JOSE MARIA AIZPURUA, JORGE NIETO, AND
JOSE RAMON URIARTE

CHOICE PROCEDURE CONSISTENT WITH SIMILARITY RELATIONS

ABSTRACT. We deal with the approach, initiated by Rubinstein, which assumes that people, when evaluating pairs of lotteries, use similarity relations. We interpret these relations as a way of modelling the imperfect powers of discrimination of the human mind and study the relationship between preferences and similarities. The class of both preferences and similarities that we deal with is larger than that considered by Rubinstein. The extension is made because we do not want to restrict ourselves to lottery spaces. Thus, under the above interpretation of a similarity, we find that some of the axioms imposed by Rubinstein are not justified if we want to consider other fields of choice theory. We show that any preference consistent with a pair of similarities is monotone on a subset of the choice space. We establish the implication upon the similarities of the requirement of making indifferent alternatives with a component which is zero. Furthermore, we show that Rubinstein's general results can also be obtained in this larger class of both preferences and similarity relations.

Keywords: Preference relations, similarity relations.

I. INTRODUCTION

To our knowledge, the issue of imperfect powers of discrimination of the human mind was first studied in economics by Georgescu-Roegen (1936, 1958) and Armstrong (1939, 1948, 1950, 1951). Georgescu-Roegen (1958) states that in a theory of choice we must consider the individual not as a perfect choosing-instrument but as a stochastic one and, bearing this imperfection in mind he proposed, in Georgescu-Roegen (1936), a model of the consumer's behaviour where the indifference relation was not transitive. In Armstrong (1939) there are arguments against the transitivity of indifference and in Armstrong (1950, p. 122), it is pointed out that

> The nontransitiveness of indifference must be recognized and explained on any theory of choice and the only explanation that seems to work is based on the imperfect powers of discrimination of the human mind whereby inequality becomes recognizable only when of sufficient magnitude.

Theory and Decision **29**: 235–254, 1990.

This subject is then axiomatized by Luce (1956), who also provides arguments, based on empirical evidence, against the transitivity of indifference. The two examples provided by Luce – one about 'adjacent' weights and the other about a cup of coffee with different, but similar, amounts of sugar – suggest that if indifference were transitive, then a subject would be unable to detect any weight or any sugar concentration differences, however great; something which is patently false. Therefore, to Luce the intransitivity of some indifference relations reflects "the inability of an instrument to discriminate relatively to an imposed discrimination task" Luce (1956, p. 179). But Luce also points out an important issue in utility theory. The theory of preferences underlying utility theory, which generally assumes that indifference is an equivalence relation, implies that utility is perfectly discriminable and therefore a rational being would be the one who would respond to any finite difference in utility, however small. But, again, it is false that people behave in this manner; utility is not perfectly discriminable and Luce shows that the imperfect response sensitivity to small changes in utility is related to intransitivities of the indifference relation.

Thus, we see that these three authors, and many others (see Fishburn, 1970), argue that nontransitivity of indifferences is due to the limited capacities of the human mind. All of them try to capture these limitations in a direct manner, by introducing a new set of axioms to represent a person's preference pattern. Georgescu-Roegen and Luce make an important distinction among the domains of discrimination tasks to be performed by an individual. They distinguish the physical domain from the utility domain. In both domains it can be assumed that there is an imperfect power of dicrimination. It is perhaps Georgescu-Roegen (1936, p. 572), who emphasizes this distinction more:

> The individual's behavior appears therefore as a resultant of two different types of measurement: a physical one, which is supposed to tell him the exact amounts of commodities, and a psychological one, which is his possibility of comparing satisfactions. The fact that these two kinds of measurements are both involved in the present scheme constitutes an important point in the problem.

Psychologists describe the difficulties of human perception by means of similarities. A similarity is expressed by a judgement of the type '*a* is like *b*'. For Tversky (1977, p. 327).

> Similarity plays a fundamental role in theories of knowledge and behavior. It serves as an organizing principle by which individuals classify objects, form concepts and make generalizations. Indeed, the concept of similarity is ubiquitous in psychological theory. It underlies the accounts of stimulus and response generalization in learning, it is employed to explain errors in memory and pattern recognition and it is central to the analysis of connotative meaning.

To our knowledge, the first application of the similarity concept to economics was done by Rubinstein (1988), who defines a similarity relation by a set of axioms different than the one used by psychologists such as Tversky (1977). Rubinstein's work deals with a particular class of economic decision problem: choice under risk. He assumes that people use a decision scheme based on the use of similarities on the two characteristics defining risky prospects: prizes and probabilities. With this approach, Rubinstein shows a possible explanation to the Allais Paradox. The final goal of his work is the construction of a descriptive theory of decision under risk by looking at the decision procedures themselves. The same methodological attitude is adopted by Tversky (1977, p. 332).

> Furthermore, the axioms are proposed as (normative) principles of rational behavior, whereas the axioms of the present theory are intended to be descriptive rather than prescriptive.

