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The Hertz contact problem with finite friction

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ABSTRACT

The indentation of an elastic half-space by an axisymmetric punch under a monotonically applied normal force is formulated as a mixed boundary value problem under the assumption of Coulomb friction with coefficient μ in the region of contact. Within an inner circle the contact is adhesive, while in the surrounding annulus the surface moves inwards with increasing load. The slip boundary between the two regions depends on μ and the Poisson ratio v, and is found uniquely as an eigenvalue of a certain integral equation.

For power law indentors of the form $z \propto r^n$, a group property of the integral operator connecting stresses and displacements makes it possible to derive the contact stress distributions from those under a flat punch by a simple quadrature, and shows that the slip radius is the same in all such cases.

An iterative numerical solution using a dual system of Volterra equations is described, and calculated distributions of surface stress presented for the cases of indentation by a flat punch and by a sphere.

ZUSAMMENFASSUNG

Das Eindringen eines axial-symetrischen Stempels in einen elastischen unendlichen Halbraum, unter Einwirkung einer monotonischen senkrechten Kraft, wird dargestellt als ein gemischtes Grenzwert problem, wobei ein Coulombscher Reibungskoeffizient μ im Kontaktvolumen angenommen wird. Innerhalb eines inneren Kreises der Kontakt is haftend während in dem umgebenden Kreisring eine nach innen gerichtete Bewegung der Ebene stattfindet, die mit der Kraft wächst. Die Gleitgrenze zwischen diesen beiden Gebieten hängt von μ und dem Poisson Verhältnis v ab und ist ein eindeutiger Eigenwert eines Integralgleichung.

Für Stempel, deren Form $z \propto r^n$ gehorcht, wird gezeigt, dass eine Gruppeneigenschaft des Integral Operators, welche Spannungen und Verschiebungen verknüpft, ermöglicht, die Kontaktspannungsverteilung in der Umgebung eines flachen Stempels durch einfache Quadratur abzuleiten und zu zeigen, dass der Gleitradius in allen Fällen der gleiche ist.

Es wird eine "iterative numerische" Lösung beschrieben, die Volterra-Gleichungen benutzt. Die Berechnungen der Oberflächenspannungsverteilung für das Eindringen eines ebenen Stempels in eine Kugel wird gegeben.

1. Introduction

The elastic contact problem presented by the indentation of a half space by a rigid axisymmetric punch has usually been treated under the assumption of zero shear stress in the contact region, in which case the mathematical solution is straightforward and well known. Several discussions of the case of fully adhesive contact have also been given, especially in the Russian literature (Galin [5], Mossakovski [8], [9], Abramian, Aratunian & Babloian [1]). For the Boussinesq problem of indentation by a flat faced punch the radial displacement is zero in the contact region under adhesive conditions, but for a curved indentor it is more difficult to formulate a boundary value problem for quasi-static treatment, since the strain history must be taken into account at each stage of the loading. The present author [13] used similarity arguments to infer the form of the surface displacements, and so obtained solutions, for indentors of polynomial shapes, when the loading is incremental and monotonic.

Physically a more realistic assumption than that of complete adhesion is to suppose a finite coefficient of friction μ between the surfaces in contact. As pointed out by Bowden & Tabor ([3], Chapter 3) there is much experimental evidence for a "locked" inner region surrounded by an annulus of slip. For a flat indentor, this slip would relieve the divergent stresses predicted near an edge under adhesive conditions by linear elastic theory (whereas with perfect bonding, plastic deformation must occur in these regions). In the present paper, axisymmetric contact with Coulomb friction under incremental loading is formulated for a flat indentor in section 2.1 as a mixed boundary-value problem governed by a coupled pair of Volterra equations, in which there appears the new complication of an unknown boundary, the radius c (expressed as a fraction of the total contact radius) dividing the zones of adhesion and slip. For monotonic loading, c is uniquely determined by the material properties, namely μ and the Poisson ratio ν. This is demonstrated formally in section 2.2 by reducing the mixed problem, with c supposed known, to a regular Fredholm equation for a quantity ϕ (defined in equation 2.11) proportional to the excess friction in the adhesive region. There is just one solution for which $\phi > 0$, and the corresponding eigenvalue provides a unique relation between the three parameters c, μ and $\gamma = (1-2\nu)/(2-2\nu)$. An expansion of the solution in powers of γ carried out in section 2.3 gives the simple limiting form $\gamma \log \{(1+c)/(1-c)\} = 2\mu c K'(c)$ for the relation.

In section 3 the corresponding Hertz problem, generalised to incremental indentation by a body of power law shape $z \propto r^n$, is considered. It is shown that similarity considerations of the type used in the author's earlier paper are still applicable, giving the functional form of the radial displacement within the adhesive region. A group property of the operator relating stresses to displacements then shows that the equations and boundary conditions can be transformed into those for a flat punch, and the dimensionless stress $p(x)$ say obtained from the corresponding flat punch stress $p_0(x)$ by the quadrature

$$p(x) = nx^{n-1} \int_x^1 t^{-n} p_0(t) dt.$$

(This transformation applies *a fortiori* to the limiting cases of frictionless indentation ($\mu = 0$) and perfect adhesion ($c = 1$) and gives the known results, e.g. for loading by a sphere, very simply in those cases). The important result follows, that the eigenvalue c is the same function of μ and ν for a curved punch of *any* power law shape as for a flat punch.

The remainder of the paper presents a numerical method, based on piecewise constant approximations to the stresses, for solving the coupled integral equations by iteration, and results are obtained for $c(\mu, \nu)$ and for the stresses. These are used to obtain corresponding solutions to the Hertz problem (indentation by a rigid sphere) using the quadrature derived earlier. The axisymmetric case is treated in section 4, and similar results are obtained in section 5 for two-dimensional indentors, for which case a check on the accuracy of the numerical methods is available.

2. The mixed boundary value problem

2.1. *Governing integral equations*

We consider an elastic half-space $z > 0$ indented by a rigid[1] axisymmetric punch exerting a normal force P over a contact circle $r < a$ of the surface, and suppose that the force is applied monotonically, starting from a state of zero stress, at a sufficiently slow rate to permit a quasistatic treatment. Suppose $\delta = (u_z)_{r=z=0}$ is the normal displacement on the axis at any instant, so that any one of the quantities P, δ or the contact radius a (except when the latter is fixed by geometry, as in the first example below) is a time-like variable describing the state of the loading and all three increase together. Write the stresses and displacements at the surface $z = 0$ non-dimensionally as

$$\sigma_{zz}(r) = -\left(\frac{G}{1-\nu}\right)\left(\frac{\delta}{a}\right)p(x), \qquad \sigma_{rz}(r) = -\left(\frac{G}{1-\nu}\right)\left(\frac{\delta}{a}\right)q(x), \tag{2.1}$$

$$u_z(r) = \delta(P)w(x), \qquad u_r(r) = \delta(P)u(x),$$

where $x = r/a$, G is the shear modulus and ν is Poisson's ratio. Then p, q, u and w are connected by the pair of coupled Volterra equations

$$\int_x^1 \frac{tp(t)dt}{\sqrt{t^2-x^2}} - \gamma\left\{\int_0^1 q(t)dt - x\int_0^x \frac{q(t)dt}{\sqrt{x^2-t^2}}\right\} = \frac{d}{dx}\int_0^x \frac{tw(t)dt}{\sqrt{x^2-t^2}} \equiv w^*(x), \tag{2.2a}$$

$$x\int_x^1 \frac{q(t)dt}{\sqrt{t^2-x^2}} - \gamma\int_0^x \frac{tp(t)dt}{\sqrt{x^2-t^2}} = \int_0^x \frac{(tu)'dt}{\sqrt{x^2-t^2}} \equiv u^*(x), \tag{2.2b}$$

where $\gamma = (1-2\nu)/(2-2\nu)$. (A derivation is given by Noble and Spence [11]). These can be written in terms of a single linear integral operator L whose structure is that of a matrix as

[1] The restriction to rigid punches simplifies the discussion, but is not necessary. The contact can be thought of as between linearly-elastic bodies of shear moduli G_1, G_2 and Poisson's ratios ν_1, ν_2, in which case the boundary conditions hold for displacements $\bar{w} = w_1 + w_2$, $\bar{u} = u_1 - u_2$, and the analysis goes through with γ replaced by the modified value

$$\bar{\gamma} = \left(\frac{1-2\nu_1}{2G_1} - \frac{1-2\nu_2}{2G_2}\right)\Bigg/\left(\frac{1-\nu_1}{G_1} + \frac{1-\nu_2}{G_2}\right),$$

as noted by Spence [15].

