *Mellin operator convention for the cone* $X^\wedge$.

6. Definition. $M_0^\mu(X)$ denotes the space of all $h(w) \in \mathscr{A}(\mathbf{C}, L^\mu_{\mathrm{cl}}(X))$ such that $h(\delta + \mathrm{i}\varrho) \in L^\mu_{\mathrm{cl}}(X; \mathbf{R}_\varrho)$, uniformly in every strip $c_1 \leqq \delta \leqq c_2, c_1, c_2 \in \mathbf{R}$. The latter condition means that the semi-norms of the Fréchet topology of $L^\mu_{\mathrm{cl}}(X; \mathbf{R}_\varrho)$ are uniformly bounded in $\delta \in [c_1, c_2]$ (cf. also [S2]).

Now $M_0^\mu(X)$ is a Fréchet space in a natural way, and we can form the functions $h(r, w) \in C^\infty(\bar{\mathbf{R}}_+, M_0^\mu(X))$.

The system of h_j in (26) gives rise to

$$h(r, w) := \sum_{j=1}^N \varphi_j \operatorname{op}_{\psi,x}(h_j)\, \tilde{\varphi}_j \in C^\infty(\bar{\mathbf{R}}_+, M_0^\mu(X)) , \tag{27}$$

cf. (14). Thus the Mellin operator convention can be viewed as a mapping

$$\{p_1, \ldots, p_N\} \to C^\infty(\bar{\mathbf{R}}_+, M_0^\mu(X))$$

such that the associated operator equals $\mathrm{op}_M^\beta\,(h)$.

Outside the conical singularity we will preserve the standard operator conventions, based on the Fourier transform. This means that the ψDO's are written as

$$A = \omega r^{-\mu} A_M^\beta \omega_1 + (1 - \omega)\, A_\psi (1 - \omega_2)\,. \tag{28}$$

Here $\omega(r)$, $\omega_i(r) \in C_0^\infty(\bar{\mathbf{R}}_+)$ are cut-off functions, i.e. $\equiv 1$ close to $r = 0$, and we assume

$$\omega\omega_1 = \omega\,, \qquad (1 - \omega)(1 - \omega_2) = 1 - \omega\,. \tag{29}$$

The latter condition ensures that $A \sim A_M^\beta$ and $A \sim A_\psi$. (28) represents also an operator convention. By changing ω, ω_i under the condition (29) we get errors in form of Green operators (cf. the notations below) if the symbols p_j satisfy the exit condition. Otherwise we get Green operators only after localizing at $r = 0$.

Let us recall the definition of totally characteristic Sobolev spaces over $X^\wedge$. For $s, \beta \in \mathbf{R}$ we denote by $\mathscr{H}^{s,\beta}(X^\wedge)$ the completion of $C_0^\infty(X^\wedge)$ with respect to the norm

$$\left\{\frac{1}{2\pi i} \int_{\Gamma_{(n+1)/2-\beta}} \|b^s(w)\,(M_{r\to w}u)\,(x, w)\|_{L^2(X)}^2\, dw\right\}^{1/2}. \tag{30}$$

Here $b^\mu((n + 1)/2 - \beta + i\varrho)$ denotes a parameter-dependent family in $L_{\mathrm{cl}}^\mu(X; \mathbf{R}_\varrho)$, such that $b^\mu(w)\colon H^s(X) \underset{\cong}{\to} H^{s-\mu}(X)$ for all $s \in \mathbf{R}$, $w \in \Gamma_{(n+1)/2-\beta}$. Further we define the space $\mathscr{K}^{s,\beta}(X^\wedge)$ by the conditions

$$\omega(r)\,\mathscr{K}^{s,\beta}(X^\wedge) = \omega(r)\,\mathscr{H}^{s,\beta}(X^\wedge)\,,$$

$$(1 - \omega(r))\,\mathscr{K}^{s,\beta}(X^\wedge) = (1 - \omega(r))\,H^s(X^\wedge)$$

for any cut-off function ω.

The space $H^s(X^\wedge)$ is defined by the conditions $H^s(X^\wedge)|_{V_0^\wedge} = H^s(V_0^\wedge)$ for every coordinate neighbourhood V on X and $V_0 \subset V$ open, ∂V_0 of class C^∞, $\bar{V}_0 \subset V$, where $H^s(V_0^\wedge) := (\pi_c)_*\, H^s(\tilde{V}_0^\wedge)$ for any diffeomorphism $\pi_c\colon \tilde{V}^\wedge \to V^\wedge$ with a conic subset $\tilde{V}^\wedge \subset \mathbf{R}^{n+1} \setminus \{0\}$, $\pi_c(\lambda\tilde{x}) = (x, \lambda r)$ for all $\lambda > 0$, $\pi_c\tilde{V}_0^\wedge = V_0^\wedge$, and $H^s(\tilde{V}_0^\wedge) := H^s(\mathbf{R}^{n+1})|_{\tilde{V}_0^\wedge}$. Finally $\mathscr{H}^{s,\beta}(\mathbf{B})$ is defined as the subspace of $H_{\mathrm{loc}}^s(\mathrm{int}\,\mathbf{B})$ with $\omega\mathscr{H}^{s,\beta}(\mathbf{B}) = \omega\mathscr{H}^{s,\beta}(X^\wedge)$ for a cut-off function ω supported by the tubular neighbourhood U, mentioned in the beginning of this section, where U and $X^\wedge$ are identified via δ.

Under the exit condition for the p_j in the global complete symbol of (28) we get continuous operators

$$A\colon \mathscr{K}^{s,\beta}(X^\wedge) \to \mathscr{K}^{s-\mu,\beta-\mu}(X^\wedge) \tag{31}$$

for all $s \in \mathbf{R}$.

The main topic of this paper is the Mellin convention for the non-smoothing part. There will occur also smoothing Mellin and Green operators. Let us briefly give the definition for the cone.

Let $\mathscr{V}$ be the system of closed subsets of C introduced in [S1], Section 1.2.2., and fix a $V \in \mathscr{V}$ which is quasi-discrete, i.e. $\Gamma_{\delta_j} \cap V = \emptyset$ for all $j \in \mathbf{Z}$ for a system $\delta_j \in \mathbf{R}$ with $\delta_j \to \pm\infty$ as $j \to \pm\infty$. Then $M_V^{-\infty}(X)$ denotes the space of all $h(w) \in \mathscr{A}(\mathbf{C} \setminus V, L^{-\infty}(X))$ such that $(\chi h)(\delta + i\varrho) \in \mathscr{S}(\mathbf{R}_\varrho, C^\infty(X \times X))$ uniformly in every strip $c_1 \leqq \delta \leqq c_2$, $c_1, c_2 \in \mathbf{R}$, for every V-excision function χ, i.e. $\chi \in C^\infty(\mathbf{C})$, $\chi \equiv 0$ close to V, $0 \leqq \chi \leqq 1$, $\chi = 1$ outside a neighbourhood of V. The space $M_V^{-\infty}(X)$ is Fréchet in a natural way. Every $V \in \mathscr{V}$ can be written as $V = V_1 + V_2$ for quasi-discrete V_i (cf. [S1], 1.1.5.) and we then define

$$M_V^{-\infty}(X) = M_{V_1}^{-\infty}(X) + M_{V_2}^{-\infty}(X) \tag{32}$$

in the Fréchet topology of the sum (cf. [S1], 1.1.2.), which is independent of the choice of V_1, V_2. Set

$$M_{\mathrm{as}}^{-\infty}(X) = \varinjlim_{V \in \mathscr{V}} M_V^{-\infty}(X) .$$

Thus the gaps of V_i can be chosen in a convenient way, according to weights, involved in the action.

If $\beta \in \mathbf{R}$ and $h \in M_{\mathrm{as}}^{-\infty}(X)$ are given, we define for every fixed $j \in \mathbf{N}$ an analogue of $\mathrm{op}_M^\beta(h)$ by

$$\mathrm{op}_M^{\beta,j}(h) = r^\beta\{\mathrm{op}_M^{\lambda_1}(T^{-\beta}h_1) + \mathrm{op}_M^{\lambda_2}(T^{-\beta}h_2)\}\, r^{-\beta}$$

for an arbitrary fixed decomposition $h = h_1 + h_2$ such that $T^{-\beta}h_i$ is holomorphic close to $\Gamma_{1/2-\lambda_i}$, and $\lambda_i \geqq 0$, $j - \lambda_i \geqq 0$, $i = 1, 2$. For $j = 0$ we have necessarily $\mathrm{op}_M^{\beta,0}(h) = \mathrm{op}_M^\beta(h)$, $\Gamma_{1/2-\beta} \cap V = \emptyset$, whereas for $j > 0$ there always exist h_i, λ_i as required. A result of the Mellin operator calculus is that $\omega_1 \mathrm{op}_M^{\beta,j}(h)\, \omega_2$ is independent of the concrete choice of decomposition data h_i, λ_i, modulo a Green operator, ω_1, ω_2 being arbitrary cut-off functions (cf. [S2], 2.1.5., Theorem 13).

Now the mentioned smoothing Mellin operators have the form

$$M = \omega_1 r^{-\mu} \sum_{j=0}^{k-1} r^j \mathrm{op}_M^{\beta-n/2,j}(h_j)\, \omega_2 \tag{33}$$

for a fixed weight interval $\theta = (-k, 0]$ (relative to β), and arbitrary $h_j \in M_{\mathrm{as}}^{-\infty}(X)$, $j = 0, \ldots, k-1$; $n = \dim X$.

Parallel to the spaces of Mellin symbols there are defined the subspaces of $\mathscr{H}^{s,\beta}(\mathbf{B})$ with (continuous) asymptotics. Fix a weight interval $\theta = (-k, 0]$, $k \in \mathbf{N} \setminus \{0\}$, which corresponds in the complex Mellin plane to the strip

$$(n+1)/2 - \beta - k < \operatorname{Re} w < (n+1)/2 - \beta . \tag{34}$$

Let $\mathscr{H}_\theta^{s,\beta}(\mathbf{B})$ denote the subspace of all $u \in \mathscr{H}^{s,\beta}(\mathbf{B})$ for which $r^{-\varkappa}u \subset \mathscr{H}^{s,\beta}(\mathbf{B})$ for all $\varkappa$ with $0 \leqq \varkappa < k$ (as above $\omega\mathscr{H}^{s,\beta}(\mathbf{B})$ is identified with $\omega\mathscr{H}^{s,\beta}(X^\wedge)$). Remember that in [S2] we have used the notation $\mathcal{H}_0^{s,\beta}(\mathbf{B})_\theta$ instead and gave another equivalent definition.

Now let $V \in \mathscr{V}$, $V \subset \{\operatorname{Re} w < (n+1)/2 - \beta\}$ be quasi-discrete. Then $M\mathcal{H}_V^{\infty,\beta}(X^\wedge)$ consists of all $h(w) \in \mathscr{A}(\mathbf{C} \setminus V, H^\infty(X))$ such that

$$\frac{1}{2\pi i} \int_{\Gamma_\delta} \|b^s(w)\,(\chi h)\,(w)\|^2_{L^2(X)}\,dw < \infty$$

for all $s \in \mathbf{R}$ and every V-excision function χ, uniformly in every strip $c_1 \leqq \delta \leqq c_2$, $c_1, c_2 \in \mathbf{R}$. Here $b^s(w)$ is of analogous structure as the order reducing family of (30) above, now in the interpretation that it is shifted for any given δ to the corresponding weight line in an obvious manner.

Denote by $\mathcal{H}_V^{\infty,\beta}(X^\wedge)$ the preimage under the inverse weighted Mellin transform (with respect to the weight line $\Gamma_{(n+1)/2-\beta}$). The space $\mathcal{H}_V^{\infty,\beta}(X^\wedge)$ is Fréchet in a natural way. For arbitrary $V \in \mathscr{V}$, $V \subset \{\operatorname{Re} w < (n+1)/2 - \beta\}$, $V = V_1 + V_2$, and V_i quasi-discrete, we set

$$\mathcal{H}_V^{\infty,\beta}(X^\wedge) = \mathcal{H}_{V_1}^{\infty,\beta}(X^\wedge) + \mathcal{H}_{V_2}^{\infty,\beta}(X^\wedge).$$

This is independent of the concrete choice of V_1, V_2 with V as sum.

It is well known that every $u \in \mathcal{H}_V^{\infty,\beta}(X^\wedge)$ for quasi-discrete V represents a sequence of $C^\infty(X)$-valued analytic functionals ζ_i on the components $V_{(i)}$ of V, i.e. $\zeta_i \in \mathscr{A}'(V_{(i)}, C^\infty(X))$. They determine the (continuous) asymptotics for u as $r \to 0$ (cf. [S2]). If Mu is meromorphic, we get the special case of the discrete asymptotics. If we are interested in the asymptotic information only in the strip (34) of the complex Mellin plane, we form the sum of Fréchet spaces

$$\mathcal{H}_{\tilde{V}}^{\infty,\beta}(X^\wedge) := \mathcal{H}_V^{\infty,\beta}(X^\wedge) + \mathcal{H}_\theta^{\infty,\beta}(X^\wedge),$$

where $\tilde{V} := V \cap \{(n+1)/2 - \beta - k < \operatorname{Re} w < (n+1)/2 - \beta\}$. The asymptotic information for a concrete $u \in \mathcal{H}_{\tilde{V}}^{\infty,\beta}(X^\wedge)$ is represented by the system of data

$$C = \{V = \cup V_{(i)}, \times_i \mathscr{A}'(V_{(i)}, C^\infty(X)); \beta, \theta\}.$$

For V in general the asymptotic information is given by an equivalence class of pairs of this type. We denote by As $(X; \mathfrak{g})$ for $\mathfrak{g} = (\beta, \theta)$ the set of (continuous) asymptotic data in this sense. In other words every $C \in$ As $(X; \mathfrak{g})$ can be represented by a pair (C_1, C_2) where C_j is associated with V_j as mentioned, $j = 1, 2$. We then set

$$\mathcal{H}_{\tilde{V}}^{\infty,\beta}(X^\wedge) := \mathcal{H}_\theta^{\infty,\beta}(X^\wedge) + \mathcal{H}_{V_1}^{\infty,\beta}(X^\wedge) + \mathcal{H}_{V_2}^{\infty,\beta}(X^\wedge).$$

For arbitrary $s \in \mathbf{R}$ we define

$$\mathcal{H}_{\tilde{V}}^{s,\beta}(X^\wedge) = \mathcal{H}_\theta^{s,\beta}(X^\wedge) + \mathcal{H}_{\tilde{V}}^{\infty,\beta}(X^\wedge)$$

and set

$$\mathscr{H}_V^{s,\beta}(\mathbf{B}) = \{u \in \mathscr{H}^{s,\beta}(\mathbf{B}) : \omega u \in \mathscr{H}_V^{s,\beta}(X^\wedge)\},$$

$$\mathscr{K}_V^{s,\beta}(X^\wedge) = \{u \in \mathscr{K}^{s,\beta}(X^\wedge) : \omega u \in \mathscr{H}_V^{s,\beta}(X^\wedge)\}.$$

In [S2] we have used the notations $\mathscr{H}_V^{s,\beta}(X^\wedge)_\theta$, $\mathscr{H}_V^{s,\beta}(\mathbf{B})_\theta$ and $\mathscr{K}_V^{s,\beta}(X^\wedge)_\theta$, respectively, where the definitions are equivalent to the present ones. Intuitively $C \in \mathrm{As}\,(X; \mathfrak{g})$ can be associated with the set $\tilde{V} = V \cap \{(n+1/2 - \beta - k < \operatorname{Re} w < (n+1)/2 - \beta\}$. The sets $\tilde{V}$ will be called carriers of asymptotics of asymptotic types in $\mathrm{As}\,(X; \mathfrak{g})$. Our notation indicates X as the base of the cone, since analogous notations apply to corners with $\mathbf{B}$ instead of X. Then the analytic functionals are $\mathscr{H}_V^{\infty,\beta}(\mathbf{B})$-valued.