We think that the similarity approach used by Rubinstein can be extended to describe the bounded capacities of perception of the human mind when faced to any choice problems. Our work may be seen as an indirect way of allowing the introduction of the imperfectness of human perception into the preference relations defined over a set of alternatives. It should be noted that it is very likely that the set of axioms by which Rubinstein defines a similarity relation should need a modification in order to capture better that imperfectness. Nevertheless, the present work accepts the definition of similarity given by Rubinstein and we shall learn from the results obtained the adequacy of the adopted definiton.

Thus, our primitive tools are a pair of similarities and the issue that we focus on is one posed by Rubinstein: what preference relations are consistent with a given pair of similarity relations? We provide an answer to this question for a class of both similarities and preferences larger than those considered by Rubinstein. The reasons for this extension are the following.

If two alternatives A and B are percieved to be similar in the sense that all their relevant characteristics are indistinguishable, then it makes no sense to say that A is preferred to B, even when there is vector dominance of the characteristics of A over that of B. When monotonicity of preferences is required, we are, in fact, assuming that all the characteristics of the objects can be perfectly discriminated. As Georgescu-Roegen (1958, p. 159), points out, the most important implication of the monotonicity assumption

> is that the quantities of all commodities involved are not estimated by the consumer's senses alone, but determined by outside scales (. . .). This means that in the model under consideration the threshold in choice is completely isolated from the ordinary sensorial threshold, that is, from a phenomenon irrelevant to the economic behavior of the consumer in a world where quantities exchanged are determined with the aid of physical instrument.

The present paper does not consider a world where agents go around carrying measuring instruments to help them in everyday choices. Therefore, we shall not make the assumption of monotonicity of preferences. But, as a negative result, we show that the decision procedure based on similarities is strong enough (together with the transitivity of the preference relation) to imply the monotonicity property in a large subset of the choice space.

In Rubinstein's work, the assumption (R.4) states that the lotteries $(x, 0)$ and $(0, y)$ are declared indifferent, for all x and y in $[0, 1]$. This assumption is not justified when we want to deal with a more general class of choice problems. Consider, for instance, the case where the alternatives are financial assets. A decision maker has to make a portfolio selection taking into account two characteristics of each asset: the expected rate of return (in real terms), x, and the degree of risk (measured, say, by its β). Would it be acceptable in this case to require $(x, 0)$ to be indifferent to $(x', 0)$ for any x and x'?

Our Theorem 1 shows that the (R.4) is equivalent to the assumption that the similarity relations on the characteristics of the alternatives are such that no point different from zero is similar to it.

Since our primitive data are the similarity relations, which describe the bounded perception of an agent facing a decision problem, two numbers are declared similar if they are perceived as undistinguishable. Therefore in this context we do not find any justification to accept that zero is more distinguishable than any other number, i.e. that zero and a sufficiently small ε can be perfectly discriminated and that α and $\alpha + \varepsilon$ cannot, for all $\alpha \neq 0$.

For these reasons, the present work does not impose the assumption (R.4) on preferences and will consider a larger family of similarity relations which may or may not distinguish zero from any other positive number less than 1. In this way, again, we open the possibility of applying the similarity approach to other fields of choice theory where zero does not receive a special treatment.

Our Theorems 2 and 3 show that results of Rubinstein can be extended to the larger set of continuous complete preorders consistent with a pair of similarities.

Thus, the present work should be viewed as a modest attempt to enlarge the domains of application of the similarity approach in which agents are not described *a priori* as perfect perceivers.

Nevertheless, our main conclusions are negative, so that further work has to be done to describe the relationships between preferences and similarities. We find, as in Rubinstein's work, that a continuous, complete preorder over the set of alternatives is 'almost' uniquely determined by a given pair of similarity relations, i.e., two people, having the same capability of perception, must have 'almost' the same preferences. And, furthermore, these preferences are monotone. Thus, using Georgescu-Roegen's terminology, having introduced sensorial thresholds in the perception of the characteristics, we find that preferences consistent with those sensorial thresholds do not recognize them.

The work is organized as follows. In Part II we present the notation and definitions needed. In Part III we present the results.

II. NOTATION AND DEFINITIONS

1. *Preferences on the Set of Alternatives*

Let D be the domain of alternatives. Any object in D is defined by two characteristics (x, y) measured by real numbers. We will normalize these values, and therefore $X = Y = [0, 1]$ will be the spaces of characteristics. Hence, the set of alternatives, which are the objects of choice, will be the square $[0,1] \times [0, 1]$. Let $\gtrsim$ denote a binary relation defined on $[0, 1]^2$. We will assume that $\gtrsim$ satisfies the following properties:

(R.1) $\gtrsim$ is complete, reflexive and transitive.

(R.3) $\gtrsim$ is continuous.

Remark 1. In the particular case in which the alternatives are lotteries which x is a prize and y is the probability of having that prize, Rubinstein (1988) assumes that $\gtrsim$ furthermore satisfies:

(R.2) Monotonicity: $x_1 > x_2$ and $y_1 > y_2$ imply that $(y_1, y_1) > (x_2, y_2)$.