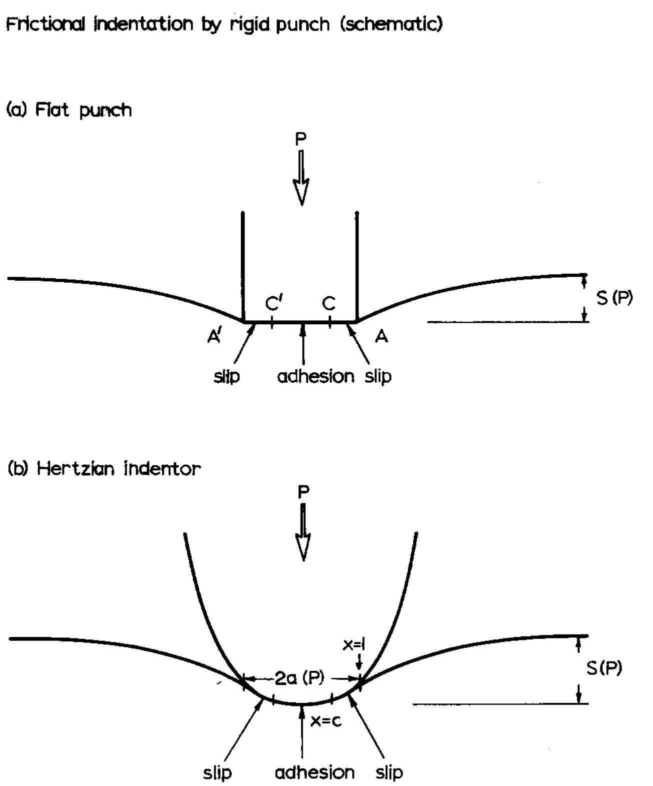
$$L\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} w^* \\ u^* \end{pmatrix}. \tag{2.3}$$

The normal compliance of the surface, i.e. the indentation δ per unit of applied force P, can be written in terms of a dimensionless compliance factor

$$P^* = \frac{\pi}{2}\int_0^1 xp(x)dx. \tag{2.4}$$

Then $P = 4GaP^*\delta/(1-\nu)$. Shield and Anderson [12] have shown from general energy considerations that, "for a given load, a rough punch penetrates less than a smooth punch and further than a perfectly rough punch"; accordingly, we expect P to increase monotonically with μ from the value for a smooth punch, namely 1, to the fully adhesive limit $(1/2\gamma)\log\{(1+\gamma)/(1-\gamma)\}$ as $\mu \to \infty$, and this is borne out by the calculations summarised later (figure 4).

For indentors of general shape, p and q would depend on a as well as x, but for the particular shapes considered in the present paper the solutions do not depend explicitly on a. Consider first indentation by a flat punch of circular cross section, for which a is constant (Figure 1a). In this case, at any instant, u_z = constant = $\delta(P)$ over the contact area, so



$$w = 1 \quad \text{on} \quad (0, 1). \tag{2.5}$$

We look for a solution in which no slip takes place over a central circle $(0, c)(c < 1)$. The frictional force required to resist slip at points within the circle cannot then exceed the limiting value μp, so

$$u = 0, \qquad \mu p - q > 0 \quad \text{on} \quad (0, c), \tag{2.6a}$$

while in the outer annulus $(c, 1)$, as δ is slowly increased slip takes place *inward*, with limiting friction acting outward, so that $(\partial/\partial\delta)u_r(r)$ is negative, i.e.

$$u < 0, \qquad \mu p - q = 0 \quad \text{on} \quad (c, 1). \tag{2.6b}$$

(μ must be looked on as the limiting value of the coefficient of sliding friction when the relative speed goes to zero. This could be lower than the coefficient of static friction, but if (2.6a) is satisfied when μ has its sliding value, it is satisfied *a fortiori* if the static value is higher.)

Altogether, therefore, the problem reduces to the solution for p and q on $(0, 1)$, and u^* on $(c, 1)$, of the equation

$$L\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 \\ 0, u^* \end{pmatrix}, \tag{2.7}$$

where the lower line of the right hand side denotes the values on $(0, c)$ and on $(c, 1)$ respectively, subject to the conditions

$$\mu p - q \begin{cases} \geqq 0 & \text{on} \quad (0, c), \\ = 0 & \text{on} \quad (c, 1), \end{cases} \tag{2.8}$$

and

$$u \leqq 0 \qquad \text{on} \quad (c, 1), \tag{2.9}$$

where u is to be obtained from the solution as the inverse of u^*, namely

$$u(x) = \frac{2}{\pi x} \int_0^x \frac{t u^*(t) dt}{\sqrt{x^2 - t^2}}. \tag{2.10}$$

In this system, c is an eigen value depending on the physical constants μ and γ. Analytically, it is more straightforward to determine a relation between μ and γ for fixed c. The existence and uniqueness of such a relation are shown by the formal solution of the next section.

2.2. *The eigenvalue problem*

The equations and boundary conditions can be reduced to a sinle Fredholm equation for

$$\phi(x) \equiv p(x) - \frac{q(x)}{\mu} \tag{2.11}$$

on the interval $(0, c)$, as follows. (ϕ is defined on the whole interval $(0, 1)$, but vanishes on $(c, 1)$, by (2.8).) First write (2.2b) as

$$\chi(x) \equiv x\int_x^1 \frac{p(t)dt}{\sqrt{t^2-x^2}} - \frac{\gamma}{\mu}\int_0^x \frac{tp(t)dt}{\sqrt{x^2-t^2}} = x\int_x^c \frac{\phi(t)dt}{\sqrt{t^2-x^2}} \tag{2.12}$$

on $(0, c)$. Since $\phi \geqq 0$ on this interval, $\chi \geqq 0$ also; and in particular

$$\chi(c) = 0. \tag{2.13}$$

In the limiting case in which p can be approximated by the frctionless distribution $(1-x^2)^{-\frac{1}{2}}$ the last condition immediately gives a value for μ/γ as a function of c, noted in equation (2.28) below. Inversion of the Abel equation (2.12) on the interval $(0, c)$ gives

$$\phi(x) = -\frac{2}{\pi}\frac{d}{dx}\int_x^c \frac{\chi(y)dy}{\sqrt{y^2-x^2}}. \tag{2.14}$$

The integral on the right can be expressed in terms of elliptic integrals, and differentiated, using standard results, leading to an expression for ϕ in terms of p on $0 < x < c$:

$$\phi(x) = p(x)+\gamma^2 \cot \pi\alpha \int_0^1 l(x, t)p(t)dt \tag{2.15}$$

in which

$$\tan \pi\alpha = \mu\gamma$$

and

$$l(x, t) \sim \frac{1}{\pi(t-x)} \tag{2.16}$$

near $x = t$, so that the integral is a Cauchy principal value. The precise form of $l(x, t)$ is given in Appendix A. The condition (2.13) has been used in the derivation of this equation.