Finally the Green operators G are characterized by the mapping properties

$$G : \mathscr{H}^{s,\beta}(\mathbf{B}) \to \mathscr{H}_{\tilde{V}}^{\infty,\beta-\mu}(\mathbf{B}),$$

$$G^* : \mathscr{H}^{s,-\beta+\mu}(\mathbf{B}) \to \mathscr{H}_{\tilde{W}}^{\infty,-\beta}(\mathbf{B})$$

for arbitrary $s \in \mathbf{R}$ with

$$\tilde{V} = V \cap \{(n+1)/2 - \beta + \mu - k < \operatorname{Re} w\},$$

$$\tilde{W} = W \cap \{(n+1)/2 + \beta - k < \operatorname{Re} w\}$$

for certain $V, W \in \mathfrak{V}$, $V \subset \{\operatorname{Re} w < (n+1)/2 - \beta + \mu\}$, $W \subset \{\operatorname{Re} w < (n+1)/2 + \beta\}$, dependent on G. Here the * refers to a fixed scalar product $(.,.)_0$ in $\mathscr{H}^0(\mathbf{B})$ and the duality

$$(.,.)_0 : \mathscr{H}^{s,\beta}(\mathbf{B}) \times \mathscr{H}^{-s,-\beta}(\mathbf{B}) \to \mathbf{C}.$$

The class $C^\mu(\mathbf{B}, \mathfrak{g})$ of cone operators over $\mathbf{B}$ for the weight data $\mathfrak{g} = (\beta, \theta)$ is altogether defined as the set of all

$$\omega r^{-\mu} A_M \omega_1 + (1 - \omega) P(1 - \omega_2) + M + G \tag{35}$$

for arbitrary $A_M = \mathrm{op}_M^{\beta - n/2}(h)$, $h(r, w) \in C^\infty(\bar{\mathbf{R}}_+, M_0^\mu(X))$ and $P \in L_{\mathrm{cl}}^\mu(\operatorname{int} \mathbf{B})$ with cut-off functions ω, ω_1, and M being a smoothing Mellin operator of the form (33), G a Green operator of the described sort. ♦

Since $C^\mu(\mathbf{B}, \mathfrak{g}) \subset L_{\mathrm{cl}}^\mu(\operatorname{int} \mathbf{B})$, we have for every $A \in C^\mu(\mathbf{B}, \mathfrak{g})$ a homogeneous principal symbol

$$\sigma_\psi^\mu(A) \in C^\infty(T^*(\operatorname{int} \mathbf{B}) \setminus 0), \tag{36}$$

which is close to $\partial \mathbf{B}$ in the coordinates (x, r, ξ, ϱ), $x \in V$, of the form $r^{-\mu} p_{(\mu)}(x, r, \xi, r\varrho)$ for a function $p_{(\mu)}(x, r, \xi, \varrho) \in C^\infty(V \times \bar{\mathbf{R}}_+ \times (\mathbf{R}^{n+1} \setminus \{0\}))$, with $p_{(\mu)}(x, r, \lambda\xi, \lambda\varrho) = \lambda^\mu p_{(\mu)}(x, r, \xi, \varrho)$ for all $(x, r, \xi, \varrho) \in V \times \bar{\mathbf{R}}_+ \times (\mathbf{R}^{n+1} \setminus \{0\})$ and $\lambda > 0$.

The Mellin symbol of A of conormal order μ is defined as

$$\sigma_M^\mu(A)(w) = h(0, w) + h_0(w) \tag{37}$$

with the mentioned $h(r, w) \in C^\infty(\bar{\mathbf{R}}_+, M_O^\mu(X))$ and $h_0(w)$ from (33). We do not recall here anything from the algebra properties and the symbolic rules that are elaborated in detail in [S2], Chapter 2, cf. also [S5]. Let us only remember that

$$\sigma_\psi^\mu(A) = 0\,, \qquad \sigma_M^\mu(A) = 0 \Rightarrow A \in C^{\mu-1}(\mathbf{B}, \mathfrak{g})\,.$$

1.3. Pseudo-Differential Operators on Manifolds with Edges

As mentioned in the beginning the corner calculus contains operators near the outgoing edges. This section studies operators away from the corner vertex. We allow the edge to be of arbitrary dimension q. Below for the corner we then have $q = 1$.

Similarly as for the cone we study the interior of the stretched wedge

$$X^\wedge \times \Omega = X \times \bar{\mathbf{R}}_+ \times \Omega \ni (x, r, y)$$

with X being a closed compact C^∞ manifold of dimension n, $\Omega \subseteq \mathbf{R}^q$ open.

In the previous section we have defined $S_{\mathrm{cl}}^\mu(\mathrm{c}V^\wedge \times \mathbf{R}_{\xi,\varrho}^{n+1})$. Analogously we obtain the space $S_{\mathrm{cl}}^\mu(\mathrm{c}V^\wedge \times \Omega \times \mathbf{R}_{\xi,\varrho}^{n+1} \times \mathbf{R}_\eta^q)$ for open $\Omega \subseteq \mathbf{R}^q$. Then we define

$$\begin{aligned}&\tilde{S}_{\mathrm{cl}}^\mu(\mathrm{c}V^\wedge \times \Omega \times \mathbf{R}_{\xi,\varrho}^{n+1} \times w\mathbf{R}_\eta^q)\\ &= \{a(x, r, y, \xi, r\varrho, r\eta)\colon a(x, r, y, \xi, \varrho, \eta) \in S_{\mathrm{cl}}^\mu(\mathrm{c}V^\wedge \times \Omega \times \mathbf{R}^{n+1} \times \mathbf{R}_\eta^q)\}\,.\end{aligned}$$

The notation $w\mathbf{R}_\eta^q$ indicates the wedge degenerate behaviour in η consisting of the combination $r\eta$. The tilda indicates $r\varrho$, the totally characteristic behaviour near $r = 0$. By

$$S_{\mathrm{cl}}^\mu(\mathrm{c}V^\wedge \times \Omega \times \mathbf{R}_\xi^n \times \Gamma_{1/2-\beta} \times \mathbf{R}_\eta^q) \tag{1}$$

we denote the space of all $h(x, r, y, \xi, w, \eta)$, defined for $w \in \Gamma_{1/2-\beta}$, such that

$$h(x, r, y, \xi, 1/2 - \beta + \mathrm{i}\varrho, \eta) \in S_{\mathrm{cl}}^\mu(\mathrm{c}V^\wedge \times \Omega \times \mathbf{R}_{\xi,\varrho}^{n+1} \times \mathbf{R}_\eta^q)\,. \tag{2}$$

Further $S_{\mathrm{cl}}^\mu(\mathrm{c}V^\wedge \times \Omega \times \mathbf{R}_\xi^n \times \Gamma_{1/2-\beta} \times w\mathbf{R}_\eta^q)$ is defined as the set of all $h(x, r, y, \xi, w, r\eta)$ for which $h(x, r, y, \xi, w, \eta)$ belongs to (1). Then

$$S_{\mathrm{cl}}^\mu(\mathrm{c}V^\wedge \times \Omega \times \mathbf{R}^n \times \mathbf{C} \times \mathbf{R}^q)_{\mathrm{hol}} \tag{3}$$

consists of all $h(x, r, y, \xi, w, \eta) \in \mathscr{A}(\mathbf{C}_w, C^\infty(V \times \bar{\mathbf{R}}_+ \times \Omega \times \mathbf{R}_{\xi,\eta}^{n+q}))$ such that (2) holds uniformly in $c_1 \leqq \beta \leqq c_2$ for all $c_1, c_2 \in \mathbf{R}$, and $S_{\mathrm{cl}}^\mu(\mathrm{c}V^\wedge \times \Omega \times \mathbf{R}^n \times \mathbf{C} \times w\mathbf{R}^q)_{\mathrm{hol}}$ is the space of all $h(x, r, y, \xi, w, r\eta)$ with $h(x, r, y, \xi, w, \eta)$ in (3).

1. Remark. *For every $h \in S_{\mathrm{cl}}^\mu(\mathrm{c}V^\wedge \times \Omega \times \mathbf{R}^n \times \Gamma_{1/2-\beta} \times \mathbf{R}^q)$ there exists an $h_{-\infty} \in S^{-\infty}(\mathrm{c}V^\wedge \times \Omega \times \mathbf{R}^n \times \Gamma_{1/2-\beta} \times \mathbf{R}^q)$ such that $h - h_{-\infty}$ extends to an element in* (3).

This follows by a kernel excision argument, analogously to 1.2., Lemma 1.

With $h \in (1)$ we form parameter-dependent operators

$$\begin{aligned}&\mathrm{op}_M^\beta\, \mathrm{op}_{\psi,x}(h)(y, \eta)\\ &:= r^\beta M_{w\to r}^{-1} F_{\xi\to x}^{-1}(T^{-\beta}h)(x, r, y, \xi, w, r\eta)\, F_{x'\to\xi} M_{r'\to w}(r')^{-\beta}\,,\end{aligned}$$

$(y, \eta) \in \Omega \times \mathbf{R}^q$ being the parameters $(T^{-\beta}h)(\ldots, w, \ldots) = h(\ldots, w - \beta, \ldots)$. (4) is analogous to 1.2., (13) but here we insert $r\eta$ instead of η. This is indicated by $\mathbf{M}$. It is clear that $\mathrm{op}_{\mathbf{M}}^{\beta} \mathrm{op}_{\psi,x}(h)(y, \eta)$ is a (y, η) dependent operator family in $L_{\mathrm{cl}}^{\mu}(V^{\wedge})$ and

$$\mathrm{op}_{\psi,y}\{\mathrm{op}_{\mathbf{M}}^{\beta} \mathrm{op}_{\psi,x}(h)\} \in L_{\mathrm{cl}}^{\mu}(V^{\wedge} \times \Omega).$$

Now let

$$p(x, r, y, \xi, \varrho, \eta) = a(x, r, y, \xi, r\varrho, r\eta) \in \tilde{S}_{\mathrm{cl}}^{\mu}(\mathrm{c}V^{\wedge} \times \Omega \times \mathbf{R}_{\xi,\varrho}^{n+1} \times w\mathbf{R}_{\eta}^{q}).$$

Then $\mathrm{op}_{\psi,(x,r)}(p)(y, \eta)$ is an operator family in $L_{\mathrm{cl}}^{\mu}(V^{\wedge})$, dependent on (y, η), and we want to pass to a representation of the form (4).

2. Theorem. *For every $\beta \in \mathbf{R}$ there exists a non-canonical mapping*

$$m^{\beta}: S_{\mathrm{cl}}^{\mu}(\mathrm{c}V^{\wedge} \times \Omega \times \mathbf{R}_{\xi,\varrho,\eta}^{n+1+q}) \to S_{\mathrm{cl}}^{\mu}(\mathrm{c}V^{\wedge} \times \Omega \times \mathbf{R}_{\xi}^{n} \times \mathbf{C}_{w} \times \mathbf{R}_{\eta}^{q})_{\mathrm{hol}} \tag{5}$$

such that

$$\mathrm{op}_{\psi,(x,r)}(p)(y, \eta) \sim \mathrm{op}_{\mathbf{M}}^{\beta} \mathrm{op}_{\psi,x}(h)(y, \eta)$$

for $h = m^{\beta}(a)$ and

$$p(x, r, y, \xi, \varrho, \eta) = a(x, r, y, \xi, r\varrho, r\eta).$$

The equivalence holds for all (y, η) in the sense of $L_{\mathrm{cl}}^{\mu}(V^{\wedge})$ but also in the sense of $L_{\mathrm{cl}}^{\mu}(V^{\wedge} \times \Omega)$ after applying $\mathrm{op}_{\psi,y}$ on both sides.

Incidentally it is convenient to use instead of (5) the mapping

$$\tilde{m}^{\beta}: \tilde{S}_{\mathrm{cl}}^{\mu}(\mathrm{c}V^{\wedge} \times \Omega \times \mathbf{R}^{n+1} \times w\mathbf{R}^{q}) \to S_{\mathrm{cl}}^{\mu}(\mathrm{c}V^{\wedge} \times \Omega \times \mathbf{R}^{n} \times \mathbf{C} \times \mathbf{R}^{q})_{\mathrm{hol}}$$

defined as $\tilde{m}^{\beta} = m^{\beta} \circ \iota$,

$$\iota: \tilde{S}_{\mathrm{cl}}^{\mu}(\mathrm{c}V^{\wedge} \times \Omega \times \mathbf{R}^{n+1} \times w\mathbf{R}^{q}) \to S_{\mathrm{cl}}^{\mu}(\mathrm{c}V^{\wedge} \times \Omega \times \mathbf{R}^{n+1+q}),$$

$$\iota: a(x, r, y, \xi, r\varrho, r\eta) \to a(x, r, y, \xi, \varrho, \eta).$$

The proof of Theorem 2 follows by a straightforward modification of the arguments for 1.2., Theorem 2. In a more special situation it is contained in [S2], Chapter 3.

3. Proposition. *Let $V_1, V_2 \subseteq \mathbf{R}^n$, $\Omega_1, \Omega_2 \subseteq \mathbf{R}^q$ be open, and $\chi: V_1 \to V_2$, $\varphi: \Omega_1 \to \Omega_2$ be diffeomorphisms, $\chi(x) = \tilde{x}$, $\varphi(y) = \tilde{y}$. Then there exists a non-canonical mapping*

$$\delta: \tilde{S}_{\mathrm{cl}}^{\mu}(\mathrm{c}V_1^{\wedge} \times \Omega_1 \times \mathbf{R}^{n+1} \times w\mathbf{R}^{q}) \to \tilde{S}_{\mathrm{cl}}^{\mu}(\mathrm{c}V_2^{\wedge} \times \Omega_2 \times \mathbf{R}^{n+1} \times w\mathbf{R}^{q})$$

such that

$$(\chi \times \mathrm{id} \times \varphi)_{*} \mathrm{op}_{\psi,(x,r,y)}(p_1) \sim \mathrm{op}_{\psi,(\tilde{x},r,\tilde{y})}(p_2)$$

for $p_2 = \delta(p_1)$ *with* $(\chi \times \mathrm{id} \times \varphi)_*$ *being the push-forward of* ψDO's *under the diffeomorphism* $\chi \times \mathrm{id} \times \varphi \colon V_1 \times \mathbf{R}_+ \times \Omega_1 \to V_2 \times \mathbf{R}_+ \times \Omega_2$.