(R.4) For any (x, y) and (x', y') in $([0, 1] \times 0) \cup (0 \times [0, 1])$, $(x, y) \sim (x', y')$.

where $>$ and $\sim$ are the asymmetric and symmetric parts of $\gtrsim$, respectively. In this work, trying to cover a more general domain of choice problems, it is assumed that $\gtrsim$ is just a continuous and complete preordering.

2. *Similarity Relation*

A binary relation S of the set $A = [0, 1]$ is a similarity relation if the following axioms are satisfied:

(S.1) Reflexity: for all $a \in A$, $a\,S\,a$.

(S.2) Symmetry: for all $a, b \in A$, if $a\,S\,b$ then $b\,S\,a$.

(S.3) Continuity: the graph of S is closed.

(S.4) Betweenness: if $a \leqslant b \leqslant c$ and $a\,S\,c$ then $a\,S\,b$ and $b\,S\,c$.

(S.5) Non-degeneracy:

(1) For all $0 < a < 1$, there are b and c. $c < a < b$ such that $b\,S\,a$ and $c\,S\,a$. For $a = 1$ there is c as above. Thus the only element which may be not similar to any other element in A is zero.
(2) $0 \not{S} 1$.

(S.6) Responsiveness: if $a\,S\,b$ and there is an $a' < a(a' > a)$ such that $a'\,S\,b$ then, if $b > 0$, there is $b' < b$ ($b' > b$, if $b < 1$), such that $a'\,S\,b'$.

Remark 2. (a) Note that (S.3), (S.4)′ and (S.5) imply (S.1). (b) The set of independent axioms which defines a similarity relation is (S.2)–(S.6).

Proof. (a) For any $a \in (0, 1]$, it is easy to see that (S.4)′ and (S.5) imply that $a\,S\,a$. Now, the continuity axiom, (S.3), guarantees that $0\,S\,0$ because $\varepsilon\,S\,\varepsilon$ for any $\varepsilon > 0$.

(b) We shall prove it by means of examples of binary relations, R, defined on $A = [0, 1]$, that satisfy only four of the above mentioned five axioms.

Example 1: $a\,R\,b$ when $1/3 \leqslant a/b \leqslant 2$. R satisfies (S.3), (S.4), (S.5), (S.6) but not (S.2) because, say, $1/3\,R\,1$ but $1 \not{R}\, 1/3$.

Example 2: $a\,R\,b$ when $1/2 < a/b < 2$. R satisfies (S.2), (S.4), (S.5), (S.6) but not (S.3) because, say, $0.3\,R\,0.6 - \varepsilon$ for all $\varepsilon > 0$ but $0.3 \not{R}$ 0.6.

Example 3: $a\,R\,b$ when (i) $a, b \in [0, 0.1]$, or (ii) $a, b \in [0.9, 1]$, or (iii) $|a - b| = 0.1$. R satisfies (S.2), (S.3), (S.5), (S.6) but not (S.4) because, say $0.8\,R\,0.9$ and $0.8 \not{R}\, 0.85$.

Example 4. $a\,R\,b$ when $1/2 \leqslant (a-1)/(b-1) \leqslant 2$. R satisfies (S.2), (S.3), (S.4) and (S.6); for any $a \in [0, 1]$ the set of points related to a is given by $[2a - 1, a + 1/2] \cap [0, 1]$. R does not satisfy (S.5) because there does not exist not exist $c < 1$ and $c\,R\,1$.

Example 5. $a\ R\ b$ when (i) $a, b \in [0, 0.5]$, or (ii) $a, b \in [0.5, 1]$. R satisfies (S.2), (S.3), (S.4), (S.5) but not (S.6) because, say $0.3\ R\ 0.5$, $0.4\ R\ 0.5$ and there is no $b' > 0.5$ such that $0.4\ R\ b'$.

Remark 3. Rubinstein (1988) defines (S.4) betweenness as follows: if $a \leqslant b \leqslant c \leqslant d$ and $a\ S\ d$ then $b\ S\ c$.

It is easy to see that (S.4)$\Leftrightarrow$(S.4)$'$:

(S.4) $\Rightarrow$ (S.4)$'$

If $a \leqslant b \leqslant c$ and $a\ S\ c$ then $a\ S\ b$ because $a \leqslant a \leqslant b \leqslant c$ and (S.4); $b\ S\ c$ because $a \leqslant b \leqslant c \leqslant c$ and (S.4).

(S.4)$'$ $\Rightarrow$ (S.4)

If $a \leqslant b \leqslant c \leqslant d$ and $a\ S\ d$ then $a\ S\ c$ because $a \leqslant c \leqslant d$ and (S.4)$'$ and therefore $b\ S\ c$ because $a \leqslant b \leqslant c$ and (S.4)$'$.

□

Given $a \in A$, we defined the set $S[a]$ as

$$S[a] = \{b \in A : b\ S\ a\}$$

Let $a^* = \max S[a]$ and $a_* = \min S[a]$ denote the maximum element and the minimum element of $S[a]$, respectively. Notice that for all a such that $a^* \neq 1$, $(a^*)_* = a$ and for all a such that $a_* \neq 0$ $(a_*)^* = a$.