Again, from (2.2a) with $w = 1$, writing y for x and operating on the equation by

$$\frac{2}{\pi}\int_0^x \frac{dy}{\sqrt{x^2-y^2}},$$

we find

$$\int_0^1 k_1\left(\frac{x}{y}\right) p(y)dy-\gamma\int_x^1 q(y)dy = 1 \tag{2.17}$$

where

$$k_1(s) = \begin{cases}(2/\pi)K(s) \\ (2/\pi s)K(1/s)\end{cases} \quad (s \lessgtr 1),$$

and K is the complete elliptic integral of the first kind.

Differentiation with respect to x and replacement of q by $\mu(p-\phi)$ gives the singular integral equation

$$0 < x < 1: \qquad p(x)+\cot \pi\alpha \int_0^1 k(x, y)p(y)dy = \phi(x) \tag{2.18}$$

where

$$k(x, y) = \frac{d}{dx}\ k_1\left(\frac{x}{y}\right) \sim \frac{1}{\pi(y-x)} \qquad \text{near } x = y \tag{2.19}$$

An analytic solution of this equation by the Wiener-Hopf technique is possible, but the details are complicated. However, because of the singular behaviour of k we can write the solution formally in terms of a resolvent kernel h having the same singular behaviour, as

$$p(x) = (\sin \pi\alpha)^2\phi(x) - \cos \pi\alpha \sin \pi\alpha \int_0^1 h(x, y)\phi(y)dy + C\bar{p}(x), \tag{2.20}$$

where

$$h(x, y) \sim \frac{1}{\pi(y-x)} \qquad \text{near } y = x \tag{2.21}$$

and $\bar{p}(x)$ is the solution of the homogeneous equation, i.e. of (2.18) with $\phi = 0$. This solution (which corresponds physically to the case of limiting friction throughout the contact region) is unbounded at $x = 0$ and must therefore be excluded by setting $C = 0$.

If now we substitute $p(x)$ from (2.20) into (2.15), and reverse the order of integration by use of the Bertrand-Poincaré lemma which shows that

$$\int_0^1 l(x, t)dt \int_0^c k(t, y)\phi(y)dy = -\phi(x) + \int_0^c \phi(y)dy \int_0^1 l(x, t)k(t, y)dt$$

we obtain an equation

$$(1-\gamma^2)(T\phi)(x) = \int_0^c m(x, y)\phi(y)dy \tag{2.22}$$

in which m, derived from k and l, is bounded at $x = y$, and T is written for the singular operator

$$T\phi \equiv \phi(x) + \tan \pi\alpha \int_0^c \frac{\phi(t)dt}{t-x}, \tag{2.23}$$

which can be inverted on $(0, c)$, to give the regular Fredholm equation

$$(1-\gamma^2)\phi(x) = \int_0^c n(x, y)\phi(y)dy, \tag{2.24}$$

where $n(x, y) = T^{-1}m(x, y)$ (the eigen-solution of (2.23), which is unbounded at $x = 0$ and c, being excluded in the inversion). $n(x, y)$ contains α and γ as parameters. It follows that ϕ is an eigenfunction and $(1-\gamma^2)$ the corresponding eigenvalue of the kernel n, i.e. that a relation

$$1-\gamma^2 = \lambda(\alpha, \gamma) \tag{2.25}$$

exists between the parameters, so that for given values of c and γ, α and therefore the friction coefficient μ is determined.

In the two-dimensional case it was possible to show explicitly for the analogous equation (Spence [14]) that $n(x, y) \geqq 0$ on $(0, c)^2$ so from the general theory of positive kernels it followed that only one eigen-function ϕ, that corresponding to the largest eigen-value of n, was such that

$$\phi \geqq 0 \quad \text{on } (0, c)$$

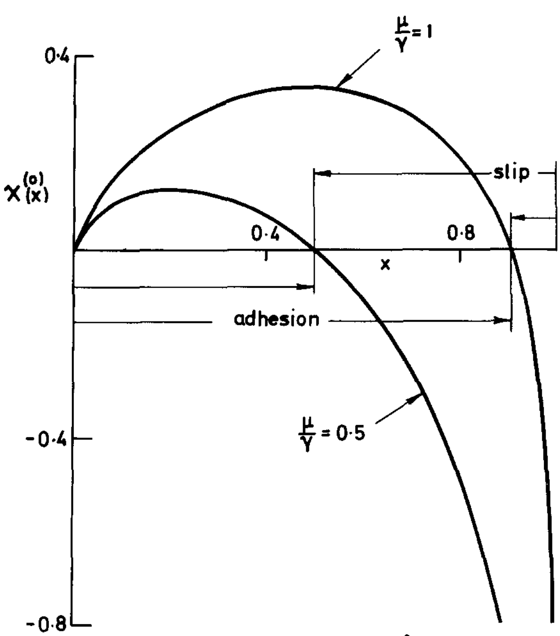
as required by (2.8), and there seems no reason to doubt that the same is true in the present case, but the detailed calculation of $n(x, y)$ has not been undertaken.

2.3. *Solution by successive approximations*

The physical parameter γ takes values moderately small compared with 1 for common materials; e.g. $\gamma = \frac{2}{7}$ when $\nu = 0.3$ (typical of steel), and even for the extreme case $\nu = 0$, $\gamma = \frac{1}{2}$, so an expansion procedure in powers of γ should be of some use. The first approximation, obtained by putting $\gamma = 0$ in (2.2a), is the classical frictionless distribution

$$p^{(0)}(x) = (2/\pi)(1-x^2)^{-\frac{1}{2}}. \tag{2.26}$$

Substitution of this expression in (2.2b) then gives an equation for the corresponding shear stress $q^{(0)}(x)$ say. To carry out the substitution, we calculate the function $\chi(x)$ defined by (2.12), with $p^{(0)}$ in place of p, as



$$\chi(x) = x\int_x^1 \frac{p(t)\,dt}{\sqrt{t^2-x^2}} - \frac{\gamma}{\mu}\int_0^x \frac{t\,p(t)\,dt}{\sqrt{x^2-t^2}} = \begin{cases} x\int_x^c \left(p-\frac{q}{\mu}\right)\frac{dt}{\sqrt{t^2-x^2}} \\ \frac{x}{\mu}\frac{d}{dx}\int_c^x \frac{u(t)\,dt}{\sqrt{t^2-x^2}} \end{cases}$$

$$\chi^{(0)}(x) = \frac{2}{\pi}\left[x\,K'(x) - \left(\frac{\gamma}{\mu}\right)\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)\right]$$