4. Proposition. *There is a non-canonical mapping*

$$\varkappa \colon S^{\mu}_{\mathrm{cl}}(\mathrm{c}V_1^{\wedge} \times \Omega_1 \times \mathbf{R}^n \times \mathbf{C} \times \mathbf{R}^q)_{\mathrm{hol}} \to S^{\mu}_{\mathrm{cl}}(\mathrm{c}V_2^{\wedge} \times \Omega_2 \times \mathbf{R}^n \times \mathbf{C} \times \mathbf{R}^q)_{\mathrm{hol}}$$

such that

$$(\chi \times \mathrm{id} \times \varphi)_* \, \mathrm{op}_{\psi,y} \, \mathrm{op}^{\beta}_{M} \, \mathrm{op}_{\psi,x} \, (h_1) \sim \mathrm{op}_{\psi,\tilde{y}} \, \mathrm{op}^{\beta}_{M} \, \mathrm{op}_{\psi,\tilde{x}} \, (h_2)$$

for $h_2 = \varkappa(h_1)$ *and*

$$\varkappa \tilde{m}_1^{\beta} \sim \tilde{m}_2^{\beta} \delta$$

with $\sim$ *being equivalence mod elements of order* $-\infty$.

Propositions 3, 4 are easy consequences of the ψD substitution rule for ψDO's on complete symbolic level.

Let Y be a closed compact C^{∞} manifold, $q = \dim Y$, which plays the role of an edge globally.

Let $\mathfrak{V} = \{V_k\}_{0 \leq k \leq M}$ be as in Section 1.2., and $\mathfrak{W} = \{\Omega_l\}_{0 \leq l \leq M}$ be an open covering of Y by coordinate neighbourhoods, with local coordinates $y \in \mathbf{R}^q \cong \Omega_l$. Then a system $\{r^{-\mu} b_{kl}\}_{k,l=0,\dots,M}$ with $b_{kl} \in \tilde{S}^{\mu}_{\mathrm{cl}}(\mathrm{c}V_k^{\wedge} \times \Omega_l \times \mathbf{R}^{n+1} \times w\mathbf{R}^q)$, $k, l = 0, \dots, M$, is called a complete symbol on $X^{\wedge} \times Y$ (with respect to $\mathfrak{V}^{\wedge}, \mathfrak{W}$), if

$$\delta_{k'l',kl} b_{jl}|_{V_k^{\wedge} \times \Omega_l \cap V_{k'}^{\wedge} \times \Omega_{l'}} \sim b_{k'l'}|_{V_k^{\wedge} \times \Omega_l \cap V_{k'}^{\wedge} \times \Omega_{l'}}$$

for the transition diffeomorphisms $V_k \times \Omega_l \cap V_{k'} \times \Omega_{l'} \to V_k \times \Omega_l \cap V_{k'} \times \Omega_{l'}$ and the associated mappings $\delta_{k'l',kl}$ from Proposition 3, for all $k, l, k', l' = 0, \dots, M$. For the moment we only need global symbols over $X^{\wedge} \times \Omega$ with respect to $\mathfrak{V}^{\wedge}$ and a coordinate neighbourhood Ω on Y. These are tuples

$$\{r^{-\mu} b_k\}_{0 \leq k \leq M} \tag{6}$$

with

$$b_k \in \tilde{S}^{\mu}_{\mathrm{cl}}(\mathrm{c}V_k^{\wedge} \times \Omega \times \mathbf{R}^{n+1} \times w\mathbf{R}^q), \qquad k = 0, \dots, M \tag{7}$$

satisfying the corresponding equivalences over $V_k^{\wedge} \times \Omega \cap V_{k'}^{\wedge} \times \Omega$. Let $b_{k,(\mu)}(x, r, y, \xi, r\varrho, r\eta)$ denote the homogeneous principal part of b_k of order μ. Then the system $\{b_{k,(\mu)}\}_{k=0,\dots,M}$ consists of the local representatives of a global function

$$b_{(\mu)} \in C^{\infty}(T^*(X^{\wedge} \times \Omega) \setminus 0), \tag{8}$$

where $T^*(\dots) \setminus 0$ means the cotangent bundle of the manifold in the brackets minus the zero section. Similarly $\{b_{kl,(\mu)}\}$ gives rise to an element

$$b_{(\mu)} \in C^{\infty}(T^*(X^{\wedge} \times Y) \setminus 0). \tag{9}$$

For given (6) we now form the operator families

$$b_\psi(y, \eta) = \sum_{k=0}^{M} r^{-\mu} \psi_k \, \mathrm{op}_{\psi,(x,r)}(b_k)(y, \eta) \, \tilde{\psi}_k \,, \tag{10}$$

$$b_{\mathbf{M}}(y, \eta) = \sum_{k=0}^{M} \psi_k \, \mathrm{op}_{\mathbf{M}}^{\beta} \, \mathrm{op}_{\psi,x}(h_k)(y, \eta) \, \tilde{\psi}_k \tag{11}$$

where $h_k = \tilde{m}^\beta(b_k)$, $k = 0, \dots, M$ (cf. the analogous expressions 1.2. (25), (26)). Then it follows

$$\begin{aligned} b_\psi(y, \eta) &\sim r^{-\mu} b_{\mathbf{M}}(y, \eta) &&\text{in} \quad L^\mu_{\mathrm{cl}}(X^\wedge) \quad \text{for all} \quad y, \eta \,, \\ \mathrm{op}_{\psi,y}(b_\psi) &\sim r^{-\mu} \, \mathrm{op}_{\psi,y}(b_{\mathbf{M}}) &&\text{in} \quad L^\mu_{\mathrm{cl}}(X^\wedge \times \Omega) \,. \end{aligned}$$

5. Definition. Let $\{r^{-\mu} b_k\}_{0 \leqq k \leqq M}$ be a complete symbol over $X^\wedge \times \Omega$ belonging to $\mathfrak{B}^\wedge$. Then the (non-canonical) mapping $\{r^{-\mu} b_k\}_{0 \leqq k \leqq M} \to r^{-\mu} b_{\mathbf{M}}(y, \eta)$ is called a *Mellin operator convention for the wedge* $X^\wedge \times \Omega$ (on edge symbolic level).

6. Definition. $M^\mu_0(X; \mathbf{R}^q)$ denotes the space of all $h(w, \eta) \in \mathscr{A}(\mathbf{C}, L^\mu_{\mathrm{cl}}(X; \mathbf{R}^q))$ such that $h(\delta + i\varrho, \eta) \in L^\mu_{\mathrm{cl}}(X; \mathbf{R}^q_\eta \times \mathbf{R}^1_\varrho)$, uniformly in every strip $c_1 \leqq \delta \leqq c_2$, $c_1, c_2 \in \mathbf{R}$.

The space $M^\mu_0(X; \mathbf{R}^q)$ has a natural Fréchet structure. Thus it makes sense to define $C^\infty(\bar{\mathbf{R}}_+ \times \Omega, M^\mu_0(X; \mathbf{R}^q))$.

For $q = 1$, z varying on $\Gamma_{1/2-\gamma}$, we set

$$M^\mu_0(X; \Gamma_{1/2-\gamma}) \ni h(w, z) \Leftrightarrow h(w, 1/2 - \gamma + i\tau) \in M^\mu_0(X; \mathbf{R}_\tau) \,.$$

From the system of h_j of (11) we can pass to the operator family

$$h(r, y, w, \eta) = \sum_{k=0}^{M} \psi_k \, \mathrm{op}_{\psi,x}(h_k)(r, y, w, \eta) \, \tilde{\psi}_k \,, \tag{12}$$

acting globally along X. Then

$$h(r, y, w, \eta) \in C^\infty(\bar{\mathbf{R}}_+ \times \Omega, M^\mu_0(X; \mathbf{R}^q_\eta)) \,. \tag{13}$$

Thus the Mellin operator convention leads to a mapping

$$\{b_k\}_{0 \leqq k \leqq M} \to C^\infty(\bar{\mathbf{R}}_+ \times \Omega, M^\mu_0(X; \mathbf{R}^q)) \tag{14}$$

and $b_{\mathbf{M}}(y, \eta) = \mathrm{op}_{\mathbf{M}}^\beta(h)(y, \eta)$.

Let $\eta \to [\eta]$ denote a strictly positive C^∞ function on $\mathbf{R}^q$ with $[\eta] = |\eta|$ for $|\eta| \geqq$ const. Choose cut-off functions $\omega(r)$, $\omega_i(r)$, $i = 1, 2$, satisfying the conditions 1.2. (29). Then we form the operator family

$$\begin{aligned} b(y, \eta) := {} & r^{-\mu} \omega(r[\eta]) \, b_{\mathbf{M}}(y, \eta) \, \omega_1(r[\eta]) \\ & + (1 - \omega(r[\eta])) \, b_\psi(y, \eta) \, (1 - \omega_2(r[\eta])) \end{aligned} \tag{15}$$

which is again equivalent to $b_\psi(y, \eta)$ and $r^{-\mu} b_{\mathbf{M}}(y, \eta)$, both with respect to $L^\mu_{\mathrm{cl}}(V^\wedge)$ for all y, η and with respect to $L^\mu_{\mathrm{cl}}(V^\wedge \times \Omega)$ after applying $\mathrm{op}_{\psi,y}$.

The operator family (15) will be interpreted in the edge ψD calculus as a (complete) operator-valued symbol. It has again a symbolic structure that we want to analyze. First we have the homogeneous principal interior symbol of order μ

$$\sigma_\psi^\mu(b)\,(x, r, y, \xi, \varrho, \eta) = r^{-\mu} b_{(\mu)}(x, r, y, \xi, r\varrho, r\eta)\,, \tag{16}$$

cf. (8). Further

$$\sigma_M^\mu(b)\,(y, w) := h_0(y, w) := h(0, y, w, 0) \tag{17}$$

is the Mellin symbol of (15) of conormal order μ (cf. (13)). It is an operator-valued function on $\Omega \times \Gamma_{1/2-\beta}$. Another operator-valued symbol is the homogeneous principal edge symbol of (15) of order μ, namely

$$\begin{aligned} \sigma_\Lambda^\mu(b)\,(y, \eta) = {} & \omega(r\,|\eta|)\, r^{-\mu}\, \mathrm{op}_M^\beta\,(h_0)\,(y, \eta)\, \omega_1(r\,|\eta|) \\ & + (1 - \omega(r\,|\eta|))\, b_{\psi,0}(y, \eta)\, (1 - \omega_2(r\,|\eta|))\,. \end{aligned} \tag{18}$$

Here

$$b_{\psi,0}(y, \eta) = \sum_{k=0}^{M} r^{-\mu} \varphi_k\, \mathrm{op}_{\psi,(x,r)}\,(b_{k,0})\,(y, \eta)\, \tilde{\varphi}_k$$

with $b_{k,0}(x, y, \xi, r\varrho, r\eta) := b_k(x, 0, y, \xi, r\varrho, r\eta)$, $(y, \eta) \in \Omega \times \mathbf{R}^q$ (cf. analogously (10)).

Note that here, in contrast to the calculus of [S1], Chapter 3, there are not only involved the homogeneous principal parts of b_k. In [S1] we have used a smaller class of interior symbols, and it was assumed $X = S^n$. Nevertheless, the leading edge symbol satisfies the analogous homogeneity condition, namely

7. Proposition.

$$\sigma_\Lambda^\mu(b)\,(y, \lambda\eta) = \lambda^\mu \varkappa_\lambda \sigma_\Lambda^\mu(b)\,(y, \eta)\, \varkappa_\lambda^{-1} \tag{19}$$

for all $y \in \Omega$, $\eta \in \mathbf{R}^q \setminus \{0\}$, $\lambda \in \mathbf{R}_+$. *Here*

$$(\varkappa_\lambda u)\,(x, r) = \lambda^\alpha u(x, \lambda r) \tag{20}$$

for arbitrary fixed $\alpha \in \mathbf{R}$.

This follows by a straightforward calculation.

The choice of α in the given calculus depends on $n = \dim X$. Here we set $\alpha = (n + 1)/2$.

We say that $a(x, r, y, \xi, \varrho, \eta) \in S_{\mathrm{cl}}^\mu(\mathrm{c}V^\wedge \times \Omega \times \mathbf{R}^{n+1+q})$ satisfies the exit condition if there exists a sequence $e_j(x, y, \xi, \varrho, \eta) \in S_{\mathrm{cl}}^\mu(\mathrm{c}V^\wedge \times \Omega \times \mathbf{R}^{n+1+q})$ such that

$$\chi(r)\, a(x, y, r, \xi, \varrho) \sim \chi(r) \sum_{j=0}^{\infty} r^{-j} e_j(x, y, \xi, \varrho, \eta) \tag{21}$$

for any r-excision function χ. More precisely we demand

$$\left| D_r^k D_{x,y}^\alpha D_{\xi,\varrho,\eta}^\beta \chi(r) \left\{ a - \sum_{j=0}^{N} r^{-j} e_j \right\} \right| \leq c(1 + |\xi, \varrho, \eta|)^{\mu - |\beta|}\, (1 + r)^{-(N+1)-k}$$

for all $k \in \mathbf{N}$, $\alpha \in \mathbf{N}^{n+q}$, $\beta \in \mathbf{N}^{n+1+q}$, with a constant $c = c(k, \alpha, \beta, K)$ for all $x \in K \Subset V$, $r \in \mathbf{R}_+$, $y \in \Omega$, $(\xi, \varrho, \eta) \in \mathbf{R}^{n+1+q}$.

$S^{\mu}_{\mathrm{cl}}(\mathrm{c}V^{\wedge} \times \Omega \times \mathbf{R}^{n+1+q})_{\mathrm{e}}$ will denote the subspace of all

$$a(x, r, y, \xi, \varrho, \eta) \in S^{\mu}_{\mathrm{cl}}(\mathrm{c}V^{\wedge} \times \Omega \times \mathbf{R}^{n+1+q})$$

satisfying the exit condition, and we set

$$\begin{aligned}&\tilde{S}^{\mu}_{\mathrm{cl}}(\mathrm{c}V^{\wedge} \times \Omega \times \mathbf{R}^{n+1+q})_{\mathrm{e}}\\ &= \{a(x, r, \xi, r\varrho, r\eta) \colon a(x, r, \xi, \varrho, \eta) \in S^{\mu}_{\mathrm{cl}}(\mathrm{c}V^{\wedge} \times \Omega \times \mathbf{R}^{n+1+q})_{\mathrm{e}}\}\,.\end{aligned}$$

If the symbols (7) that are involved in (15) satisfy the exit condition, then

$$b(y, \eta) \colon \mathscr{K}^{s,\beta}(X^{\wedge}) \to \mathscr{K}^{s-\mu,\beta-\mu}(X^{\wedge})$$

is continuous for every y, η, and all $s \in \mathbf{R}$. This is true in particular of $\sigma^{\mu}_{\Lambda}(b)$. Thus

$$\sigma^{\mu}_{\Lambda}(b) \in C^{\infty}\left(T^{*}Y, \bigcap_{s \in \mathbf{R}} \mathscr{L}(\mathscr{K}^{s,\beta}(X^{\wedge}), \mathscr{K}^{s-\mu,\beta-\mu}(X^{\wedge}))\right).$$

In compositions and parametrix constructions for the operator families of the type (15) there will occur also smoothing Mellin and Green operator families, similarly as for the cone in the previous section. For describing the smoothing Mellin families it is adequate to choose a conormal order ν with $\mu - \nu \in \mathbf{N}$. Then we consider

$$m(y, \eta) = \omega_1(r[\eta])\, r^{-\nu} \sum_{j=0}^{k-1} r^j \sum_{|\alpha| \le j} \eta^{\alpha}\, \mathrm{op}_M^{\beta - n/2, j}(h_{j\alpha})(y)\, \omega_2(r[\eta])\,. \tag{23}$$

Here $k \in \mathbf{N} \setminus \{0\}$ is the length of the given weight interval (cf. 1.2. (22)), and $h_{j\alpha} \in C^{\infty}(\Omega, M^{-\infty}_{V_{j\alpha}}(X))$ for certain $V_{j\alpha} \in \mathscr{V}$. In the definition of (23) a parameter-dependent analogue of 1.2., (32) is used, namely

$$C^{\infty}(\Omega, M_V^{-\infty}(X)) = C^{\infty}(\Omega, M_{V_1}^{-\infty}(X)) + C^{\infty}(\Omega, M_{V_2}^{-\infty}(X))\,,$$

cf. [S1].