Example. An example of a similarity relation is the λ-ratio similarity, which is defined by $a\ S\ b$ if $1/\lambda \leqslant a/b \leqslant \lambda$. Take $\lambda = 2$ and $a = 0.3$, then $S[a] = [0.15, 0.6]$, thus $a^* = 0.6$, $a_* = 0.15$ and $(a^*)_* = (a_*)^* = 0.3$. Another example is the ε-difference similarity, which is defined by $a\ S\ b$ if $|a - b| \leqslant \varepsilon$. Take $\varepsilon = 0.2$, then $S[0] = [0, 0.2]$.

We introduce now the definition of a procedure for preference determination, based on the use of similarity relations. The procedure is in fact a description of the decision scheme which an individual is supposed to follow in order to determine his/her preferred alternative.

3. Condition $**$

We say that $\succsim$ satisfies the condition $**$ with respect to the similarities S_x and S_y on characteristics X and Y, respectively when for all $(x, y) \in [0, 1]2$, if (x', y') is such that:

(1) $y' \not S_y\, y$, $y' > y$ and $x' \; S_x \; x$ but $x' \neq 0$ whenever $x = 0$, then $(x', y') \succ (x, y)$.

(2) $y' \; S_y \; y$, but $y' \neq 0$ whenever $y = 0$, and $x' \not S_x\, x$ with $x' > x$, then $(x', y') \succ (x, y)$.

4. Definitions. The sets $\hat{C}$, T_R, T_L and T)

$\hat{C} = \{(S_x, S_y, \succsim)$: $\succsim$ satisfies (R.1), (R.3) and $**$ relative to S_x
and S_y satisfying (S.1, 2, 3, 4′, 5, 6)$\}$

$$T_R = \{(x, y) \in [0, 1]^2 : x \in (0, 1_*], \; y \in [0^*, 1)\}$$

$$T_L = \{(x, y) \in [0, 1]^2 : x \in [0^*, 1], \; y \in (0, 1_*]\}$$

$$T = T_R \cup T_L \, .$$

Elements of set $\hat{C}$ are triples formed by two similarities over characteristics and a continuous complete preorder 'consistent' with these similarities. T_R and T_L are subsets of the choice space allowing 'movements to the right' and 'to the left', respectively, as will be seen in the results of the next section.

III. THE RESULTS

The following result, which is the first connection between similarities and preferences, is obtained as a direct implication of $**$ and (R.3).

LEMMA *1. Let $(S_x, S_y, \succsim) \in \hat{C}$, then*:

(a) *If $(x, y) \in T_R$ then $(x, y) \sim (x^*, y^*)$.*

(b) *If $(x, y) \in T_L$ then $(x, y) \sim (x^*, y^*)$.*

Proof. Part (a) *Case 1.* $y \geqslant 0^* > 0$. In this case $y_* > 0$. By $**$, for any $0 < \varepsilon < x$, $(x - \varepsilon, y) \prec (x^*, y_*)$ because $y\ S_y\ y_*$, $x^* > x - \varepsilon$ and $x^*\ \not S_x\ x - \varepsilon$, by (S.6) and the definition of x^*. By (R.3) $(x, y) \precsim (x^*, y_*)$. Similarly, by $**$, $(x, y) \succ (x^*, y_* - \varepsilon)$ for any $0 < \varepsilon < y_*$. Thus, by (R.3) $(x, y) \succsim (x^*, y_*)$.

Case 2: $y = 0^* \geqslant 0$. In this case $y_* = 0$ and by (S.5) $1 > 0^*$.

When $0^* > 0$, by $**$ $(x - \varepsilon, y) \prec (x^*, y_*)$ and by (R.3), $(x, y) \precsim (x^*, y_*)$. Similarly, by $**$ $(x, y + \varepsilon) \succ (x^*, y_*$ and by (R.3), $(x, y) \succsim (x^*, y_*)$.

When $0^* = 0$, by $**$ $(x, y + \varepsilon) \succ (x^*, y_*)$ and by (R.3), $(x, y) \succsim (x^*, y_*)$. Similarly, by $* *$, $(x, y) \prec (x^*, y_* + \varepsilon)$ because $y = y_* = 0$ and by (R.3), $(x, y) \precsim (x^*, y_*)$.

Part (b): the same as in part (a), changing x for y.

Remark 4. Lemma 1 implies Rubinstein's Lemma 2 for the particular case in which alternatives are the simple lotteries described before, but the converse is not true in general.

Proof. Take any x and y such that $x_* > 0$ and $y_* > 0$. Then $x_* \in (0, 1_*]$ and $y_* \in (0, 1_*]$. Now call z to x_*, then $(z, y) \in T_R$ because $y \in (0^*, 1]$. Thus, by Lemma 1 $(z, y) \sim (y^*, y_*)$, therefore $(x_*, y) \sim (x, y_*)$ as Rubinstein's Lemma 2.