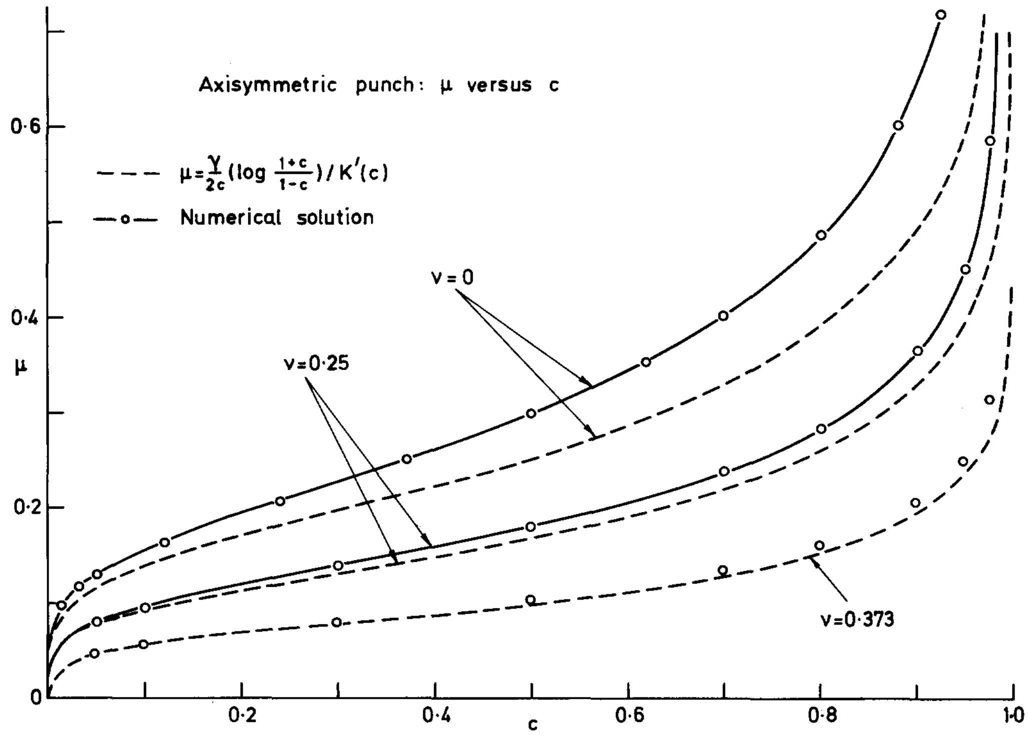
$$\chi^{(0)}(x) = \frac{2}{\pi}\left[xK'(x) - \left(\frac{\gamma}{\mu}\right)\tfrac{1}{2}\log\frac{1+x}{1-x}\right], \tag{2.27}$$

where $K'(x) = K(\sqrt{1-x^2})$. $\chi^{(0)}(x)$ is the difference between two strictly monotonic increasing functions and therefore has just one root $x = c$ between 0 and 1, $\chi^{(0)}$ being $\gtrless 0$ on the intervals $x \lessgtr c$ as required by the signs of ϕ and u respectively. $\chi^{(0)}$ is plotted for two values of μ/γ in figure 2.

Setting $x = c$, we have

$$\frac{\mu}{\gamma} = \frac{1}{2c}\left(\log\frac{1+c}{1-c}\right)/K'(c) \tag{2.28}$$

as the limiting form of the relation between μ, c and γ when γ is small. This expression is plotted in figure 3 along with values of μ obtained from a numerical solution to the



general problem, and provides a good approximation when ν is close to $\frac{1}{2}(\gamma \ll 1)$, although less accurate for $\nu = 0$. From the limiting behaviour of $K'(c)$ as $c \to 0, 1$, we obtain the asymptotic expressions

$$c \sim \begin{cases} 4\exp(-\gamma/\mu) & \text{as } \mu \to 0 \\ 1\text{–}2\exp(-\pi\mu/\gamma) & \text{as } \mu \to \infty \end{cases} \quad \text{for fixed } \gamma. \tag{2.29}$$

These indicate the general shape of the curve of c against μ. With $\chi^{(0)}$ given by (2.27), the integral (2.14) has the value

$$\phi^{(0)}(x) = \frac{2}{\pi}(1-x^2)^{-\frac{1}{2}}\psi(x, c), \tag{2.30}$$

where

$$\psi(x, c) = \Lambda_0\left(\sin^{-1}\frac{\omega}{c}, \sqrt{1-c^2}\right) - \frac{2c}{\pi x}K'(c)\left[\frac{\omega}{c} - \left(\log\frac{1+\omega}{1-\omega}\right)\Big/\left(\log\frac{1+c}{1-c}\right)\right],$$

Λ_0 being the Heuman function (Byrd & Friedman [4] p 35), and $\omega = [(c^2-x^2)/(1-x^2)]^{\frac{1}{2}}$. Thus the shear stress is given in this approximation by

$$\frac{q^{(0)}}{\mu p^{(0)}} = 1-\psi. \tag{2.31}$$

For all c, $1-\psi$ increases monotonically from 0 to 1 as x increases from 0 to c (and also, for fixed $x < c$, decreases monotonically as c increases). $q/\mu p$ as found from this formula is indistinguishable on a graph from the full numerical solution plotted in figure 5.

A further approximation to the pressure could be obtained by inserting $q^{(0)}$ in (2.2*a*). Then

$$p(x) = p^{(0)}(x) + \gamma p^{(1)}(x), \tag{2.32}$$

where $p^{(1)}$ satisfies the Abel equation

$$\int_x^1 \frac{tp^{(1)}(t)dt}{\sqrt{t^2-x^2}} = \int_0^1 \left(1 - \frac{x}{\sqrt{x^2-t^2}}\right) q^{(0)}(t)dt. \qquad (t < x) \tag{2.33}$$

The solution cannot be calculated in closed terms, but could be found numerically. Also, without being solved in full the equation provides an expression for the normal compliance of the surface:

$$P^* = \frac{\pi}{2}\int_0^1 xp(x)dx = 1 + \gamma\int_0^1 (1-\sqrt{1-t^2})q^{(0)}(t)dt \tag{2.34}$$

suitable for numerical quadrature.

Having found $p^{(1)}$ from (2.33), it would be necessary to refine the approximation to c by means of equation (2.13), and the labour involved in proceeding to a further approximation would be prohibitive.

The adhesive limit. Some further light is thrown on the accuracy of this approximation procedure by applying it to the case of full adhesion (i.e. $c = 1$), for which the exact solution is quoted in appendix B.

Eliminating μ from (2.31) by means of (2.28), we have

$$q^{(0)} = \frac{\gamma}{2c}p^{(0)}\left(\log\frac{1+c}{1-c}\right)(1-\psi)/K'(c).$$

As $c \to 1$, the right hand side tends uniformly to

$$-\frac{2\gamma}{\pi^2}\frac{\log(1-x^2)}{x\sqrt{1-x^2}} \tag{2.35}$$

in any interval $(0, 1-\delta)$, which is precisely the limit of the expression (B3) for q when $\gamma \ll 1$. When this expression is used to evaluate (2.34) the result is

$$P^* = 1+\tfrac{1}{3}\gamma^2, \tag{2.36}$$

in agreement to order γ^2 with the exact value $(1/2\gamma)\log\{(1+\gamma)/(1-\gamma)\}$. For $\gamma = \frac{1}{2}$, (2.36) gives 1.0833 compared with the exact value 1.0988.

3. The Hertzian indentor

3.1. *Self similar solutions*

Consider now the situation indicated in figure 1b of progressive indentation by a body whose shape is given by the power law

$$z = r^n/nb^{n-1}$$

with z measured into the body, b being a typical body length; e.g. for a sphere of radius R the Newtonian approximation, valid provided $R \gg a$ is of the above form with $n = 2$ and $b = R$.

As the normal force P is progressively increased the contact radius $a(P)$ and the maximum indentation $\delta(P)$ increase; a can now be treated as the time-coordinate. The dimensionless normal displacement is now

$$w = 1-(a^n/nb^{n-1}\delta)x^n \qquad \text{on } (0, 1). \tag{3.1}$$

For a self-similar solution of the form implied by (1.2) to exist, w must be independent of a, so we can write

$$\delta = Aa^n/b^{n-1}, \tag{3.2}$$

where A is a constant which depends only on n and on the material constants, ν, μ. Moreover, since there is to be no relative slip between the indentor and the half space within the circle $x < c$, the value of u_r within this region cannot change with a, so that

$$\frac{\partial}{\partial a} u_r(r; a) = 0 \qquad \text{for } 0 < \frac{r}{a} < c, \tag{3.3}$$

i.e. u_r must be a function of r only, whereas from (3.2) and (2.1) we deduce that u_r equals a^n times a function of $x = r/a$. The only function u_r satisfying both these requirements is a multiple of r^n, which can be expressed non-dimensionally as

$$u(x) = Cx^n \qquad (0 < x < c) \tag{3.4}$$

where C, like A, is a material constant to be determined. For $c < x < 1$, $u(x)$ is unknown.