If we set

$$\sigma^{\nu}_{\Lambda}(m)(y, \eta) = \omega_1(r|\eta|)\, r^{-\nu} \sum_{j=0}^{k-1} r^j \sum_{|\alpha| = j} \eta^{\alpha}\, \mathrm{op}_M^{\beta - n/2, j}(h_{j\alpha})(y)\, \omega_2(r|\eta|)\,, \tag{24}$$

then we have obviously

$$\sigma^{\nu}_{\Lambda}(m)(y, \lambda\eta) = \lambda^{\nu} \varkappa_{\lambda} \sigma^{\nu}_{\Lambda}(m)(y, \eta)\, \varkappa_{\lambda}^{-1} \tag{25}$$

for all $y \in \Omega$, $\eta \in \mathbf{R}^q \setminus \{0\}$, $\lambda \in \mathbf{R}_+$.

Changing of the cut-off functions ω_i in (23) or of the decomposition data of the Mellin symbols leads to remainders of Green type. They are defined as follows. First introduce for every $B = W \cap \{(n+1)/2 - \beta - k < \operatorname{Re} w\}$ for $W \in \mathscr{V}$, $W \subset \{\operatorname{Re} w < (n+1)/2 - \beta\}$, $\theta = (-k, 0]$, the subspace of all $u \in \mathscr{K}^{s,\beta}(X^{\wedge})$ with $\omega u \in \mathscr{K}^{s,\beta}_{B}(X^{\wedge})$ for any cut-off function $\omega(r)$.

The Schwartz space $\mathscr{S}_B^\beta(X^\wedge)$ on $X^\wedge$ with weight β and asymptotics near $r = 0$ is defined by

$$\omega\mathscr{S}_B^\beta(X^\wedge) = \omega\mathscr{K}_B^{\infty,\beta}(X^\wedge),$$

$$(1-\omega)\,\mathscr{S}_B^\beta(X^\wedge) = (1-\omega)\,\mathscr{S}(X\times\mathbf{R})|_{X^\wedge},$$

with $\mathscr{S}(X\times\mathbf{R}) = C^\infty(X)\otimes_\pi\mathscr{S}(\mathbf{R})$ and $\mathscr{S}(\mathbf{R})$ being the standard Schwartz space on $\mathbf{R}$.

Note that there exists a sequence of Hilbert spaces $\mathscr{S}_B^\beta(X^\wedge)^{(j)}$, $j\in\mathbf{N}$, such that

$$\mathscr{S}_B^\beta(X^\wedge) = \varprojlim_{j\in\mathbf{N}} \mathscr{S}_B^\beta(X^\wedge)^{(j)}. \tag{26}$$

These spaces can be chosen in such a way that they are closed under the action of $\varkappa_\lambda$ for all $\lambda\in\mathbf{R}_+$. An analogous statement holds for $\mathscr{K}_B^{s,\beta}(X^\wedge)$ for all s, β and B.

Let E be a Banach space and $\{\varkappa_\lambda\}_{\lambda\in\mathbf{R}_+}\in C(\mathbf{R}_+,\mathscr{L}_\sigma(E))$ be a group of isomorphisms ($\varkappa_\lambda\varkappa_{\lambda'} = \varkappa_{\lambda\lambda'}$, $\varkappa_\lambda^{-1} = \varkappa_{\lambda^{-1}}$ for all $\lambda,\lambda'\in\mathbf{R}_+$) with $\mathscr{L}_\sigma(\ldots)$ being the space $\mathscr{L}(\ldots)$ in the strong topology. Write $\varkappa(\eta) = \varkappa_{[\eta]}$. For a second Banach space $\tilde{E}$ we fix analogously a group $C(\mathbf{R}_+,\mathscr{L}_\sigma(\tilde{E}))$. For $\Omega\subseteq\mathbf{R}^p$ open, $\mu\in\mathbf{R}$,

$$S^\mu(\Omega\times\mathbf{R}^q; E,\tilde{E}) \tag{27}$$

denotes the space of all $a(y,\eta)\in C^\infty(\Omega\times\mathbf{R}^q,\mathscr{L}(E,\tilde{E}))$ such that

$$\|\tilde{\varkappa}^{-1}(\eta)\,(D_y^\alpha D_\eta^\beta a(y,\eta))\,\varkappa(\eta)\|_{\mathscr{L}(E,\tilde{E})} \leqq c[\eta]^{\mu-|\beta|}$$

for all multi-indices $\alpha\in\mathbf{N}^p$, $\beta\in\mathbf{N}^q$ and $y\in K\Subset\Omega$, $\eta\in\mathbf{R}^q$, with a constant $c = c(\alpha,\beta,K) > 0$.

The operator-valued functions in (27) will be used as the amplitude functions of a ψD calculus with operator-valued symbols, where in particular $p = q$ or $p = 2q$ (and $\Omega^2 := \Omega\times\Omega$ in the latter case for open $\Omega\subseteq\mathbf{R}^q$). In [S1], Chapter 3, the standard elements of the calculus were proved.

In particular we have the notion of homogeneity and of classical amplitude functions. $S^{(\mu)}(\Omega\times\mathbf{R}^q; E,\tilde{E})$ denotes the space of all $a(y,\eta)\in C^\infty(\Omega\times\mathbf{R}^q,\mathscr{L}(E,\tilde{E}))$ with

$$a(y,\lambda\eta) = \lambda^\mu\tilde{\varkappa}_\lambda a(y,\eta)\,\varkappa_\lambda^{-1} \tag{28}$$

for all $y\in\Omega$, $|\eta|\geqq$ const, $\lambda\geqq 1$. Further $S_{\mathrm{cl}}^\mu(\Omega\times\mathbf{R}^q; E,\tilde{E})$ is defined as the subclass of all $a(y,\eta)\in(27)$ with

$$a\sim\sum_{j=0}^\infty a_{\mu-j}\quad\text{for a sequence}\quad a_{\mu-j}\in S^{(\mu-j)}(\Omega\times\mathbf{R}^q; E,\tilde{E}).$$

For every $a(y,\eta)\in S_{\mathrm{cl}}^\mu(\Omega\times\mathbf{R}^q; E,\tilde{E})$ there is a unique $a_{(\mu)}(y,\eta)\in C^\infty(\Omega\times(\mathbf{R}^q\setminus\{0\}),\mathscr{L}(E,\tilde{E}))$ satisfying (28) for all $y\in\Omega$, $\eta\neq 0$, $\lambda\in\mathbf{R}_+$, and $a(y,\eta)-\chi(\eta)\,a_{(\mu)}(y,\eta)\in S_{\mathrm{cl}}^{\mu-1}(\Omega\times\mathbf{R}^q; E,\tilde{E})$, $\chi(\eta)$ being any excision function (i.e. $\chi\in C^\infty(\mathbf{R}^q)$, $\chi = 0$ close to $\eta = 0$, $\chi = 1$ for $|\eta| >$ const). As in the scalar calculus it is called the homogeneous principal part of $a(y, \eta)$ of order μ. Analogous definitions make sense for $\tilde{E} = \varprojlim_{j \in \mathbf{N}} \tilde{E}^{(j)}$ for a sequence $\tilde{E}^{(j)}$ of Banach spaces. The symbol estimates are then required for all $j \in \mathbf{N}$.

8. Definition. Let $\mu, \nu, \beta \in \mathbf{R}$, $\mu - \nu \in \mathbf{N}$, $\beta' := \beta - \mu$, $\theta = (-k, 0]$, $k \in \mathbf{N} \setminus \{0\}$.
A $g(y, y', \eta) \in \bigcap_{s \in \mathbf{R}} C^\infty(\Omega^2 \times \mathbf{R}^q, \mathscr{L}(\mathscr{K}^{s,\beta}(X^\wedge), \mathscr{K}^{\infty,\beta'}(X^\wedge))$ is called a *Green edge symbol* of the class $\mathfrak{R}^\nu_G(\Omega^2 \times \mathbf{R}^q, \mathfrak{g})$ for $\mathfrak{g} = (\mu; \beta, \theta)$, with the carriers of asymptotics

$$B_1 = W_1 \cap \{(n+1)/2 - \beta' - k < \operatorname{Re} w\},$$

$$B_2 = W_2 \cap \{(n+1)/2 + \beta - k < \operatorname{Re} w\}$$

for $W_i \in \mathscr{V}$, $W_1 \subset \{\operatorname{Re} w < (n+1)/2 - \beta'\}$, $W_2 \subset \{\operatorname{Re} w < (n+1)/2 + \beta\}$ if

$$g(y, y', \eta) \in S^\nu_{\mathrm{cl}}(\Omega^2 \times \mathbf{R}^q; \mathscr{K}^{s,\beta}(X^\wedge), \mathscr{S}^{\beta'}_{B_1}(X^\wedge)),$$

$$g^*(y, y', \eta) \in S^\nu_{\mathrm{cl}}(\Omega^2 \times \mathbf{R}^q; \mathscr{K}^{s,-\beta'}(X^\wedge), \mathscr{S}^{-\beta}_{B_2}(X^\wedge))$$

for all $s \in \mathbf{R}$.

The operator families

$$p(y, y', \eta) = b(y, \eta) + m(y, \eta) + g(y, y', \eta)$$

with b being given by (15) for ν instead of μ, $\mu - \nu \in \mathbf{N}$ (and under the exit condition for the b_k), m by (23) and g by the latter definition form a subspace of

$$\bigcap_{s \in \mathbf{R}} S^\nu(\Omega^2 \times \mathbf{R}^q; \mathscr{K}^{s,\beta}(X^\wedge), \mathscr{K}^{s-\nu,\beta'}(X^\wedge))$$

that we call $\mathfrak{R}^\nu(\Omega^2 \times \mathbf{R}^q, \mathfrak{g})$. This space has a rich internal structure with respect to algebra properties that are compatible with the symbolic rules, analogously as in the more special case of [S2], Chapter 3. $\mathfrak{R}^\nu(\Omega^2 \times \mathbf{R}^q, \mathfrak{g})$ is the space of cone operator-valued symbols that give rise to the class

$$Y^\nu(X^\wedge \times \Omega, \mathfrak{g}) \tag{29}$$

of wedge ψDO's by applying $\mathrm{op}_{\psi,y}$ and adding negligible Green operators (of analogous structure as those of [S2]). For $A \in Y^\nu(X^\wedge \times \Omega, \mathfrak{g})$ we have the interior symbol

$$\sigma^\nu_\psi(A) := \sigma^\nu_\psi(b), \tag{30}$$

$$\sigma^\nu_\Lambda(A) := \sigma^\nu_\Lambda(b) + \sigma^\nu_\Lambda(m) + \sigma^\nu_\Lambda(g), \tag{31}$$

where $\sigma^\nu_\Lambda(g)$ is the homogeneous principal part of $g(y, y', \eta)$ of order ν in the sense of the homogeneity (28).

Moreover, we can define

$$\sigma^\nu_M(A)(w) := \sigma^\nu_M(b)(w) + \sigma^\nu_M(m)(w), \tag{32}$$

where $\sigma^\nu_M(b)$ was given by (17) with μ instead of ν, and $\sigma^\nu_M(m) = h_{00}$ (cf. (23)). The symbolic level (32) is subordinate to (31) in the sense that it is uniquely determined by (31).

It is not our aim here to formulate a complete generalization of the wedge operator calculus from [S2] to the class (29). This will be subject of a separate paper, devoted to the problem of branching asymptotics. Let us only mention that for $A \in Y^\nu(X^\wedge \times \Omega, \mathfrak{g})$

$$\sigma_\psi^\nu(A) = 0, \sigma_\Lambda^\nu(A) = 0 \quad \Rightarrow \quad A \in Y^{\nu-1}(X^\wedge \times \Omega, \mathfrak{g}). \tag{33}$$

The role of (29) for the corner is that the localization of the corner class outside the vertex along the outgoing edge $\mathbf{R}_+$ coincides with $Y^\nu(X^\wedge \times \mathbf{R}_+, \mathfrak{g})$. Let us finally note that the highest order $\mu = \nu$ refers to the ellipticity whereas $\nu < \mu, \mu - \nu \in \mathbf{N}$, is generated by symbol relations like (33). If $\mathbf{B}$ is a stretched manifold with conical singularities that corresponds to $\overline{X^\wedge}$ close to $\partial\mathbf{B}$, then the class

$$Y^\nu(\mathbf{B} \times \Omega, \mathfrak{g}) \subset L_{\mathrm{cl}}^\mu(\mathrm{int}\, \mathbf{B} \times \Omega) \tag{34}$$

is defined as the subspace of operators A + G with $\varphi_0 A \tilde{\varphi}_0 \in Y^\nu(X^\wedge \times \Omega, \mathfrak{g})$ for every $\varphi_0, \tilde{\varphi}_0 \in C^\infty(\mathbf{B})$, $\varphi_0, \tilde{\varphi}_0 \equiv 1$ close to $\partial\mathbf{B}$, supported by a small tubular neighbourhood of $\partial\mathbf{B}$, and G being a global negligible Green operator (cf. [S2]). On $Y^\nu(\mathbf{B} \times \Omega, \mathfrak{g})$ we have the symbolic levels $\sigma_\psi^\nu, \sigma_\Lambda^\nu, \sigma_M^\nu$ analogously as above ($\sigma_\Lambda^\nu, \sigma_M^\nu$ depend only on the restriction of the operator to a neighbourhood of $\partial\mathbf{B}$).

1.4. Polar Coordinates in Pseudo-Differential Operators

A motivation for studying the symbol classes 1.2.(4) may be the behaviour of ψDO's in $\mathbf{R}^{n+1} \ni x$ under introducing polar coordinates (x, r), $x = \tilde{x}/|\tilde{x}|$, $r = |\tilde{x}|$. We will give a formulation in terms of non-canonical symbol mappings. By substituting polar coordinates several times we then obtain the corner symbol classes. They will be in Chapter 2 the starting point for the repeated Mellin conventions.