Now take any x and y such that $x \in (0, 1^*]$ and $y \in (0^*, 1]$, i.e. $(x, y) \in T_R$. Then

$$0 < x^* \leqslant 1$$

and

$$y_* > 0$$

Let us all z to x^*. Since $z_* = (x^*)_* = x > 0$ and $y_* > 0$ we may apply Rubinstein's Lemma 2, i.e.

$$(z, y_*) \sim (z_*, y)$$

Therefore

$$(x^*, y_*) \sim (x, y)\ .$$

But when $x \in (0, 1_*]$ and $y = 0^*$, $(x, y) \in T_R$ and $y_* = 0$. This case is not considered in Rubinstein's Lemma 2. □

The next result presents the necessary and sufficient condition that a pair of similarities must satisfy in order that the set of alternatives $([0, 1] \times 0) \cup (0 \times [0, 1])$, be an indifference class for any preference consistent with that pair of similarities. This result also motivates the necessity of relaxing assumption (R.4). The theorem shows that if (R.4) is maintained, the similarities on both characteristics should be of a certain type. Therefore assumption (R.4) puts some limits on the admissible class of similarities. This assumption is empirically justified in the case treated by Rubinstein (1988) where the alternatives are lotteries. But in the general case, there is no reason to limit the acceptable type of similarity relations.

THEOREM 1. *Let* $(S_x, S_y, \succsim) \in \hat{C}$, *then for all* x *and* y *in* $[0, 1]$ $(x, 0) \sim (0, y)$ *if and only if* $S_x[0] = S_y[0] = \{0\}$.

The following two lemmata will be helpful in proving this theorem.

LEMMA 2. *Let* $(S_x, S_y, \succsim) \in \hat{C}$: (a) *If* S_y *is such that* $S_y[0] = (0)$, *then for all* $x \in [0, 1]$ *and for all* $x' \in S_x[x]$, $(x, 0) \sim (x', 0)$. (b) *If* S_x *is such that* $S_x[0] = (0)$, *then for all* $y \in [0, 1]$ *and for all* $y' \in S_y[y]$, $(0, y) \sim (0, y')$.

Proof. (*a*) Let us consider in $Y = [0, 1]$ a sequence (η^k) defined as follows: $\eta^0 = \eta > 0$ and $\eta^{k+1} = (\eta^k)_*$. Notice that (η^k) goes to zero as k increases and that $\eta^k > 0$ for all k because, by assumption, S_y is such that $0^* = 0$. By property $**$, for all k

$$(x, \eta^k) \succ \left(x', \frac{\eta^{k+1}}{2}\right)$$

and by (R.3) it follows that $(x, 0) \succsim (x', 0)$. Equivalently, it must be true that for all k

$$\left(x', \frac{\eta^{k+1}}{2}\right) \prec (x', \eta^k)$$

and by (R.3) $(x, 0) \precsim (x', 0)$.

(b) Proceed as in part (a) changing x for y.

LEMMA 3. *Let* $(S_x, S_y, \succsim) \in \hat{C}$: (a) *If* S_y *is such that* $S_y[0] = \{0\}$, *then for all* $x \in [0, 1]$, $(x, 0) \sim (1, 0)$. (b) *If* S_x *is such that* $S_x[0] = \{0\}$, *then for all* $y \in [0, 1]$, $(0, y) \sim (0, 1)$.

Proof. (a) Consider the sequence (x^k) defined as $x^0 = 1$ and $x^{k+1} = (x^k)_*$. Notice that x^k goes to zero as k increases and that $x^k > 0$ for all k when $0 \not S_x\, x$ for all $x \in (0, 1]$. We are going to show that for any given k, if $x \geqslant x^k$, then $(x, 0) \sim (1, 0)$.

When $k = 1$, $x \geqslant x^k$ implies that $x\ S_x\ 1$ and therefore, by Lemma 2 $(x, 0) \sim (1, 0)$.

Now imagine that, when $k = N$, for an arbitrary N, it is true that $x \geqslant x^N$ implies that $(x, 0) \sim (1, 0)$. Then, for $k = N + 1$ if $x^k \geqslant x \geqslant x^{k+1}$, by Lemma 2, $(x, 0) \sim (x^k, 0)$ and by (R.1) $(x, 0) \sim (1, 0)$. Therefore we can conclude that for an arbitrary k, $x \geqslant x^k$ implies that $(x, 0) \sim (1, 0)$. Since $(x^k) \to 0$, there is a k such that $x^k < x$ for all $x > 0$ and thus $(x, 0) \sim (1, 0)$ for all $x > 0$. Hence by (R.3), $(0, 0) \sim (1, 0)$.

(b) Proceed as in part (a) changing x for y.

Proof of Theorem 1. We must show first that: (a) For all $x \in [0, 1]$, $(0, 0) \sim (x, 0) \Leftrightarrow S_y[0] = \{0\}$. (b) For all $y \in [0, 1]$, $(0, 0) \sim (0, y) \Leftrightarrow S_x[0] = \{0\}$.

We only have to prove the necessity part of both (a) and (b) because Lemma 3 is the sufficiency part.

(a) Suppose that $0_y^* > 0$, where $0_y^* = \max S_y[0]$; by (S.5) on S_x, $1_* \neq 0$ and by ** $(1, 0) \succ (1_* - \eta, 0^*)$ for all $\eta \in (0, 1_*)$. But $(1_* - \eta, 0^*) \in T_R$, therefore by Lemma 1, $(1_* - \eta, 0^*) \sim ((1_* - \eta)^*, 0)$. Transitivity of $\succsim$ implies that $((1_* - \eta)^*, 0) \prec (1, 0)$ which is in contradiction with the initial assumption.