Substitution of (3.1) and (3.4) puts (3.3) in the form

$$L\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1-A^*x^n \\ C^*x^n,\ u^*(x) \end{pmatrix} \tag{3.5}$$

where

$$A^* = \Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)\Big/ 2A\Gamma\left(\frac{n+1}{2}\right), \qquad C^* = \left[\Gamma\left(\frac{n+3}{2}\right)\Gamma\left(\frac{1}{2}\right)\Big/ \Gamma\left(1+\frac{n}{2}\right)\right]C,$$

and, as before, the two expressions on the lower line of the right hand side are the values on $(0, c)$ and $(c, 1)$ respectively.

We again have an eigen value problem for c, subject to the same constraints (1.8) on the stresses, while in place of (1.9) the condition that slip takes place inwards is

$$\frac{\partial}{\partial a} u_r(r, a) < 0. \tag{3.6}$$

The constants A and C have to be determined as part of the solution, with the aid of the further boundary condition that the stresses at the edge of the contact region must vanish, i.e.

$$p(1) = 0 = q(1). \tag{3.7}$$

This defines the point where contact between the indentor and the half space is lost. It does not apply when the contact radius is determined by the geometry, as in figure 1a; singularities in the edge stresses must then be admitted.

3.2. *Transformation to equations for a flat punch*

However, even without knowledge of A and C, the solution of (3.5) subject to the frictional constraint (2.8) can be expressed in terms of the solution for the flat punch case. To show this, apply the differential operator

$$D \equiv 1 - \frac{x}{n}\frac{d}{dx} \tag{3.8}$$

to both sides of (1.15), obtaining

$$DL\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 \\ 0,\ Du^* \end{pmatrix}. \tag{3.9}$$

Now integration by parts followed by differentiation with respect to x shows that for functions p and q satisfying (3.7) the operators D and L commute according to the rule

$$DL\begin{pmatrix} p \\ q \end{pmatrix} = L\begin{pmatrix} \tilde{D}p \\ \tilde{D}q \end{pmatrix}, \qquad \tilde{D} = D - \frac{1}{n}. \tag{3.10}$$

Therefore if we define

$$p_0 = \tilde{D}p, \qquad q_0 = \tilde{D}q, \qquad u_0^* = Du^*, \tag{3.11}$$

the equations become

$$L\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0,\ u_0^* \end{pmatrix}. \tag{3.12}$$

These are exactly the equations (2.7) that hold for the flat punch. Once this problem has been solved, therefore, the stress distributions for the power law case are given by quadrature of (3.11) with the boundary condition (3.7) as

$$p(x) = nx^{n-1}\int_x^1 t^{-n}p_0(t)dt, \qquad q(x) = nx^{n-1}\int_x^1 t^{-n}q_0(t)dt, \tag{3.13}$$

provided these distributions and the associated radial displacement $u_r(r, a)$ satisfy the constraints (2.8, 2.9). To confirm that they do, we note first that

$$\mu p - q = nx^{n-1}\int_x^1 (\mu p_0 - q_0)t^{-n}dt \begin{cases} > 0 & \text{on } (0, c) \\ = 0 & \text{on } (c, 1) \end{cases} \tag{3.14}$$

since the integrand is positive and zero respectively on the two intervals by (2.5) and (2.6). From $u_0^* = Du^*$ we obtain the differential equation

$$D(xu)' = (xu_0)'$$

for u, with solution expressible in terms of the physical coordinates as

$$u_r(r, a) = \frac{Ar^n}{b^{n-1}}\left[C - n\int_0^{r/a} t^{-n-1}u_0(t)dt\right], \tag{3.15}$$

whence

$$\frac{\partial}{\partial a}u_r(r, a) = 0 \quad \text{on } \left(0 < \frac{r}{a} < c\right), = nA\left(\frac{a}{b}\right)^{n-1} u_0\left(\frac{r}{a}\right) \quad \text{on } \left(c < \frac{r}{a} < 1\right).$$

Since $u_0(r/a)$ is negative on $(c, 1)$, by (2.9), $\partial u_r/\partial a$ is zero or negative as required by (3.3) and (3.6). We are therefore able to conclude that the slip radius c as a function of μ, v is the same for the power law body as for the flat indentor.

The constants A and C can now be calculated by substitution of the stress distributions (3.13) into equations (3.5). In the numerical work described in later sections, this was straightforward, but a quadrature that gives A directly and so provides a numerical check is obtained by multiplying the first equation by $(2/\pi)(1-x^2)^{-\frac{1}{2}}$ and integrating with respect to x from 0 to 1. This gives

$$1 - \frac{1}{nA} = \frac{2}{\pi}\int_0^1 tK(t)p(t)dt, \tag{3.16}$$

where $K(t)$ is the elliptic integral.

Substitution for $p(x)$ from (3.13) and integration by parts shows that the normal compliance (2.4) is now

$$P^* = \left(\frac{n}{n+1}\right)P_0^*, \tag{3.17}$$

where $P_0^* = (\pi/2)\int_0^1 xp_0(x)dx$ is the flat punch value.

It may also be noted by setting $x = 0$ in (3.11) that the pressure at the centre is related to that at the centre of the flat punch (provided $n > 1$), by

$$p(0) = \left(\frac{n}{n-1}\right)p_0(0). \tag{3.18}$$

Limiting cases

This transformation has been developed with the frictional indentation problem in mind, but applies *a fortiori* in the limiting cases of (a) frictionless and (b) fully adhesive indentation. In the frictionless case, the flat punch problem has the Boussinesq solution $p_0(x) = (2/\pi)(1-x^2)^{-\frac{1}{2}}$, whence for the Hertz problem of indentation by a sphere, from (3.13) with $n = 2$ we immediately obtain the classical result

$$p(x) = (4/\pi)(1-x^2)^{\frac{1}{2}}. \tag{3.19}$$

The corresponding value of the constant A is 1. Similarly, for a conical indentor with semiangle close to $\pi/2$, by putting $n = 1$ in (3.13) we find

$$p(x) = (2/\pi)\log\left[(1+\sqrt{1-x^2})/x\right]. \tag{3.20}$$

4. Numerical solution for axisymmetric flat punch

A numerical scheme for computing the solution of equations (2.2) was constructed by use of piecewise constant approximations to $p_0(t)$ and $t^{-1}q_0(t)$ over N sub-intervals $h_i = \{t : t_{i-1} < t < t_i\}$ spanning the full interval (0, 1), the resulting equations being satisfied at the N mid points $x_i = \frac{1}{2}(t_{i-1}+t_i)$. The values make up the elements of a $2N$ vector $\{p_i\}$ defined by

$$p_0(t) = p_i, \qquad t^{-1}q_0(t) = p_{i+N} \qquad \text{for } t \in h_i,\ i = (1, N) \tag{4.1}$$

($i = (1, N)$ is written throughout for $i = 1, 2, \ldots, N$), and the slip radius c is chosen as the M^{th} point of subdivision, so

$$c = t_M, \qquad M < N. \tag{4.2}$$

The equations can be written, with $t_j - t_{j-1} = |h_j|$, in the partitioned form

$$\begin{aligned} &\sum_j A_{ij}p_j + \gamma\sum_j (x_iB_{ij} - x_j|h_j|)p_{j+N} = 1, \quad i = (1, N), \\ &-\gamma\sum_j B_{ij}p_j + x_i\sum_j A_{ij}p_{j+N} = \begin{cases} 0 & i = (1, M), \\ u_i^* & i = (M+1, N), \end{cases} \end{aligned} \tag{4.3}$$

where all summations extend from $j = 1$ to N, and i is in the range indicated. The $\{A_{ij}\}$, $\{B_{ij}\}$ form $N \times N$ matrices of upper and lower triangular form respectively, with elements

$$\left.\begin{matrix} A_{ij} \\ B_{ij} \end{matrix}\right\} = \int_{h_j} \frac{t\,dt}{|t^2 - x_i^2|^{\frac{1}{2}}} \begin{cases} (t > x_i), \\ (t < x_i), \end{cases} \tag{4.4}$$

(so that $A_{ij} = 0$ for $j < i$, $B_{ij} = 0$ for $j > i$).