Let V be a coordinate neighbourhood on the unit sphere S^n of $\mathbf{R}^{n+1}$ with local coordinates $x \in \mathbf{R}^n$. Set

$$\tilde{V}^\wedge = \{\tilde{x} \in \mathbf{R}^{n+1} \setminus \{0\} : \tilde{x}/|\tilde{x}| \in V\}.$$

Denote by

$$S_{\mathrm{cl}}^\mu(\mathrm{c}\tilde{V}^\wedge \times \mathbf{R}^{n+1}) \tag{1}$$

the subspace of all $a(\tilde{x}, \tilde{\xi}) \in S_{\mathrm{cl}}^\mu(\tilde{V}^\wedge \times \mathbf{R}^{n+1})$ with $a = a_1|_{\tilde{V}^\wedge \times \mathbf{R}^{n+1}}$ for some $a_1 \in S_{\mathrm{cl}}^\mu(\mathbf{R}_{\tilde{x}}^{n+1} \times \mathbf{R}_{\tilde{\xi}}^{n+1})$. The mapping

$$\pi_{\mathrm{c}}: \tilde{V}^\wedge \to V^\wedge; \qquad \tilde{x} \to (x, r) \tag{2}$$

gives rise to a push-forward of ψDO's

$$(\pi_{\mathrm{c}})_*: L_{\mathrm{cl}}^\mu(\tilde{V}^\wedge) - L_{\mathrm{cl}}^\mu(V^\wedge).$$

1. Proposition. *There exists a non-canonical mapping*

$$\sigma_{\mathrm{c}}: S_{\mathrm{cl}}^\mu(\mathrm{c}\tilde{V}_{\tilde{x}}^\wedge \times \mathbf{R}_{\tilde{\xi}}^{n+1}) \to S_{\mathrm{cl}}^\mu(\mathrm{c}V_{x,r}^\wedge \times \mathbf{R}_{\xi,\varrho}^{n+1})$$

such that for $b = \sigma_c(a)$, $b = b(x, r, \xi, \varrho)$, *and* $\tilde{b}(x, r, \xi, \varrho) = b(x, r, \xi, r\varrho)$

$$(\pi_c)_* \operatorname{op}_{\psi,\tilde{x}}(a) \sim r^{-\mu} \operatorname{op}_{\psi,(x,r)}(\tilde{b}). \tag{3}$$

Proof: The proof is a simple consequence of the substitution rule on complete symbolic level, cf. 1.2. (20). In the present case we have $(\tilde{x}, \tilde{\xi}) = (y, \eta)$, $(x, r, \xi, \varrho) = (\tilde{y}, \tilde{\eta})$, $\pi_c = \varkappa$. Then ${}^t d\pi_c(\tilde{x}) = r^{-1}R^{-1}(x)(E_n \oplus rE_1)$ with E_n being the $n \times n$-unit matrix and $R^{-1}(x)$ an $(n+1)\times(n+1)$-matrix with C^∞ coefficients. The function 1.2. (21) follows in the form $r^{-|\alpha|}P_\alpha(\xi, r\varrho)$ with a polynomial P_α in $(\xi, r\varrho)$ of degree $\leqq |\alpha|/2$. Thus the resulting symbol has the asymptotic expansion

$$\sum_\alpha \frac{1}{\alpha!} a^{(\alpha)} \left(\pi_c^{-1}(x, r), r^{-1}R^{-1}(x) \begin{pmatrix} \xi \\ r\varrho \end{pmatrix}\right) r^{-|\alpha|} P_\alpha(\xi, r\varrho). \tag{4}$$

Let $a_j^{(\alpha)}(\tilde{x}, \tilde{\xi}) \in C^\infty(\tilde{V}^\wedge \times (\mathbf{R}^{n+1} \setminus \{0\}))$ be the unique homogeneous component of $a^{(\alpha)}(\tilde{x}, \tilde{\xi})$ of order $\mu - |\alpha| - j$. Then

$$a_j^{(\alpha)}(\tilde{x}, r^{-1}R^{-1}\zeta) = r^{-\mu+|\alpha|+j} a_j^{(\alpha)}(\tilde{x}, R^{-1}\zeta)$$

for every $r > 0$, $\zeta = \begin{pmatrix} \xi \\ r\varrho \end{pmatrix}$.

There exists a sequence of constants $c_{\alpha j}$, increasing sufficiently fast as $|\alpha| + j \to \infty$, such that (4) is equivalent to

$$r^{-\mu} \sum_{\alpha, j} \frac{1}{\alpha!} \chi(\zeta/c_{\alpha j})\, r^j a_j^{(\alpha)}(\pi_c^{-1}(x, r), R^{-1}(x)\,\zeta)\, P_\alpha(\zeta) \tag{5}$$

in the sense of $S^\mu_{\mathrm{cl}}(V^\wedge \times \mathbf{R}^{n+1})$, χ being an excision function in ζ. In fact, this is certainly true for $1 \geqq r > 2^{-k}$ for every k with constants $c_{\alpha j}(k)$. But then a diagonal argument yields appropriate constants for all $k \in \mathbf{N}$. For $r >$ const > 0 the sum can be carried out in the standard way without extra precautions. The sum in (5) can be made convergent also in $S^\mu_{\mathrm{cl}}(\mathrm{c}V^\wedge \times \mathbf{R}^{n+1}_\zeta)$ by enlarging (if necessary) the constants $c_{\alpha j}$, where now ζ plays the role of a formal covariable. This yields just $b(x, r, \zeta)$ as desired. □

We can also talk about the mapping

$$\tilde{\sigma}_c\colon S^\mu_{\mathrm{cl}}(\mathrm{c}\tilde{V}^\wedge \times \mathbf{R}^{n+1}) \to \tilde{S}^\mu_{\mathrm{cl}}(\mathrm{c}V^\wedge \times \mathbf{R}^{n+1}) \tag{6}$$

such that (3) holds for $\tilde{b} = \tilde{\sigma}_c(a)$.

Note that (6) is not surjective modulo elements of order $-\infty$. The cone calculus of operators 1.2. (35) with interior symbols being in the image under (6) (close to the conical singularities) corresponds to a subalgebra of the class with symbols in $r^{-\mu}\,\tilde{S}^\mu_{\mathrm{cl}}(\mathrm{c}V^\wedge \times \mathbf{R}^{n+1})$ in general. It is also closed under parametrix construction for elliptic operators.

Now let U be a coordinate neighbourhood on $S^{n+1} = \{\tilde{y} \in \mathbf{R}^{n+2} : |\tilde{y}| = 1\}$, and

$$\tilde{U}^\wedge = \{\tilde{y} \in \mathbf{R}^{n+2} \setminus \{0\} : \tilde{y}/|\tilde{y}| \in U\}.$$

Let

$$\pi_e\colon \tilde{U}^\wedge \to U^\wedge = U \times \mathbf{R}_+ \tag{7}$$

denote the diffeomorphism, induced by polar coordinates. On U we fix local coordinates $\tilde{x} \in \mathbf{R}^{n+1} \cong U$ such that a prescribed point $\tilde{y}_0 \in U$ corresponds to the origin $\tilde{x} = 0$. For the above V we can identify $\tilde{V}^\wedge$ with a subset of U. Let

$$\tilde{V}^{2\wedge} = \{\tilde{y} \in \mathbf{R}^{n+2} \setminus \{0\} : \tilde{y}/|\tilde{y}| \in \tilde{V}^\wedge\}.$$

Then

$$\tilde{V}^{2\wedge} \subset \tilde{U}^\wedge . \tag{8}$$

(7), (2) induce a diffeomorphism

$$\pi : \tilde{V}^{2\wedge} \to V^{2\wedge} := V \times \mathbf{R}_+ \times \mathbf{R}_+ ,$$

defined as the composition

$$\tilde{V}^{2\wedge} \to \tilde{V}^\wedge \times \mathbf{R}_+ \to V^{2\wedge}, \qquad \tilde{y} \to (\tilde{x}, t) \to (x, r, t),$$

where $t = |\tilde{y}|$, $r = |\tilde{x}|$. Again we can look at the push-forward of ψDO's $\pi_* : L^\mu_{\mathrm{cl}}(\tilde{V}^{2\wedge}) \to L^\mu_{\mathrm{cl}}(V^{2\wedge})$. Denote by

$$S^\mu_{\mathrm{cl}}(\mathrm{c}\tilde{V}^{2\wedge} \times \mathbf{R}^{n+2})$$

the subspace of all $a(\tilde{y}, \tilde{\eta}) \in S^\mu_{\mathrm{cl}}(\tilde{V}^{2\wedge} \times \mathbf{R}^{n+2})$ with $a = a_1|_{\tilde{V}^{2\wedge} \times \mathbf{R}^{n+2}}$ for some $a_1 \in S^\mu_{\mathrm{cl}}(\mathbf{R}^{n+2}_{\tilde{y}} \times \mathbf{R}^{n+2}_{\tilde{\eta}})$. Moreover, define

$$S^\mu_{\mathrm{cl}}(\mathrm{c}\tilde{V}^{2\wedge} \times \mathbf{R}^{n+2})$$

as the subspace of all $b(x, r, t, \xi, \varrho, \tau) \in S^\mu_{\mathrm{cl}}(V^{2\wedge} \times \mathbf{R}^{n+2})$ with $b = b_1|_{V^{2\wedge} \times \mathbf{R}^{n+2}}$ for some $b_1 \in S^\mu_{\mathrm{cl}}(V \times \mathbf{R}^2 \times \mathbf{R}^{n+2})$, $\mathbf{R}^2 = \mathbf{R}_r \times \mathbf{R}_t$. Analogously to 1.2. (4) we introduce

$$\tilde{S}^\mu_{\mathrm{cl}}(\mathrm{c}V^{2\wedge} \times \mathbf{R}^{n+2}) = \{b(x, r, t, \xi, r\varrho, rt\tau) : \\ b(x, r, t, \xi, \varrho, \tau) \in S^\mu_{\mathrm{cl}}(\mathrm{c}V^{2\wedge} \times \mathbf{R}^{n+2})\}. \tag{9}$$

2. Proposition. *There exists a non-canonical mapping*

$$\sigma : S^\mu_{\mathrm{cl}}(\mathrm{c}\tilde{V}^{2\wedge} \times \mathbf{R}^{n+2}) \to S^\mu_{\mathrm{cl}}(\mathrm{c}V^{2\wedge} \times \mathbf{R}^{n+2}) \tag{10}$$

such that for $b = \sigma(a)$, $b = b(x, r, t, \xi, \varrho, \tau)$ *and* $\tilde{b}(x, r, t, \xi, \varrho, \tau) = b(x, r, t, \xi, r\varrho, rt\tau)$

$$\pi_* \operatorname{op}_{\psi, \tilde{y}}(a) \sim t^{-\mu} r^{-\mu} \operatorname{op}_{\psi, (x, r, t)}(\tilde{b}). \tag{11}$$

Proof: The assertion follows by iterated application of Proposition 1. The substitution $\tilde{y} \to (\tilde{x}, t)$ gives rise to a mapping $a(\tilde{y}, \tilde{\eta}) \to t^{-\mu} c(\tilde{x}, t, \tilde{\xi}, t\tau)$ with $c(\tilde{x}, t, \tilde{\xi}, \tau) \in S^\mu_{\mathrm{cl}}(\mathrm{c}\tilde{V}^\wedge \times \mathbf{R}^{n+2}_{\tilde{\xi}, \tau})$. For $\tilde{x} = (x, r)$ we have to apply a parameter-dependent variant of Proposition 1, where the given symbol contains extra variables t and covariables τ that are untouched by the diffeomorphism. This leads in the corresponding terms of (4) to a dependence on the covariables in the combination $\zeta = (\xi, r\varrho, rt\tau)$. The other elements of the proof are analogous as above. □

We can also define a mapping

$$\tilde{\sigma}: S^{\mu}_{\mathrm{cl}}(c\tilde{V}^{2\wedge} \times \mathbf{R}^{n+2}) \to \tilde{S}^{\mu}_{\mathrm{cl}}(c\tilde{V}^{2\wedge} \times \mathbf{R}^{n+2}) \tag{12}$$

such that (11) holds for $\tilde{b} = \tilde{\sigma}(a)$. Similarly as above the image under (12) defines an interesting subclass of the final algebra of corner ψDO's.

It is now clear how to introduce the interior symbols for higher corners which are locally close to the highest vertex of the form $V^{p\wedge} := V \times (\mathbf{R}_+)^p$ for some $p \in \mathbf{N}$, $V \subseteq \mathbf{R}^n$ open. Let $t = (t_1, \ldots, t_p) \in (\mathbf{R}_+)^p$ and $\tau = (\tau_1, \ldots, \tau_p)$ be the associated covariable. Then the adequate structure is

$$t_1^{-\mu} t_2^{-\mu} \cdot \ldots \cdot t_p^{-\mu} b(x, t, \xi, t_1\tau_1, t_1 t_2 \tau_2, \ldots, t_1 \cdot \ldots \cdot t_p \tau_p)$$

for $b(x, t, \xi, \tau) \in S^{\mu}_{\mathrm{cl}}(cV^{p\wedge} \times \mathbf{R}^{n+p})$ with obvious notations. In the present paper we content ourselves with $p = 2$. The complex symbolic effects make it necessary to study separately the singularities of small orders. For arbitrary $p > 2$ it seems to be advisable to establish a more axiomatic approach.

2. Corner Mellin Operators

2.1. Mellin Conventions Near Corners

1. Definition. Let $\beta, \gamma \in \mathbf{R}$, and denote by

$$S^{\mu}_{\mathrm{cl}}(cV^{2\wedge} \times \mathbf{R}^n \times \Gamma_{1/2-\beta} \times \Gamma_{1/2-\gamma})$$

the space of all $h(x, r, t, \xi, w, z)$, defined for $w \in \Gamma_{1/2-\beta}$, $z \in \Gamma_{1/2-\gamma}$, such that

$$h(x, r, t, \xi, 1/2 - \beta + i\varrho, 1/2 - \gamma + i\tau) \in S^{\mu}_{\mathrm{cl}}(cV^{2\wedge} \times \mathbf{R}^{n+2}_{\xi,\varrho,\tau}) \,. \tag{1}$$

Moreover,

$$S^{\mu}_{\mathrm{cl}}(cV^{2\wedge} \times \mathbf{R}^n \times \mathbf{C} \times \mathbf{C})_{\mathrm{hol}} \tag{2}$$

denotes the space of all $h(x, r, t, \xi, w, z) \in \mathscr{A}(\mathbf{C}^2_{w,z}, C^{\infty}(V \times \bar{\mathbf{R}}_+ \times \bar{\mathbf{R}}_+ \times \mathbf{R}^n_{\xi}))$ such that (1) holds uniformly in $c_1 \leqq \beta \leqq c_2$, $d_1 \leqq \gamma \leqq d_2$ for all $c_i, d_i \in \mathbf{R}$, $i = 1, 2$. By

$$\tilde{S}^{\mu}_{\mathrm{cl}}(cV^{2\wedge} \times \mathbf{R}^n \times \mathbf{C} \times \mathbf{C})_{\mathrm{hol}}$$

we denote the set of all $h(x, r, t, \xi, w, rz)$ such that $h(x, r, t, \xi, w, z)$ belongs to (2).