(b) Proceed as in part (a) changing x for y.

Thus, we have that for all $x \in [0, 1]$ and for all $y \in [0, 1]$, $(x, 0) \sim (0, 0) \sim (0, y)$, therefore by (R.1) $(x, 0) \sim (0, y)$. Q.E.D.

A priori, it would seem that, given any alternative (x', y'), the indifference classes would be thick on the set $S_x[x'] \times S_y[y']$, because one cannot discriminate in $S_x[x']$ and $S_y[y']$. Nevertheless, we show now that if a preference satisfies both (R.1), (R.2) and the ** property relative to the similarities S_x and S_y then the preference satisfies the monotonicity property (R.2) restricted to a subset of alternatives.

LEMMA 4. *Let* $(S_x, S_y, \succsim) \in \hat{C}$. *Given any* $(x, y) \in T = T_R \cup T_L$, *if* $x' > x$ *and* $y' > y$ *then* $(x', y') \succ (x, y)$; *i.e.*, $\succsim$ *satisfies the monotonicity property*, (R.2), *when restricted to* T.

Proof. Let $(x, y) \in T_R$; by Lemma 1, $(x, y) \sim (x^*, y^*)$. Let $(x', y') \in [0, 1]^2$ such that $x' > x$ and $y' > y$.

If $x' \leqslant x^*$, then by ** $(x', y') \succ (x^*, y_*)$, thus, by (R.1) $(x', y') \succ (x, y)$.

If $x' > x^*$, ** implies both $(x', y') \succ (x, y')$ and $(x, y') \succ (x^*, y_*)$. By (R.1), $(x', y') \succ (x^*, y_*)$; hence $(x', y') \succ (x, y)$.

When $(x, y) \in T_L$, we use part (b) of Lemma 1 and proceed as above.

The next result establishes the 'almost' uniqueness of a preference relation consistency with a given pair of similarities. It is the extension of Rubinstein's Proposition 2 to a general framework in which neither R.4 nor monotonicity of preferences are assumed.

THEOREM 2. *Let* S_x *and* S_y *be a pair of arbitrary similarities on the set* $[0, 1]$. *Then there are functions* $u: [0, 1] \rightarrow R^1$ *and* $v: [0, 1] \rightarrow R^1$ *such that*: (a) *The function* $v(y)u(x)$ *represents a preference on* $[0, 1]^2$ *satisfying* (R.1), (R.2), (R.3) *and* ** *relative to* S_x *and* S_y. (b) *If* $(S_x, S_y, \succsim) \in \hat{C}$, *then for all* (x_1, y_1) *and* (x_2, y_2) *satisfying* $v(y_1)u(x_1) > v(y_2)u(x_2)$, *there are* $x'_i\ S_x\ x_i$ *and* $y'_i\ S_y\ y_i$, $(i = 1, 2)$, *such that* $(x'_1, y'_1) \succ (x'_2, y'_2)$.

Proof. (a) By Rubinstein's Proposition 1 given any $\lambda > 1$ there are continuous and strictly increasing functions u and v which represent S_x and S_y, respectively; i.e. for all $x, x' \in [0, 1]$, $x\ S_x\ x' \Leftrightarrow u(x)/u(x') \in [1/\lambda, \lambda]$; and the same applies for v and S_y.

Let $\succsim$ be the preference relation represented by $v(y)u(x)$; then $\succsim$ is a continuous and complete preordering which satisfies the monotonicity property (R.2). We show now that $\succsim$ also satisfies ** relative to S_x and S_y. Assume that $x_1\ S_x\ x_2$, $x_2 \neq x_1$ if $x_1 = 0$, $y_1 \not S_y\ y_2$ and $y_1 > y_2$. Then $1/\lambda \leqslant u(x_1)/u(x_2) \leqslant \lambda$ and $v(y_1) > \lambda v(y_2)$. Therefore $u(x_1) \times v(y_1) > (1/\lambda)u(x_2)\lambda v(y_2) = u(x_2)v(y_2)$.

When $y_1\ S_y\ y_2$, $y_2 \neq y_1$ if $y_1 = 0$, $x_1 \not S_x\ x_2$ and $x_1 > x_2$, the same reasoning, changing $u(x_1)$ for $v(y_1)$ and $u(x_2)$ for $v(y_2)$, leads to the conclusion that $u(x_1)v(y_1) > u(x_2)v(y_2)$.

(b) Since v and u are strictly increasing, the inequality $v(y_1) \times u(x_1) > v(y_2)u(x_2)$ means that the inequalities $y_1 \leqslant y_2$ and $x_1 \leqslant x_2$ cannot occur at the same time. We must have either $y_1 > y_2$ or $x_1 > x_2$ or both. In the last case we cannot conclude that $(x_1, y_1) \succ (x_2, y_2)$ since we are not assuming that $\succsim$ is monotone and (x_2, y_2) could be outside T. We ought to show that the ranking of (x_1, y_1) *vis-à-vis* (x_2, y_2) may be deduced from the knowledge that it satisfies (R.1), (R.3) and $**$ with respect S_x and S_y. The proof must take into account that ε-difference similarities are also considered because $\succsim$ does not necessarily satisfy (R.4).