The most convenient procedure for solving the eigenvalue problem was to fix c and iterate to find μ, given the material constant γ. Then (2.6) gives

$$x_jp_{j+N} = \mu p_j \qquad j = (M+1, N), \tag{4.5}$$

and the system (2.2) can be written

$$\sum_{l=1}^{N+M} D_{kl}p_l = e_k, \tag{4.6}$$

where D_{kl} is a $2N \times (N+M)$ matrix whose elements are linear combinations of the A_{ij}, B_{ij} containing μ and γ as parameters. With i, j running from 1 to N, the elements of D are

$$D_{ij} = \begin{cases} A_{ij} \\ A_{ij}+\mu\gamma\left(\frac{x_i}{x_j}B_{ij}-|h_j|\right) \end{cases} \qquad D_{i+N,j} = \begin{cases} \gamma B_{ij} & j=(1, M), \\ \gamma B_{ij}+\frac{\mu x_i}{x_j}A_{ij} & j=(M+1, N), \end{cases} \tag{4.7}$$

$$D_{i,j+N} = \gamma(x_i B_{ij} - x_j|h_j|), \qquad D_{i+N\, j+N} = x_i A_{ij} \qquad j = (1, M),$$

and

$$e_k = \begin{cases} 1 & k = (1, N), \\ 0 & k = (N+1, N+M). \end{cases} \tag{4.8}$$

For $k = (1, N+M)$, (4.6) therefore provides a set of $N+M$ linear equations for the p_l, $l = (1, N+M)$ which were solved using a standard sub-routine.

The compliance P^* was evaluated from the solution as

$$P^* = \frac{\pi}{2}\sum_{j=1}^{N} |h_j|x_j p_j. \tag{4.9}$$

As a guide in selecting the mesh size, the method was first applied to the case of fully adhesive contact, in which no slip takes place over any part of the interval (0, 1). Then $c = 1$, i.e. $M = N$, and μ does not enter the calculation. Exact expressions are derivable for this case (see Appendix B) for P^* and for the limiting behaviour of $p(x)$ and $q(x)$ as $x \to 0$, namely

$$P^* = \frac{\pi\kappa}{2\gamma} = 1+\frac{\nu^2}{3}+\ldots, p(x) = p(0)+0(x^2), \qquad \frac{q(x)}{x} = \tfrac{1}{2}\kappa p(0)+0(x^2), \tag{4.10}$$

where

$$\kappa = \frac{1}{\pi}\log\left(\frac{1+\gamma}{1-\gamma}\right), \qquad p(0) = \left(\frac{\kappa}{\gamma}\right)\left(\cosh\frac{\pi\kappa}{2}\right).$$

Four mesh sizes were tested

(i) $N = 25$, all $|h_i| = .04$

(ii) $N = 50$, all $|h_i| = .02$

(iii) $N = 55$, $|h_i| = \begin{cases} .02 & (i = 1, 45) \\ .01 & (46, 55) \end{cases}$

(iv) $N = 60$, $|h_i| = \begin{cases} .02 & (i = 1, 40) \\ .01 & (41, 60) \end{cases}$

the last two being refined in the outer region because of the known singularity of p and q at $x = 1$ in the adhesive case. Results obtained for $\nu = 0$ ($\gamma = \frac{1}{2}$) were

N	25	50	55	60	Exact
P^*	1.09652	1.09769	1.09834	1.09826	1.09861
$p(0)$	.80926	.80881	.80778	.80784	.80760

For different values of ν, the comparison when $N = 55$ was

ν	0.	.125	.25	.375
P^* computed	1.09834	1.06884	1.03964	1.01366
P^* exact	1.09861	1.06900	1.03972	1.01366

On this basis, $N \sim 55-60$ was accepted as a satisfactory sub-division. This involves inverting a matrix of up to 120 rows, which is the effective limit for single precision arithmetic. Improved accuracy might, however, have been obtained for the same N by use of a more sophisticated integration technique. The present integration method, however, appears to give P^* correct to 3 in 10^4 in the most severe case $\nu = 0$, and much more accurately as ν approaches $\frac{1}{2}$. In the limit $\nu = \frac{1}{2}$, $\gamma = 0$ and q is zero. The exact solution in this case is $p_i = (2/\pi)(1-x_i^2)^{-\frac{1}{2}}$ which was reproduced with errors only in the fourth figure except close to the outer boundary, where the last three p_i were fairly seriously in error. In the frictional cases, however, the calculation is believed to be accurate even in this region, since the oscillatory behaviour due to adhesion can be shown analytically to be absent.

4.1. *Iteration to find* μ

The solution gives (i) the stress ratio $q/\mu p$ in the adhesive region $(0, c)$ as

$$q(x_i)/\mu p(x_i) = \frac{x_i p_{i+N}}{\mu p_i}, \qquad i = (1, M), \tag{4.11}$$

(ii) the displacement u in the slip region $(c, 1)$ as

$$u(x_i) = \frac{2}{\pi x_i} \sum_{j=1}^{N} B_{ij} e_{j+N} \qquad i = (M+1, N), \tag{4.12}$$

where e_{j+N} is calculated from (4.6) using the p_i already found. This equation is the numerical version of (4.10), since $e_{j+N} = u_j^*$. (In the last sum e_{j+N} is zero in $(1, M)$ and B_{ij} is zero in $(i+1, N)$.)

Two types of curve were found for these quantities, for fixed c, γ, according to whether μ was larger or smaller than the final μ_c. For $\mu < \mu_c$ the calculated distribution of $q/\mu p$ rises to values greater than 1 thus violating the inequality (2.5), while for $\mu > \mu_c$ the distribution decreases as $x \to c$, in fact representing a negative singularity at this point, while the distribution of u is initially positive in the slip region $x \to c$, contrary to (2.6). It therefore appears that the correct value $\mu = \mu_c$ to satisfy all the conditions is that for which $q/\mu p \to 1$ as $x \to c$. For any other value there would certainly be a discontinuity in stress at $x = c$.

To find μ_c by iteration, the value of $(q/p)_{x=c} = \mu^*$ say was estimated by Gaussian extrapolation from q/p at the mid points of the last 3 intervals.

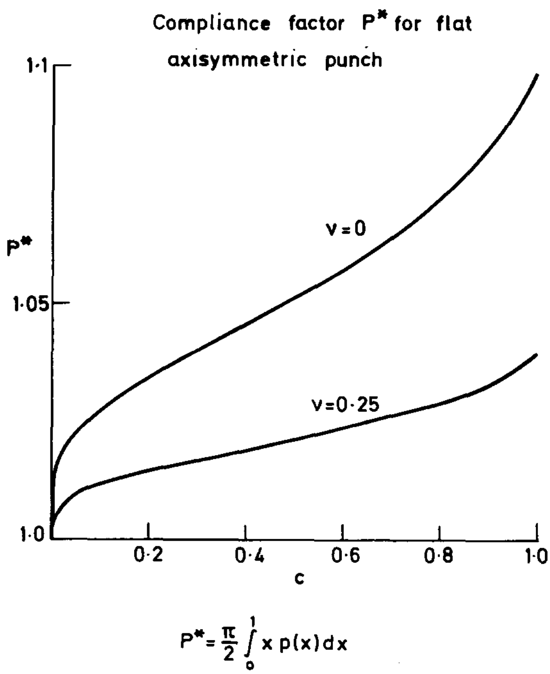
For different values of μ this gives a curve $\mu^*(\mu)$ cutting the line $\mu^* = \mu$ at μ_c. For a given mesh, the intersection was found with an accuracy of 10^{-5} by use of *regula falsi* in approximately five iterations.