2. Lemma. *For every* $h \in S^{\mu}_{\mathrm{cl}}(cV^{2\wedge} \times \mathbf{R}^n \times \Gamma_{1/2-\beta} \times \Gamma_{1/2-\gamma})$ *there exists an* $h_{-\infty} \in S^{-\infty}(cV^{2\wedge} \times \mathbf{R}^n \times \Gamma_{1/2-\beta} \times \Gamma_{1/2-\gamma})$ *such that* $h - h_{-\infty}$ *extends to an element in* $S^{\mu}_{\mathrm{cl}}(cV^{2\wedge} \times \mathbf{R}^n \times \mathbf{C} \times \mathbf{C})_{\mathrm{hol}}$.

The proof is completely analogous to 1.2., Lemma 1, and will be dropped. For

$$\tilde{h}(x, r, t, \xi, w, z) = h(x, r, t, \xi, w, rz) \,,$$

with $h \in S^{\mu}_{cl}(cV^{2\wedge} \times \mathbf{R}^n \times \mathbf{C} \times \mathbf{C})_{hol}$, and $(T_w^{-\beta} T_z^{-\gamma}) \tilde{h}(x, r, t, \xi, w, z) = \tilde{h}(x, r, t, \xi, w - \beta, z - \gamma)$, we define

$$(\mathrm{op}^{\gamma}_{M,t}\, \mathrm{op}^{\beta}_{M,r}\, \mathrm{op}_{\psi,x}(h)\, u)(x, r, t)$$
$$= t^{\gamma} M^{-1}_{z \to t} r^{\beta} M^{-1}_{w \to r} F^{-1}_{\xi \to x} (T_w^{-\beta} T_z^{-\gamma} \tilde{h})(x, r, t, \xi, w, z)$$
$$\times F_{x' \to \xi} M_{r' \to w} M_{t' \to z} (r')^{-\beta} (t')^{-\gamma} u(x', r', t'),$$

$u \in C_0^{\infty}(V^{2\wedge})$.

3. Theorem. *For every $\beta, \gamma \in \mathbf{R}$ there exists a non-canonical mapping*

$$m^{\beta\gamma}: S^{\mu}_{cl}(cV^{2\wedge} \times \mathbf{R}^{n+2}_{\xi,\varrho,\tau}) \to S^{\mu}_{cl}(cV^{2\wedge} \times \mathbf{R}^n \times \mathbf{C} \times \mathbf{C})_{hol} \tag{3}$$

such that

$$\mathrm{op}_{\psi,(x,r,t)}(\tilde{a}) \sim \mathrm{op}^{\gamma}_{M,t}\, \mathrm{op}^{\beta}_{M,r}\, \mathrm{op}_{\psi,x}(h)$$

for $h = m^{\beta\gamma}(a)$ and $\tilde{a}$ defined by $\tilde{a}(x, r, t, \xi, \varrho, \tau) = a(x, r, t, \xi, r\varrho, rt\tau)$.

Proof: Let us assume first $\beta = \gamma = 1/2$. The method of proving 1.2., Theorem 2, can be applied separately with respect to $r\varrho$ and $t\tau$. For simplicity we shall neglect again the dependence on x, ξ. If $h(r, t, w, z)$ is given with $h(r, t, \mathrm{i}\varrho, \mathrm{i}\tau) \in S^{\mu}_{cl}((\bar{\mathbf{R}}_+)^2 \times \Gamma_0^2)$, then the associated Mellin ψDO has the form

$$A = \mathrm{op}^{1/2}_{M,t}\, \mathrm{op}^{1/2}_{M,r}(h)\, u(r, t)$$
$$= \int e^{\mathrm{i}\varrho(\log r - \log r') + \mathrm{i}\tau(\log t - \log t')} b(r, t, \varrho, r\tau)\, u(r', t')\, dr'/r'\, đ\varrho\, dt'/t'\, đ\tau$$

with $b(r, t, \varrho, r\tau) = h(r, t, -\mathrm{i}\varrho, -\mathrm{i}r\tau)$. For the diffeomorphism $\varkappa: \mathbf{R}^2 \to (\mathbf{R}_+)^2$, $\varkappa(y, \tilde{y}) = (e^y, e^{\tilde{y}}) = (r, t)$ it follows

$$Au = (\varkappa^*)^{-1}\, \mathrm{op}_{\psi,(y,\tilde{y})}(f)\, \varkappa^* u$$

for $f(y, \tilde{y}, \eta, \tilde{\eta}) = b(e^y, e^{\tilde{y}}, \eta, e^y \tilde{\eta})$. Similarly as in 1.2. we obtain an $f_0(r, t, r\varrho, rt\tau)$ with $\varkappa_*\, \mathrm{op}_{\psi,(y,\tilde{y})}(f) \sim \mathrm{op}_{\psi,(r,t)}(f_0)$, where now

$$f_0(r, t, r\varrho, rt\tau) \sim \sum_{\alpha} \frac{1}{\alpha!} f^{(\alpha)}(y, \tilde{y}, r\varrho, rt\tau)\, Q_{\alpha}(r, r\varrho, rt\tau)$$

for $y = \log r$, $\tilde{y} = \log t$. Here $\alpha = (\alpha_1, \alpha_2) \in \mathbf{N}^2$, $Q_{\alpha}(r, \zeta)$ is a polynomial in $\zeta = (\zeta_1, \zeta_2)$ of order $\leqq |\alpha|/2$, with coefficients in $C^{\infty}(\bar{\mathbf{R}}_+)$,

$$Q_{\alpha}(r, r\varrho, rt\tau) = P_{\alpha_1}(r\varrho)\, r^{\alpha_2} P_{\alpha_2}(t\tau)$$

with polynomials $P_{\alpha_1}(r\varrho), P_{\alpha_2}(t\tau)$ of analogous sort as in 1.2. With $\zeta = (r\varrho, rt\tau)$, interpreted as a covariable, we can form the convergent sum

$$g(y, \tilde{y}, \zeta) = \sum_{\alpha} \frac{1}{\alpha!} f^{(\alpha)}(y, \tilde{y}, \zeta)\, Q_{\alpha}(r, \zeta)\, \chi\left(\frac{\zeta}{c_{\alpha}}\right),$$

$r = e^y$. Now we define $f_0(r, t, \zeta) = g(\log r, \log t, \zeta)$. Then $f_0(r, t, \zeta) \in S^{\mu}_{cl}((\bar{\mathbf{R}}_+)^2 \times \mathbf{R}^2_{\zeta})$.

Moreover

$$f_0(r,t,\varrho,\tau) - h(r,t,-i\varrho,-i\tau) \in S_{cl}^{\mu-1}((\bar{\mathbf{R}}_+)^2 \times \mathbf{R}^2).$$

Set $\tilde{f}_0(r,t,\varrho,\tau) = f_0(r,t,r\varrho,rt\tau)$. Inserting $h_0(r,t,i\varrho,ir\tau) = a(r,t,-\varrho,-r\tau)$ for the above h it follows

$$\mathrm{op}_{M,t}^{1/2}\,\mathrm{op}_{\mathbf{M},r}^{1/2}(h_0) \sim \mathrm{op}_{\psi,(r,t)}(\tilde{f}_0) = \mathrm{op}_{\psi,(r,t)}(\tilde{a}) - \mathrm{op}_{\psi,(r,t)}(\tilde{a}_1)$$

with $a_1(r,t,\varrho,\tau) = f_0(r,t,\varrho,\tau) - a(r,t,\varrho,\tau) \in S_{cl}^{\mu-1}((\bar{\mathbf{R}}_+)^2 \times \mathbf{R}^2)$. This procedure can be continued analogously as in 1.2. It yields a sequence h_k, $k \in \mathbf{N}$, ord $h_k = \mu - k$, such that for $h' \sim \sum_{k=0}^{\infty} h_k$

$$\mathrm{op}_{M,t}^{1/2}\,\mathrm{op}_{\mathbf{M},r}^{1/2}(h') \sim \mathrm{op}_{\psi,(r,t)}(\tilde{a}).$$

Now Lemma 2 allows to replace h' by $h \in S_{cl}^{\mu}((\bar{\mathbf{R}}_+)^2 \times \mathbf{C}^2)_{hol}$. For finishing the proof we show the following.

4. Lemma. *Let $h(x,r,t,\xi,w,z) \in S_{cl}^{\mu}(cV^\wedge \times \mathbf{R}_\xi^n \times \Gamma_0^2)$ be arbitrary, and $\beta, \gamma \in \mathbf{R}$. Then there exists an $h_{\beta\gamma}(x,r,t,\xi,w,z) \in S_{cl}^{\mu}(cV^\wedge \times \mathbf{R}_\xi^n \times \Gamma_{1/2-\beta} \times \Gamma_{1/2-\gamma})$ such that*

$$\mathrm{op}_{M,t}^{1/2}\,\mathrm{op}_{\mathbf{M},r}^{1/2}\,\mathrm{op}_{\psi,x}(h) \sim \mathrm{op}_{M,t}^{\gamma}\,\mathrm{op}_{\mathbf{M},r}^{\beta}\,\mathrm{op}_{\psi,x}(h_{\beta\gamma}).$$

Proof: For simplicity we consider the (x,ξ) independent case. By definition we have

$$\mathrm{op}_M^{1/2}\,\mathrm{op}_{\mathbf{M}}^{1/2}(h) = r^{1/2}t^{1/2}\,\mathrm{op}_M\,\mathrm{op}_{\mathbf{M}}(f_0)\,(r')^{-1/2}\,(t')^{-1/2},$$

$$\mathrm{op}_M^{\gamma}\,\mathrm{op}_{\mathbf{M}}^{\beta}(h_{\beta\gamma}) = r^{\beta}t^{\gamma}\,\mathrm{op}_M\,\mathrm{op}_{\mathbf{M}}(f_1)\,(r')^{-\beta}\,(t')^{-\gamma},$$

with $f_0 = T_w^{-1/2}T_z^{-1/2}h$, $f_1 = T_w^{-\beta}T_z^{-\gamma}h_{\beta\gamma}$. Thus it suffices to find an $f_1 \in S_{cl}^{\mu}((\bar{\mathbf{R}}_+)^2 \times \Gamma_{1/2}^2)$ such that

$$r^{1/2-\beta}t^{1/2-\gamma}\,\mathrm{op}_M\,\mathrm{op}_{\mathbf{M}}(f_0)\,(r')^{-1/2+\beta}\,(t')^{-1/2+\gamma} \sim \mathrm{op}_M\,\mathrm{op}_{\mathbf{M}}(f_1).$$

The left-hand side is a Mellin ψDO with the r, r', t, t'-dependent amplitude function

$$f(r,r',t,t',w,rz) = r^{1/2-\beta}t^{1/2-\gamma}f_0(r,t,w,rz)\,(r')^{-1/2+\beta}\,(t')^{-1/2+\gamma}.$$

Applying the Mellin operator calculus (cf. [S1]; 2.2. Theorem 5), we can pass to an equivalent r', t'-independent amplitude function. The corresponding formula is

$$\sum_{\alpha}\frac{1}{\alpha!}\left(-r'\frac{\partial}{\partial r'}\right)^{\alpha_1}\left(-t'\frac{\partial}{\partial t'}\right)^{\alpha_2} D_\varrho^{\alpha_1}D_\tau^{\alpha_2} f(r,r',t,t',w,rz)|_{r'=r,t'=t}.$$

The dependence on z in the combination rz makes no extra problem. The summands are of the form $g_\alpha(r,t,w,rz)$ with $g_\alpha(r,t,w,z) \in S_{cl}^{\mu-|\alpha|}((\bar{\mathbf{R}}_+)^2 \times \Gamma_{1/2}^2)$. After carrying out the asymptotic sum $g \sim \sum g_\alpha$ in $S_{cl}^{\mu}((\bar{\mathbf{R}}_+)^2 \times \Gamma_{1/2}^2)$ we can set $f_1(r,t,w,z) = g(r,t,w,z)$. □

Applying once again Lemma 2 to f_1, we can complete the proof of Theorem 3. □

Instead of (3) we can also talk about

$$\tilde{m}^{\beta\gamma}\colon \tilde{S}^{\mu}_{\mathrm{cl}}(\mathrm{c}V^{2\wedge} \times \mathbf{R}^{n+2}_{\xi,\varrho,\tau}) \to S^{\mu}_{\mathrm{cl}}(\mathrm{c}\tilde{V}^{2\wedge} \times \mathbf{R}^{n}_{\xi} \times \mathbf{C}_{w} \times \mathbf{C}_{z})_{\mathrm{hol}} \tag{4}$$

with $\tilde{m}^{\beta\gamma}(\tilde{a}) := m^{\beta\gamma}(a)$.

As an analogue of 1.2., Remark 4, we obtain for every diffeomorphism $\chi\colon V_1 \to V_2$, $\chi(x) = y$, a non-canonical mapping

$$\delta'\colon \tilde{S}^{\mu}_{\mathrm{cl}}(\mathrm{c}V_1^{2\wedge} \times \mathbf{R}^{n+2}_{\xi,\varrho,\tau}) \to \tilde{S}^{\mu}_{\mathrm{cl}}(\mathrm{c}V_2^{2\wedge} \times \mathbf{R}^{n+2}_{\eta,\varrho,\tau}) \tag{5}$$

such that for $\tilde{a}_2 = \delta'(\tilde{a}_1)$

$$(\chi \times \mathrm{id})_* \operatorname{op}_{\psi,(x,r,t)}(\tilde{a}_1) \sim \operatorname{op}_{\psi,(y,r,t)}(\tilde{a}_2),$$

$\chi \times \mathrm{id}\colon V_1 \times \mathbf{R}_+ \times \mathbf{R}_+ \to V_2 \times \mathbf{R}_+ \times \mathbf{R}_+$. Moreover there is a non-canonical mapping

$$\varkappa'\colon S^{\mu}_{\mathrm{cl}}(\mathrm{c}V_1^{2\wedge} \times \mathbf{R}^{n}_{\xi} \times \mathbf{C}_{w} \times \mathbf{C}_{z})_{\mathrm{hol}} \to S^{\mu}_{\mathrm{cl}}(\mathrm{c}V_2^{2\wedge} \times \mathbf{R}^{n}_{\eta} \times \mathbf{C}_{w} \times \mathbf{C}_{z})_{\mathrm{hol}}$$

such that for $f = \varkappa'(h)$

$$(\chi \times \mathrm{id})_* \operatorname{op}^{\gamma}_{M,t} \operatorname{op}^{\beta}_{\mathbf{M},r} \operatorname{op}_{\psi,x}(h) \sim \operatorname{op}^{\gamma}_{M,t} \operatorname{op}^{\beta}_{\mathbf{M},r} \operatorname{op}_{\psi,y}(f).$$

Then

$$\varkappa' \tilde{m}_1^{\beta\gamma} \sim \tilde{m}_2^{\beta\gamma} \delta'$$

in the sense of equivalence modulo elements of order $-\infty$, $\tilde{m}_i^{\beta\gamma}$ being the mappings (3) belonging to V_i, $i = 1, 2$.