The cases compatible with $v(y_1)u(x_1) > v(y_2)u(x_2)$ are the following:

Case I	$x_1 > x_2$	and	$y_1 > y_2$
Case II	$x_1 \leqslant x_2$	and	$y_1 > y_2$
Case III	$x_1 > x_2$	and	$y_1 \leqslant y_2$

Case III is equivalent to Case II; hence we shall only consider Case I and Case II.

Case I. $x_1 > x_2$ and $y_1 > y_2$. If either $x_1 \; S_x \; x_2$ and $y_1 \; \not S_y \; y_2$ or $x_1 \; \not S_x \; x_2$ and $y_1 \; S_y \; y_2$ then, by $**$, $(x_1, y_1) \succ (x_2, y_2)$. Therefore the cases to be studied are the following.

Ia: $\quad x_1 \; S_x \; x_2$ and $y_1 S_y y_2$.

Ib: $\quad x_1 \; \not S_x \; x_2$ and $y_1 \; \not S_y \; y_2$.

Case Ia. $x_1 \; S_x \; x_2$ and $y_1 \; S_y \; y_2$. $y_1 > y_2$ means that $(y_1)^* > (y_2)_*$. Also $(y_1)^* \; \not S_y \; (y_2)_*$ because if $(y_2)_* \neq 0$ $(((y_2)_*)^* = y_2 < (y_1)^*$; if $(y_2)_* = 0$, by (S.5) $1 \geqslant (y_1)^* > 0^*$; thus, $(y_1)^* \; \not S_y \; y_2$ and we may conclude that $(x_1, (y_1)^*) \succ (x_2, (y_2)_*)$.

Case Ib. $x_1 \; \not S_x \; x_2$ and $y_1 \; \not S_y \; y_2$. (i) Suppose that $x_2 > 0$, then, by $**$, $(x_1, y_1) \succ (x_2, y_1)$ and $(x_2, y_1) \succ (x_2, y_2)$. By (R.1), $(x_1, y_1) \succ (x_2, y_2)$.

(ii) Suppose that $y_2 > 0$, then by $**$ $(x_1, y_1) \succ (x_1, y_2)$ and $(x_1, y_2) \succ (x_2, y_2)$. By (R.1) $(x_1, y_1) \succ (x_2, y_2)$.

(iii) Suppose that $x_2 = y_2 = 0$. In this case we cannot apply, as in the previous cases, property **. Here we can distinguish two subcases:

(iiia) $S_x[0] = S_y[0] = \{0\}$. Take any ε in $(0, (x_1)_*)$; by (i), $(x_1, y_1) \succ (\varepsilon, 0)$ and by Theorem 1, $(\varepsilon, 0) \sim (0, 0)$. Thus, $(x_1, y_1) \succ (0, 0)$.

(iiib) $S_x[0] \neq \{0\}$ or $S_y[0] = \{0\}$ or both. Let us consider the case $S_x[0] \neq \{0\}$. Take $\varepsilon < \min\{(x_1)_*, 0_x^*\}$, then $(x_1, y_1) \succ (\varepsilon, 0)$.

Case II. $x_1 \leqslant x_2$ and $y_1 > y_2$. We must consider the following substances.

IIa:	$x_1\ S_x\ x_2$	and	$y_1\ S_y\ y_2$
IIb:	$x_1\ S_x\ x_2$	and	$y_1\ \not S_y\ y_2$
IIc:	$x_1\ \not S_x\ x_2$	and	$y_1\ \not S_y\ y_2$
IId:	$x_1\ \not S_x\ x_2$	and	$y_1\ S_y\ y_2$

Case IId is not compatible with the conditions $x_1 \leqslant x_2$, $y_1 > y_2$ and $v(y_1)u(x_1) > v(y_2)u(x_2)$, hence we do not need to deal with it.

Case IIa: $x\ S_x\ x_2$ and $y_1\ S_y\ y_2$. As shown in the Case Ia, $(y_1)^*\ \not S_y\ (y_1)_*$ and $(y_1)^* > (y_2)_*$. Since $x_1 \leqslant x_2$, $(x_1)^*\ S_x\ x_2$. Therefore $((x_1)^*, (y_1)^*) \succ (x_2, (y_2)_*)$, as a direct application of **.

Case IIb: $x_1\ S_x\ x_1$ and $y_1\ \not S_y\ y_2$. By **, $(x_1, y_1) \succ (x_2, y_2)$; this is true for $x_2 \neq x_1 \geqslant 0$. When $x_1 = x_2 = 0$, $v(y_1)u(x_1) > v(y_2)u(x_2)$ implies that $S_x[0] \neq \{0\}$. Hence by ** $((0)^*, y_1) \succ (0, y_2)$.