In the light of the results for the fully adhesive case, all calculations in the case of finite friction were done with a grid in which $|h_i| = .02$ except for 8 intervals on either side of t_M, which were subdivided to $|h_i| = .01$ for increased accuracy of interpolation in the region in which q varies rapidly, i.e.

$$|h_i| = \begin{cases} .02 & i = (1, M-8), (M+8, N) \\ .01 & i = (M-7, M+8) \end{cases}$$

(These are modified slightly when c is close to 0 or 1.)

The calculation was carried out for $\nu = 0, .125, .25$ and $.375$ for a range of values of c. The results are shown in figure 3 and the compliance factor P^* is plotted in figure 4. For the two-dimensional calculations described in section 5 which proceeds on identical lines, an analytic solution (Spence [14]) shows that the computed values of μ as a function of c and ν are accurate at least to 4 places, and the same is almost certainly true of the present values.



4.2. *The Hertz problem*

The distribution $p_i^{(0)}$, say, resulting from the final iteration of the flat punch calculation was used to calculate the pressure and shear stress distributions for the case of Hertzian indentation by a sphere of radius R, using the fact proved in the section 3 that, for given values of μ and ν, c is the same in both cases. If

$$p(x_i) = p_i^{(2)}, \qquad x_i^{-1}q(x_i) = p_{i+N}^{(2)} \tag{4.13}$$

are written for the Hertzian stresses at the mid-point of the i^{th} interval, their values are given by (3.13) with $n = 2$ as

$$p_i^{(2)} = 2x_i \int_{x_i}^{1} t^{-2}p^{(0)}(t)dt, \qquad p_{i+N}^{(2)} = 2\int_{x_i}^{1} t^{-2}q_0(t)dt. \tag{4.14}$$

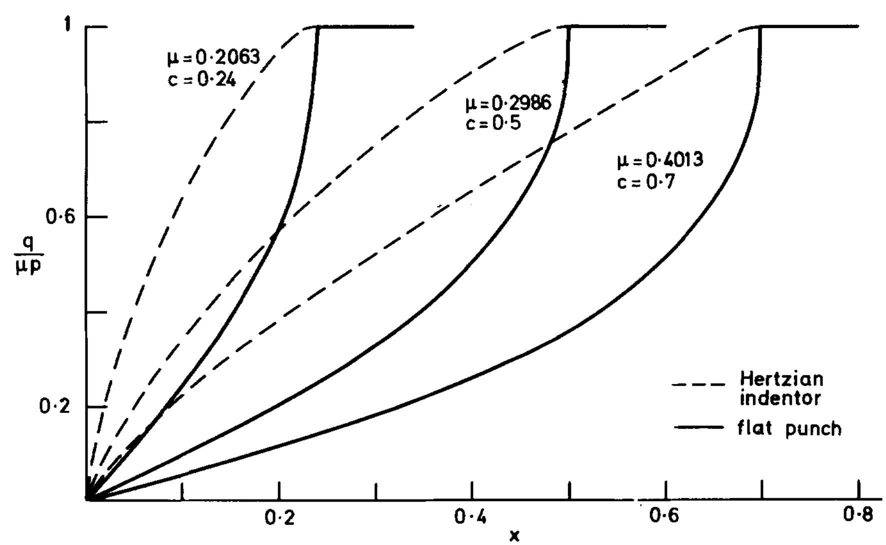
A quadratic interpolation

$$p^{(0)}(t) = a_i t^2 + b_i t + c_i \qquad (t \in h_i) \tag{4.15}$$

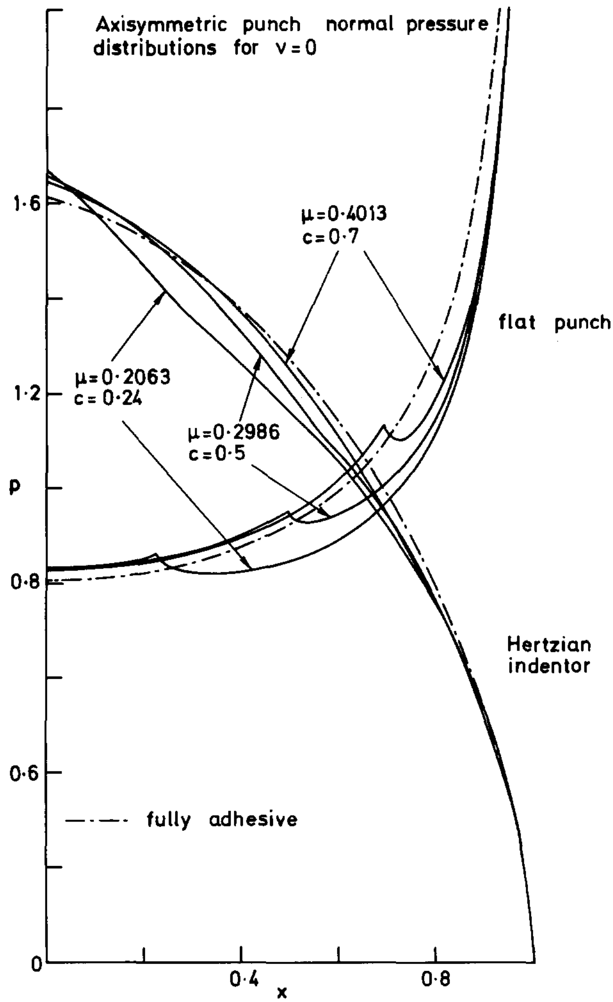
fitting the flat punch values $p_{k-1}^{(0)}$, $p_k^{(0)}$ and $p_{k+1}^{(0)}$ at the three nearest mid-points (i.e. $k = 2$ for $i = 1$, $k = i$ for $i = (2, N-1)$, $k = N-1$ for $i = N$) was multiplied by t^{-2} and integrated exactly in each interval, to obtain $p_i^{(2)}$; a similar interpolation fitting $p_{k+N}^{(0)}$ when multiplied by t^{-1} and integrated gave $p_{i+N}^{(2)}$ [For $k > M$, $p_{k+N}^{(0)} = \mu p_k^{(0)}/x_k$, by (2.5)].

The resulting distributions of shear stress and normal pressure, together with the corresponding flat punch distributions, are shown in figures 5 and 6 for the case $\nu = 0$ and a range of values of μ.

Axisymmetric punch, ν=0: shear stress in adhesive region



A test of accuracy is provided by the case of the fully-adhesive punch, for which as noted in Appendix B the pressure at the centre and its derivative can be calculated from the results of the author's earlier paper [13].



For the case $\nu = 0$, we find from (B6)

$$p^{(2)}(0) = 2p^{(0)}(0) = 1.615192, \qquad \left[\frac{d}{dx}\,p^{(2)}(x)\right]_{x=0} = -.282415,$$

while the quadratic fit to the computed values is

$$p = 1.615725 - 0.2801x - 0.781x^2.$$

This suggests that the quadrature is accurate to about 5×10^{-4}.

The normal compliance factor P^* can be calculated either as $\frac{2}{3}$ of the flat punch value, because of (1.28), or by direct numerical integration using (2.9). The former method has been seen in the table on page 19 to be accurate to about 3 in 10^4, whereas direct quadrature of the $p^{(2)}$ distribution gave 2.91840 against an exact value of 2.92963, accurate only to 4 in 10^3. This might have been improved by use of a polynomial interpolation to the $p_i^{(2)}$.