Let $U \subset V \times \mathbf{R}_+$ be open, and $(x, r) \in \bar{U} \Rightarrow r > 0$. Then we have a natural restriction mapping

$$\tilde{S}^{\mu}_{\mathrm{cl}}(\mathrm{c}V^{2\wedge} \times \mathbf{R}^{n+2}) \to \tilde{S}^{\mu}_{\mathrm{cl}}(\mathrm{c}U^{\wedge} \times \mathbf{R}^{n+2}).$$

By introducing local coordinates $\tilde{x}$ in U we can pass (non-canonically) to the transformed symbol spaces, according to 1.2., Remark 4. If $V^{2\wedge}$ is interpreted as a piece of a corner and $U^{\wedge}$ as a piece of a cone (where in general we only assume $U^{\wedge} \cap V^{2\wedge} \neq \emptyset$ but not necessarily $U^{\wedge} \subset V^{2\wedge}$), then we can talk about a diffeomorphism $V^{\wedge} \cap U \to V^{\wedge} \cap U$ from the (x, r)-coordinates of $V^{\wedge}$ to the $\tilde{x}$-coordinates of U. This corresponds to an equivalence

$$r^{-\mu} q|_{V^{2\wedge} \cap U^{\wedge}} \sim p|_{V^{2\wedge} \cap U^{\wedge}} \tag{6}$$

of given $q \in \tilde{S}^{\mu}_{\mathrm{cl}}(\mathrm{c}V^{2\wedge} \times \mathbf{R}^{n+2})$, $p \in \tilde{S}^{\mu}_{\mathrm{cl}}(\mathrm{c}U^{\wedge} \times \mathbf{R}^{n+2})$. The weight factor $r^{-\mu}$ is motivated by polar coordinates $\tilde{x} \to (x, r)$, though $r > 0$ over the intersection.

We will have to employ a variant of 1.2., Theorem 2, for symbols in $S^{\mu}_{\mathrm{cl}}(\mathrm{c}V^{2\wedge}_{x,r,t} \times \mathbf{R}^{n+2}_{\xi,\varrho,\tau})$ with respect to t, where r remains untouched. Let

$$S^{\mu}_{\mathrm{cl}}(V \times \mathbf{R}_+ \times \bar{\mathbf{R}}_+ \times \mathbf{R}^{n+2}_{\xi,\varrho,\tau})$$

be the space of all $a(x, r, t, \xi, \varrho, \tau) \in S^{\mu}_{\mathrm{cl}}(V \times \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}^{n+2})$ with $a = a_1|_{V \times \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}^{n+2}}$ for an $a_1(x, r, t, \xi, \varrho, \tau) \in S^{\mu}_{\mathrm{cl}}(V \times \mathbf{R}_+ \times \mathbf{R} \times \mathbf{R}^{n+2})$. If

$$\tilde{S}^{\mu}_{\mathrm{cl}}(V \times \mathbf{R}_+ \times \bar{\mathbf{R}}_+ \times \mathbf{R}^{n+2}_{\xi,\varrho,\tau}) = \{a(x, r, t, \xi, \varrho, t\tau)\colon a(x, r, t, \xi, \varrho, \tau) \in S^{\mu}_{\mathrm{cl}}(V \times \mathbf{R}_+ \times \bar{\mathbf{R}}_+ \times \mathbf{R}^{n+2}_{\xi,\varrho,\tau})\}$$

then we get a natural embedding

$$\tilde{S}^{\mu}_{\mathrm{cl}}(\mathrm{c}V^{2\wedge}_{x,r,t} \times \mathbf{R}^{n+2}_{\xi,\varrho,\tau}) \to \tilde{S}^{\mu}_{\mathrm{cl}}(V \times \mathbf{R}_+ \times \bar{\mathbf{R}}_+ \times \mathbf{R}^{n+2}_{\xi,\varrho,\tau}). \tag{7}$$

Define

$$S^{\mu}_{\mathrm{cl}}(V \times \mathbf{R}_+ \times \bar{\mathbf{R}}_+ \times \mathbf{R}^{n+1}_{\xi,\varrho} \times \Gamma_{1/2-\gamma})$$

by the usual scheme as the space of all $h(x, r, t, \xi, \varrho, z)$ such that

$$h(x, r, t, \xi, \varrho, 1/2 - \gamma + i\tau) \in S^{\mu}_{\mathrm{cl}}(V \times \mathbf{R}_+ \times \mathbf{R} \times \mathbf{R}^{n+2}_{\xi,\varrho,\tau}). \tag{8}$$

Moreover,

$$S^{\mu}_{\mathrm{cl}}(V \times \mathbf{R}_+ \times \bar{\mathbf{R}}_+ \times \mathbf{R}^{n+1}_{\xi,\varrho} \times \mathbf{C}_z)_{\mathrm{hol}}$$

is the space of all $h(x, r, t, \xi, \varrho, z) \in \mathscr{A}(\mathbf{C}, C^{\infty}(V \times \mathbf{R}_+ \times \bar{\mathbf{R}}_+ \times \mathbf{R}^{n+1}_{\xi,\tau}))$ such that (8) holds uniformly in $d_1 \leqq \gamma \leqq d_2$ for all $d_1, d_2 \in \mathbf{R}$.

5. Theorem. *For every $\gamma \in \mathbf{R}$ there exists a non-canonical mapping*

$$\tilde{m}^{\gamma}_t \colon \tilde{S}^{\mu}_{\mathrm{cl}}(V \times \mathbf{R}_+ \times \bar{\mathbf{R}}_+ \times \mathbf{R}^{n+2}_{\xi,\varrho,\tau}) \to S^{\mu}_{\mathrm{cl}}(V \times \mathbf{R}_+ \times \bar{\mathbf{R}}_+ \times \mathbf{R}^{n+1}_{\xi,\varrho} \times \mathbf{C}_z)_{\mathrm{hol}}$$

such that for $h = \tilde{m}^{\gamma}_t(\tilde{a})$

$$\mathrm{op}_{\psi,(x,r,t)}(\tilde{a}) \sim \mathrm{op}^{\gamma}_{M,t}\, \mathrm{op}_{\psi,(x,r)}(h), \tag{9}$$

$\tilde{a} = \tilde{a}(x, r, t, \xi, r\varrho, rt\tau)$ *for*

$$a(x, r, t, \xi, \varrho, t\tau) \in \tilde{S}^{\mu}_{\mathrm{cl}}(V \times \mathbf{R}_+ \times \bar{\mathbf{R}}_+ \times \mathbf{R}^{n+2}).$$

Proof: The assertion is a simple modification of 1.2., Theorem 2. It is to be applied here with respect to the t variable, and the role of V of 1.2., Theorem 2, now plays $V \times \mathbf{R}_+ \ni (x, r)$. □

According to (7) we get by restriction also

$$\tilde{m}^{\gamma}_t \colon \tilde{S}^{\mu}_{\mathrm{cl}}(\mathrm{c}V^{2\wedge} \times \mathbf{R}^{n+2}) \to S^{\mu}_{\mathrm{cl}}(V \times \mathbf{R}_+ \times \bar{\mathbf{R}}_+ \times \mathbf{R}^{n+1} \times \mathbf{C})_{\mathrm{hol}} \tag{10}$$

such that (9) holds for $h = \tilde{m}^{\gamma}_t(\tilde{a})$.

Now let us look at the stretched corner globally close to the vertex. $\mathbf{B}$ will denote (as in 1.2.) the stretched manifold with conical singularities which is the base of the stretched corner $\bar{\mathbf{B}}^{\wedge} = \mathbf{B} \times \bar{\mathbf{R}}_+$. The components of $\partial\mathbf{B}$ correspond to the conical points. Every point of $\partial\mathbf{B}$ has a tubular neighbourhood $\cong X \times [0, 1)$ with X being the base of a corresponding cone. Let us assume for simplicity that $\partial\mathbf{B}$ only consists of one connection component. The generalization to several connection components is completely trivial and will be dropped.

Choose a finite covering $\mathfrak{U} = \{U_0, \dots, U_N\}$ of int $\mathbf{B}$ by open sets where $U_0 \cong X \times \mathbf{R}_+$, $\bar{U}_j \cap \partial\mathbf{B} = \emptyset$ for all $j > 0$. Then the sets $U_j^{\wedge} = U_j \times \mathbf{R}_+$ form an open covering $\mathfrak{U}^{\wedge}$ of int $\mathbf{B} \times \mathbf{R}_+$ (the interior of the stretched corner), where $U_0 \times \mathbf{R}_+$ is the local model along the outgoing edge.

Moreover, let $\mathfrak{B} = \{V_0, \ldots, V_M\}$ be an open covering of X. Then the sets $V_k \times \mathbf{R}_+$ form an open covering $\mathfrak{B}^\wedge$ of U_0, and $\mathfrak{B}^{2\wedge} = \{V_k^{2\wedge}\}_{0 \leqq k \leqq M}$ is an open covering of $U_0^\wedge$. The pair $\{\mathfrak{B}^{2\wedge}, \mathfrak{U}^\wedge\}$ will always be understood in this meaning.

For simplifying notations the local coordinates in $V_k(U_j)$ will be denoted by $x \in \mathbf{R}^n$ ($\tilde{x} \in \mathbf{R}^{n+1}$). Clearly they are different for different $k(j)$.

6. Definition. A system $\{\{t^{-\mu}r^{-\mu}q_k\}_{0 \leqq k \leqq M}, \{t^{-\mu}p_j\}_{1 \leqq j \leqq N}\}$ with

$$q_k(x, r, t, \xi, r\varrho, rt\tau) \in \tilde{S}^\mu_{\mathrm{cl}}(\mathrm{c}V_k^{2\wedge} \times \mathbf{R}^{n+2}_{\xi,\varrho,\tau}), \qquad 0 \leqq k \leqq M, \tag{11}$$

$$p_j(\tilde{x}, t, \tilde{\xi}, t\tau) \in \tilde{S}^\mu_{\mathrm{cl}}(\mathrm{c}U_j^\wedge \times \mathbf{R}^{n+2}_{\tilde{\xi},\tau}), \qquad 1 \leqq j \leqq N \tag{12}$$

is called a *global complete symbol over* int $\mathbf{B}^\wedge$ *with respect to* $\{\mathfrak{B}^{2\wedge}, \mathfrak{U}^\wedge\}$ if

$$q_k|_{V_k^{2\wedge} \cap V_l^{2\wedge}} \sim q_l|_{V_k^{2\wedge} \cap V_l^{2\wedge}} \quad \text{for} \quad 0 \leqq k, l \leqq M, \tag{13}$$

$$p_i|_{U_i^\wedge \cap U_j^\wedge} \sim p_j|_{U_i^\wedge \cap U_j^\wedge} \quad \text{for} \quad 1 \leqq i, j \leqq N, \tag{14}$$

$$r^{-\mu}q_k|_{V_k^{2\wedge} \cap U_j^\wedge} \sim p_j|_{V_k^{2\wedge} \cap U_j^\wedge} \quad \text{for} \quad 0 \leqq k \leqq M, 1 \leqq j \leqq N. \tag{15}$$

(13) refers to the transition diffeomorphisms $V_k \cap V_l \to V_k \cap V_l$ and the associated mappings (5), further (14) is to be interpreted analogously to 1.2. (12), and (15) according to (6).

Let $\{\varphi_j\}_{0 \leqq j \leqq N}$ denote a partition of unity belonging to the covering $\tilde{\mathfrak{U}}$ of $\mathbf{B}$, formed by $\bar{U}_0$ and U_j, $j = 1, \ldots, N$. In other words, $\varphi_0 \in C_0^\infty(\bar{U}_0)$, $\varphi_j \in C_0^\infty(U_j)$, $j \neq 0$, $\sum \varphi_j = 1$. Further let $\{\tilde{\varphi}_j\}_{0 \leqq j \leqq N}$ be a system of functions, $\tilde{\varphi}_0 \in C_0^\infty(\bar{U}_0)$, $\tilde{\varphi}_j \in C_0^\infty(U_j)$ for $j \neq 0$, and $\varphi_j\tilde{\varphi}_j = \varphi_j$ for all j. For notational convenience we shall interprete φ_j, $\tilde{\varphi}_j$ also as functions on $\bar{U}_0^\wedge$, $U_j^\wedge$, and denote by the same letters the pull-backs under various coordinate diffeomorphisms.

Further $\{\psi_k\}_{0 \leqq k \leqq M}$ denotes a partition of unity on X, belonging to $\mathfrak{B}$, and $\{\tilde{\psi}_k\}_{0 \leqq k \leqq M}$ a system of functions $\tilde{\psi}_k \in C_0^\infty(V_k)$ with $\psi_k\tilde{\psi}_k = \psi_k$ for all k. For convenience we shall interprete ψ_k, $\tilde{\psi}_k$ also as functions on $V_k^\wedge$, $V_k^{2\wedge}$ as well as on diffeomorphic images of these sets.

With the symbols of Definition 6 we can form the operators

$$B_k = t^{-\mu} r^{-\mu} \operatorname{op}_{\psi,(x,r,t)}(q_k) \in L^\mu_{\mathrm{cl}}(V_k^{2\wedge}), \tag{16}$$

$$A_j = t^{-\mu} \operatorname{op}_{\psi,(x,t)}(p_j) \in L^\mu_{\mathrm{cl}}(U_j^\wedge), \tag{17}$$

$0 \leqq k \leqq M$, $1 \leqq j \leqq N$, and set

$$A = \varphi_0 \left\{ \sum_{k=0}^{M} \psi_k B_k \tilde{\psi}_k \right\} \tilde{\varphi}_0 + \sum_{j=1}^{N} \varphi_j A_j \tilde{\varphi}_j \in L^\mu_{\mathrm{cl}}(\operatorname{int} \mathbf{B}^\wedge).$$

It is then clear that

$$A|_{V_k^{2\wedge}} \sim B_k, \qquad A|_{U_j^\wedge} \sim A_j \quad \text{for all} \quad k, j.$$

Now we apply the mappings

$$\tilde{m}^{\beta\gamma}: \tilde{S}^{\mu}_{\mathrm{cl}}(\mathrm{c}V_k^{2\wedge} \times \mathbf{R}^{n+2}_{\xi,\varrho,\tau}) \to S^{\mu}_{\mathrm{cl}}(\mathrm{c}V_k^{2\wedge} \times \mathbf{R}^n_{\xi} \times \mathbf{C}_w \times \mathbf{C}_z)_{\mathrm{hol}},$$

$$\tilde{m}^{\gamma}: \tilde{S}^{\mu}_{\mathrm{cl}}(\mathrm{c}U_j^{\wedge} \times \mathbf{R}^{n+2}_{\xi,\tau}) \to S^{\mu}_{\mathrm{cl}}(\mathrm{c}U_j^{\wedge} \times \mathbf{R}^{n+1}_{\xi} \times \mathbf{C}_z)_{\mathrm{hol}},$$

cf. (4) and 1.2. (18), and

$$\tilde{m}^{\gamma}_t: \tilde{S}^{\mu}_{\mathrm{cl}}(\mathrm{c}V^{2\wedge} \times \mathbf{R}^{n+2}_{\xi,\varrho,\tau}) \to S^{\mu}_{\mathrm{cl}}(V \times \mathbf{R}_+ \times \bar{\mathbf{R}}_+ \times \mathbf{R}^{n+1}_{\xi,\varrho} \times \mathbf{C}_z)_{\mathrm{hol}},$$

cf. (10). Moreover, we choose cut-off functions $\omega(r), \omega_i(r), i = 1, 2$, satisfying 1.2. (29). Set

$$h_{1,k} = \tilde{m}^{\beta\gamma}(q_k), \qquad h_{2,k} = \tilde{m}^{\gamma}_t(q_k), \qquad f_j = \tilde{m}^{\gamma}(p_j).$$

Then we can form the (t, z) dependent operator family

$$\begin{aligned} h(t, z) = \varphi_0 \Bigg\{ \sum_{k=0}^{M} \psi_k r^{-\mu}[\omega(r[\tau]) \operatorname{op}^{\beta}_{\mathbf{M}} \operatorname{op}_{\psi,x}(h_{1,k})(t, z)\, \omega_1(r[\tau]) \\ + (1 - \omega(r[\tau])) \operatorname{op}_{\psi,(x,r)}(h_{2,k})(t, z)\, (1 - \omega_2(r[\tau]))]\, \tilde{\psi}_k \Bigg\} \tilde{\varphi}_0 \\ + \sum_{j=1}^{N} \varphi_j \operatorname{op}_{\psi,\tilde{x}}(f_j)(t, z)\, \tilde{\varphi}_j, \end{aligned} \tag{18}$$

$\tau = \operatorname{Im} z$. An immediate consequence of 1.2., Theorem 2, and Theorems 3, 5 is that

$$A \sim t^{-\mu} \operatorname{op}^{\gamma}_M(h) \tag{19}$$

in the sense of $L^{\mu}_{\mathrm{cl}}(\operatorname{int} \mathbf{B}^{\wedge})$.