Evaluation of the constants A and C

These are given in principle by insertion of the new distributions $p_i^{(2)}$ on the left hand sides of the equations (2.2). Writing

$$\sum_{l=1}^{N+M} D_{kl} p_l^{(2)} = e_k^{(2)} \tag{4.16}$$

we should have, if the equations were exactly satisfied by the piecewise constant distribution,

$$e_k^{(2)} = 1 - x_k^2/A \quad k = (1, N), \qquad e_{k+N}^{(2)} = \left(\frac{3\pi C}{4}\right) x_k^2 \quad k = (1, M). \tag{4.17}$$

In fact, quadratics could be fitted to the $e_k^{(2)}$ with high accuracy; typically the third differences, with $|h_i| = .02$, were of the order of 5×10^{-7}.

The values of A and C were estimated by means of least squares fits to the $e_k^{(2)}$ of the form $\alpha_0 - \alpha_1 x^2$, $\alpha_2 x^2$ respectively. For the fully adhesive case, the values of A and C are known from equation (4.12) of Spence (1968) as

$$A = (1 - \kappa^2 \Psi_1(\kappa))^{-1}, \qquad C = -\tfrac{2}{3}\kappa^2 \sqrt{3-4\nu}/(1-2\nu), \tag{4.18}$$

where $\kappa = (1/\pi) \log(3 - 4\nu)$ as before, and

$$\Psi_1(\kappa) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{\kappa}{2}\right)^{2n} \eta(2n+1) \;\; (\Psi_1(0) = \log 2).$$

The values obtained in the test case $\nu = 0$ were $\alpha_0 = .99437$, $\alpha_1 = .91345$, $\alpha_2 = .32167$ compared with the exact values 1, .91849, .33271, suggesting that α_1 gives $1/A$ with an accuracy of approximately half a percent. A summary of the results for $\nu = 0$ and $\nu = .25$ is:

	c	.24	.3	.5	.7	.8	1.0	(exact)
$\nu = 0$	μ	.2063		.2986	.4013	.4862		
	$1/A$	.9684		.9578	.9479	.9417	.9190	(.9185)
$\nu = .25$	μ		.1387	.1801		.2843		
	$1/A$		.9827	.9794		.9728	.9635	(.9668)

Similar calculations for the corresponding two-dimensional problem are presented in a University of Wisconsin Mathematics Research Centre Report (1209) by the author.

Appendix A. Kernels in equations (2.15), (2.19)

(i) $l(x, t)$ is defined in the rectangle $0 < x < c$, $0 < t < 1$, by the expressions

(1) $t < x < c$:

$$l(x, t) = \frac{2t}{\pi(t^2 - x^2)} \left[\frac{x}{c} \left(\frac{c^2 - x^2}{c^2 - t^2} \right)^{\frac{1}{2}} + E\left(\frac{t}{x}\right) - E\left(\xi, \frac{t}{x}\right) \right],$$

where $\xi = \sin^{-1}(x/c)$, $E(t/x)$ and $E(\xi, t/x)$ being the complete and incomplete elliptic integrals of the second kind.

(2) $x < t < c$:

$$l(x, t) = \frac{2t}{\pi(t^2 - x^2)}\left[\frac{x}{c}\left(\frac{c^2 - x^2}{c^2 - t^2}\right)^{\frac{1}{2}} + \frac{t}{x}\left\{E\left(\frac{x}{t}\right) - E\left(\phi, \frac{x}{t}\right)\right\}\right] - \frac{2}{\pi x}\left\{K\left(\frac{x}{t}\right) - F\left(\phi, \frac{x}{t}\right)\right\},$$

where $\phi = \sin^{-1}(t/c)$, $K(x/t)$ and $F(\phi, x/t)$ being the corresponding elliptic integrals of the first kind.

(3) $x < c < t$:

$$l(x, t) = -\frac{2\tan \pi\alpha}{\pi\gamma^2(t^2 - x^2)}\left(\frac{c^2 - x^2}{t^2 - c^2}\right)^{\frac{1}{2}}.$$

Since $E(1) - E(\xi, 1) = 1 - \sin \xi$, the first two expressions both show that $l \sim 1/\pi(t - x)$ near $x = t$ $(t < c)$. In $t > c$, the kernel is regular since l is defined only for $x < c$.

(ii) $k(x, y)$ is defined in the interior of the unit square $0 < x, y < 1$ by

$$k(x, y) = \begin{cases} \dfrac{2y^2}{\pi x} \dfrac{E\left(\frac{x}{y}\right)}{y^2 - x^2} - \dfrac{2}{\pi x} K\left(\dfrac{x}{y}\right) & (x < y), \\[2ex] \dfrac{2y}{\pi} \dfrac{E\left(\frac{y}{x}\right)}{y^2 - x^2} & (x > y). \end{cases}$$

Since $E(1) = 1$, both expressions give $k \sim 1/\pi(y - x)$ near the singularity at $y = x$. On the left of this line, the next term in the kernel also has the logarithmic singularity of K.

Appendix B. The adhesive solution

The stresses in the case of adhesive indentation by a flat axisymmetric punch were obtained in the author's earlier paper ([13] equations 4.3 and 4.4, p. 68) in the alternative forms

$$\left.\begin{matrix} xp_0(x) \\ q_0(x) \end{matrix}\right\} = -\left(\frac{2}{\pi}\cosh\frac{\pi\kappa}{2}\right)\frac{d}{dx}\int_x^1 \begin{pmatrix} w\cos\kappa\theta \\ \sin\kappa\theta \end{pmatrix} \frac{dw}{\sqrt{w^2 - x^2}} \tag{B1}$$

$$= -\left(\frac{2}{\pi\gamma}\cosh\frac{\pi\kappa}{2}\right)\frac{d}{dx}\int_0^x \begin{pmatrix} -w\sin\kappa\theta \\ \cos\kappa\theta \end{pmatrix} \frac{dw}{\sqrt{x^2 - w^2}}, \tag{B2}$$

where $\kappa = (1/\pi)\log\{(1+\gamma)/(1-\gamma)\}$ and $\theta(w) = \frac{1}{2}\log\{(1+\mathrm{w})/(1-\mathrm{w})\}$. These results were also obtained by Keer [6].

From the second expression by integration by parts followed by differentiation with respect to x we get a third form

$$\left.\begin{matrix} p_0(x) \\ q_0(x) \end{matrix}\right\} = p_0(0)\left(\frac{2}{\pi}\right)\int_0^x \begin{pmatrix} \cos\kappa\theta \\ \frac{w}{x}\sin\kappa\theta \end{pmatrix} \frac{dw}{(1-w^2)\sqrt{x^2-w^2}}, \tag{B3}$$

where $p_0(0) = (\kappa/\gamma)\cosh\pi\kappa/2$. The limiting expressions as $x \to 0$ quoted in equation (2.10) follow immediately from (A3) on putting $\theta \doteqdot w$, while the compliance P_0^* defined by (1.27) is found from (A1) as

$$P_0^* = \frac{\pi}{2}\int_0^1 x p_0(x)dx = \left(\cosh\frac{\pi\kappa}{2}\right)\int_0^1 \cos\kappa\theta dw = \frac{\pi\kappa}{2\gamma}. \tag{B4}$$

From (B3), the pressure in the Hertzian case is given by (1.23) with $n = 2$ as

$$p(x) = \frac{4}{\pi}p_0(0)x\int_x^1 \frac{dy}{y^2}\int_0^y \frac{\cos\kappa\theta(w)}{1-w^2}\frac{dw}{\sqrt{y^2-w^2}}, \tag{B5}$$

and examination of the behaviour as $x \to 0$ shows that

$$p(x) = 2p_0(0)[1-\kappa\gamma x+\ldots], \tag{B6}$$

which gives the values quoted in section 2.2.

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