Similarly to 1.3. (13) we can form the operator-valued Mellin symbols

$$h_1(r, t, w, z) \in C^{\infty}(\bar{\mathbf{R}}_{+,r} \times \bar{\mathbf{R}}_{+,t}, M^{\mu}_{\mathcal{O}}(X; \Gamma_{1/2-\gamma})) \tag{20}$$

by

$$h_1(r, t, w, z) = \sum_{k=0}^{M} \psi_k \operatorname{op}_{\psi,x}(h_{1,k})(r, t, w, z)\, \tilde{\psi}_k$$

and

$$h_2(r, t, \varrho, z) \in C^{\infty}(\bar{\mathbf{R}}_{+,r} \times \bar{\mathbf{R}}_{+,t}, L^{\mu}_{\mathrm{cl}}(X; \mathbf{R}_{\varrho} \times \Gamma_{1/2-\gamma})) \tag{21}$$

by

$$h_2(r, t, \varrho, z) = \sum_{k=0}^{M} \psi_k \operatorname{op}_{\psi,x}(h_{2,k})(r, t, \varrho, z)\, \tilde{\psi}_k.$$

Note that h_1, h_2 are holomorphic in z and that (20), (21) hold for all γ, uniformly in every strip parallel to the imaginary z axis. Set

$$\begin{aligned} e(t, z) = r^{-\mu}\omega(r[\tau]) \operatorname{op}^{\beta}_{\mathbf{M}}(h_1)(t, z)\, \omega_1(r[\tau]) \\ + r^{-\mu}(1 - \omega(r[\tau])) \operatorname{op}_{\psi,r}(h_2)(t, z)\, (1 - \omega_2(r[\tau])). \end{aligned} \tag{22}$$

Then (18) takes the form

$$h(t, z) = \varphi_0 e(t, z)\, \tilde{\varphi}_0 + \sum_{j=1}^{N} \varphi_j \operatorname{op}_{\psi, \tilde{x}}(f_j)\,(t, z)\, \tilde{\varphi}_j\,. \tag{23}$$

Note that (18) as well as $\varphi_0 e(t, z)\, \tilde{\varphi}_0$ are families of operators in $C^\mu(\mathbf{B}, \mathfrak{g})$, $\mathfrak{g} = (\tilde{\beta}, \theta)$, $\tilde{\beta} = \beta + n/2$, $\theta = (-k, 0]$, for every k (cf. the notations at the end of 1.2.), and the leading Mellin symbol 1.2. (37)

$$\sigma_M^\mu(h)\,(t, w) = \sigma_M^\mu(\varphi_0 e \tilde{\varphi}_0)\,(t, w) \tag{24}$$

is independent of z.

7. Proposition. *The operator family* (22) *satisfies*

$$\operatorname{op}_M^\gamma(\varphi_0 e \tilde{\varphi}_0) \in Y^\mu(X^\wedge \times \mathbf{R}_+, \tilde{\mathfrak{g}})$$

$\tilde{\mathfrak{g}} = (\mu; \beta + n/2, \theta)$. *The leading Mellin symbol in the sense of* 1.3. (32) *coincides with* (24). *Moreover,* $\operatorname{op}_M^\gamma(\varphi_0 e \tilde{\varphi}_0)$ *has a complete symbol* $\{r^{-\mu} b\}_{0 \leq k \leq M}$ *in the sense of* 1.3. (6), *with* $b_k = t^{-\mu} q_k'$ *and* q_k' *being again of the sort* (11).

Proof: This result follows by analogous arguments as for 1.2., Theorem 2, now applied to operator-valued symbols, acting as operators along $X^\wedge$. A relation of leading symbols modulo lower order terms, expressed by the replacement $-z \to it\tau$, follows also in the operator-valued set-up. Then the leading Mellin symbols remain untouched. □

It would be a nice analogue of 1.2. (27), here for the corner, to have holomorphy of (23) in z.

This is, of course, not the case, since the $\omega(r[\tau])$, $\omega_i(r[\tau])$ factors are not holomorphic. On the other hand, in [S3] we shall see that holomorphy can be achieved modulo a $C_G(\mathbf{B}, \mathfrak{g})$-valued error $g(t, z)$. Thus the choice of (18) for the operator (19) (or equivalently for the given complete symbol of Definition 6) is a first important step of an operator-valued Mellin operator convention for the corner theory which plays a completely analogous role as 1.2. (27), with $\mathbf{B}$ being the base of the cone.

In the present paper we do not construct anyway the complete algebra of corner ψDO's containing the full asymptotic information. This will be done in [S4]. So for the moment we may disregard the non-holomorphy of $h(t, z)$ and complete the operator convention by something along the outgoing edge far from the vertex as a counterpart of the second item on the right of 1.2. (28). To this end we apply the definitions of 1.3. for $\Omega = \mathbf{R}_{+,t}$, choose a system $\{t^{-\mu} r^{-\mu} q_k'\}_{0 \leq k \leq M}$ with q_k' of analogous sort as (11), and set $b_k = t^{-\mu} q_k'$, $k = 0, \ldots, M$.

Then $b_k \in \tilde{S}_{\mathrm{cl}}^\mu(\mathrm{c}V_k^\wedge \times \mathbf{R}_+ \times \mathbf{R}^{n+1} \times w\mathbf{R})$, and $\{r^{-\mu} b_k\}_{0 \leq k \leq M}$ is global complete symbol over $\bar{X}^\wedge \times \mathbf{R}_+$ in the edge sense, cf. 1.3. (7). The variables (t, τ) play here the role of (y, η) from Section 1.3. According to 1.3. (15) we obtain the operator family $b(t, \tau)$. In view of Proposition 7 the symbols q_k' can be chosen in such a way that

$$t^{-\mu} \operatorname{op}_M^\gamma(\varphi_0 e \tilde{\varphi}_0) \sim \operatorname{op}_{\psi, t}(b) \quad \text{in} \quad L_{\mathrm{cl}}^\mu(X^\wedge \times \mathbf{R}_+)\,.$$

In an analogous manner we can proceed with the interior parts $\{t^{-\mu} p_j\}_{1 \leq j \leq N}$ of the complete symbols and switch the Mellin action along t to an $\operatorname{op}_{\psi, t}$-action, cf. 1.2., Theorem 2. It follows altogether that a sum like

$$\omega(t)\ \{operator\ in\ \mathrm{op}_{M,t}\text{-}convention\}\ \omega_1(t)$$
$$+\ (1-\omega(t))\ \{operator\ in\ \mathrm{op}_{\psi,t}\text{-}convention\}\ (1-\omega_2(t))$$

is independent of ω, ω_i modulo smoothing operators, provided 1.2. (29) holds. The more precise analysis of [S3], [S4] will show that the errors are even in the class of Green corner operators. If

$$\{\{t^{-\mu}r^{-\mu}q_k\}_{0\leqq k\leqq M}, \{t^{-\mu}p_j\}_{1\leqq j\leqq N}\}$$

is a complete symbol as in Definition 6 we form $\mathbf{A} \sim A$ by

$$\mathbf{A} = \omega(t)\, t^{-\mu}\, \mathrm{op}_M^{\gamma}(h)\, \omega_1(t)$$
$$+\ (1-\omega(t))\,\{\varphi_0\, \mathrm{op}_{\psi,t}(b)\, \tilde{\varphi}_0 + \sum_{j=1}^{N} \varphi_j A_j \tilde{\varphi}_j\}\,(1-\omega_2(t))\,. \tag{25}$$

Here h is given by (18), A_j by (17). The cut-off functions ω, ω_i are assumed to satisfy 1.2. (29). As emphasized above the final operator convention for the corner will contain a (smoothing) correction of h in order to achieve holomorphy in z. In addition we will allow extra smoothing Mellin operators and Green operators, similarly as for the cone with smooth base (cf. 1.2. and [S3], [S4]).

Note that h depends on β, γ and b on β. In the final calculus we will replace γ by $\gamma - (n+1)/2$, and β by $\beta - n/2$ for $n = \dim X$.

2.2. Symbolic Levels of Corner Operators

This section will have a look at the symbolic structure of the operators of the form 2.1. (25).

First we have the homogeneous principal interior symbol of order μ

$$\sigma_{\psi}^{\mu}(\mathbf{A}) \in C(T^*(\mathrm{int}\,\mathbf{B}^{\wedge}) \setminus 0) \tag{1}$$

which is defined by

$$\sigma_{\psi}^{\mu}(A)|_{T^*V_k^{\wedge}\setminus 0} = t^{-\mu}r^{-\mu}q_{k,(\mu)}(x, r, t, \xi, \varrho, \tau)\,, \qquad 0 \leqq k \leqq M\,,$$
$$\sigma_{\psi}^{\mu}(A)|_{T^*U_j^{\wedge}\setminus 0} = t^{-\mu}p_{j,(\mu)}(\tilde{x}, t, \tilde{\xi}, \tau)\,, \qquad 1 \leqq j \leqq N\,,$$

with $q_{k,(\mu)}$ $(p_{j,(\mu)})$ being the homogeneous principal part of q_k (p_j) of order μ.

Furthermore, there is the corner Mellin symbol of highest conormal order μ

$$\sigma_M^{\mu}(\mathbf{A})\,(z) := h(0, z)\,, \tag{2}$$

$h(t, z)$ being defined by 2.1. (18) and z varying on $\Gamma_{1/2-\gamma}$. (2) is an operator family

$$\sigma_M^{\mu}(\mathbf{A})\,(z) \colon \mathscr{H}^{s,\beta+n/2}(\mathbf{B}) \to \mathscr{H}^{s-\mu,\beta+n/2}(\mathbf{B}) \tag{3}$$

for all $s \in \mathbf{R}$ and belongs to the parameter-dependent class (with the parameter $z \in \Gamma_{1/2-\gamma}$) of cone operators $C^{\mu}(\mathbf{B}, \mathfrak{g}; w\Gamma_{1/2-\gamma})$, that will be studied in detail in [S3], $\mathfrak{g} = (\beta, \theta)$, $\theta = (-k, 0]$. As a family of cone operators it has a leading Mellin symbol with respect to the base X of the cone which is independent of z. Let us talk about the subordinate

Mellin symbol

$$\sigma_M^\mu \sigma_M^\mu(\mathbf{A})\,(w)\colon H^s(X) \to H^{s-\mu}(X)\,, \tag{4}$$

$s \in \mathbf{R}$, with w varying on $\Gamma_{1/2-\beta}$. For $a(z) \in C^\mu(\mathbf{B}, \mathrm{g}; w\mathbf{R})$, $\mathbf{R} \cong \Gamma_{1/2-\gamma}$, we have canonically a parameter-dependent homogeneous principal symbol $\sigma_\psi^\mu(a) \in C^\infty((T^* \operatorname{int} \mathbf{B}) \times \mathbf{R}_\tau \setminus 0)$ (0 corresponds to $(\tilde{\xi}, \tau) = 0$ with $\tilde{\xi}$ being the fibre variable in $T^* \operatorname{int} \mathbf{B}$) and then

$$t^\mu \sigma_\psi^\mu(\mathbf{A})|_{t=0} = \sigma_\psi^\mu \sigma_M^\mu(\mathbf{A})\,, \tag{5}$$

where σ_ψ^μ on the right is used in the latter meaning and on the left as in (1).

The operator family $b(t, \tau)$ of the preceding section and the definition of 1.3. (18) give rise to the homogeneous principal edge symbol of order μ

$$\sigma_\Lambda^\mu(\mathbf{A})\,(t, \tau) := \sigma_\Lambda^\mu(b)\,(t, \tau)\,. \tag{6}$$

It satisfies

$$\sigma_\Lambda^\mu(\mathbf{A})\,(t, \lambda\tau) = \lambda^\mu \varkappa_\lambda \sigma_\Lambda^\mu(\mathbf{A})\,(t, \tau)\,\varkappa_\lambda^{-1}$$

for all $t \in \mathbf{R}_+$, $\tau \in \mathbf{R} \setminus \{0\}$, $\lambda \in \mathbf{R}_+$, where $(\varkappa_\lambda u)\,(x, r) = \lambda^{(n+1)/2} u(x, \lambda r)$, $u \in \mathscr{K}^{s,\beta}(X^\wedge)$ (cf. 1.3. Proposition 7).

The edge symbol (6) is an operator family

$$\sigma_\Lambda^\mu(\mathbf{A})\,(t, \tau)\colon \mathscr{K}^{s,\beta+n/2}(X^\wedge) \to \mathscr{K}^{s-\mu,\beta+n/2-\mu}(X^\wedge)$$

for all $s \in \mathbf{R}$, parametrized by $T^*\mathbf{R}_+ \setminus \{0\} \ni (t, \tau)$. Remember that by 1.3. (17) we have a subordinate Mellin symbol of (6)

$$\sigma_M^\mu \sigma_\Lambda^\mu(\mathbf{A})\,(t, w) = h_0(t, w) \tag{7}$$

which is the image of $\{t^{-\mu} q_k\}_{0 \leq k \leq M}$ under the mapping 1.3. (14), restricted to $r = 0$. It is a family of operators

$$\sigma_M^\mu \sigma_\Lambda^\mu(\mathbf{A})\,(t, w)\colon H^s(X) \to H^{s-\mu}(X)\,,$$

$s \in \mathbf{R}$, smoothly depending on $t \in \mathbf{R}_+$, $w \in \Gamma_{1/2-\beta}$, which can be interpreted as an element in $C^\infty(\mathbf{R}_+, L_{\mathrm{cl}}^\mu(X; \Gamma_{1/2-\beta}))$. Clearly $t^\mu \sigma_M^\mu \sigma_\Lambda^\mu(\mathbf{A})\,(t, w)$ is C^∞ up to $t = 0$. Then

$$t^\mu \sigma_M^\mu \sigma_\Lambda^\mu(\mathbf{A})\,(t, w)|_{t=0} = \sigma_M^\mu \sigma_M^\mu(\mathbf{A})\,(w)\,, \tag{8}$$

cf. (4).

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