Case IIc: $x_1\ \not S_x\ x_2$ and $y_1\ \not S_y\ y_2$. It must be the case that $x_1 < x_2$. Furthermore $(x_1)^* < x_2$, $x_1 < (x_2)_*$, $y_1 > (y_2)^*$ and $(y_1)_* > y_2$.

The proof for this case is a modified version of Rubinstein's proof for his Proposition 2. The modification is needed to cover the case in which $y_2 = 0$ or, more generally, $(y_2)_* = 0$.

Define the sequence (x_1^k, y_1^k) as follows: $x_1^0 = (x_1)^*$ and $y_1^0 = y_1$; then $x_1^k = (x_1^{k-1})^*$ and $y_1^k = (y^{k-1})_*$. Thus, the first element of the sequence, (x_1^0, y_1^0) has the property that

$$v(y_1^0)u(x_1^0) = \lambda v(y_1)u(x_1)$$

$$v(y_1^0)u(x_1^0) > \lambda v(y_2)u(x_2)$$

Let us denote by I the element of the sequence with the property that

$$(x_2)_* \leqslant x_1^I < x_2$$

This element must exist. Otherwise, by (S.5), we may have an index H such that

$$x_1^{H-1} < (x_2)_* < x_2 \leqslant x_1^H$$

and this would violate (S.4). The existence of only one such an I follows from the way the sequence has been defined (see Figure 1).

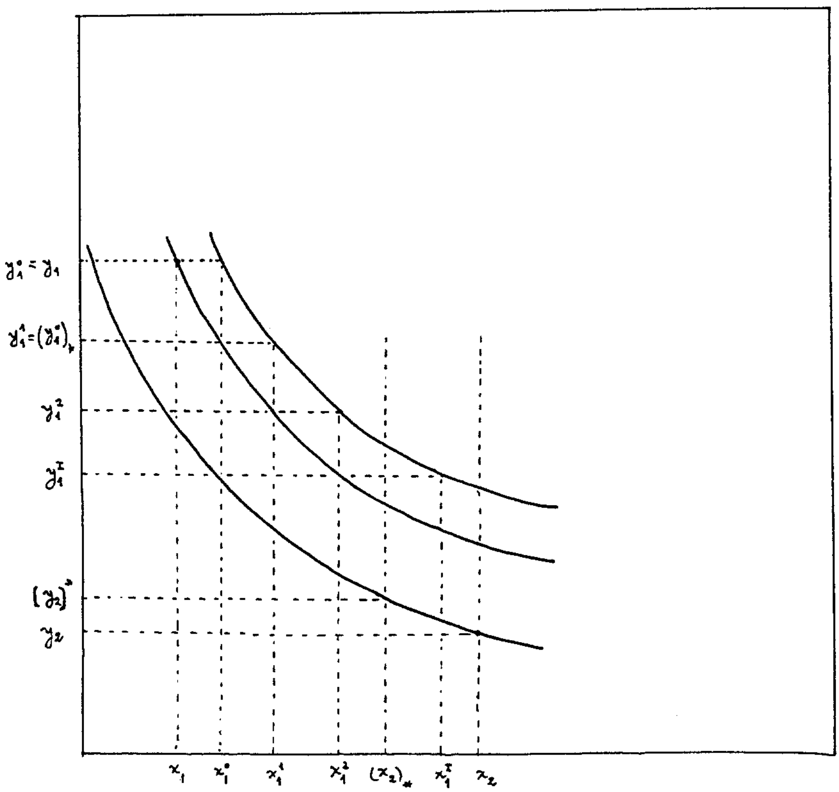

Fig. 1.

The functions $v(y)$ and $u(x)$ must satisfy, by Rubinstein's Proposition 1, $v(y_1^k) = v(y_1^0)/\lambda^k$ if $y_1^k \neq 0$ and $u(x_1^k) = \lambda^k u(x_1^0)$ if $x_1^k \neq 1$.

Let us denote by J the element of the sequence with the property that $(y_2)^* \geqslant y_1^J > y_2$. We say that $J = \infty$ when $y_2 = 0 = (y_2)^*$, because in that case no finite J will have the desired property.

Now we claim that $I < J$. If $J \leqslant I$ then the initial inequality, $v(y_1^0) \times u(x_1^0) > \lambda v(y_2)u(x_2)$ will not be satisfied; i.e. if $J \leqslant I$

$$(y_2)^* \geqslant y_1^J > y_2 \quad \text{and} \quad x^J < x_2$$

therefore $v(y_1^J) \leqslant \lambda v(y_2)$ and $u(x^J) < u(x_2)$.

Since $v(y_1^0)u(x_1^0) = v(y_1^J)u(x_1^J)$, $J \leqslant I$ would imply that $v(y^0)u(x^0) \leqslant \lambda v(y_2)u(x_2)$ which contradicts the above mentioned property.

$I < J$ means that (x_1^I, y_1^I) is such that:

$$(x_2)_* < x_1^I < x_2, \quad \text{i.e.} \quad x_1^I \; S_x \; x_2$$
$$y_1^I > (y_2)^*, \qquad \text{i.e.} \quad y_1^I \; \not{S}_y \; y_2$$

Therefore, by ** $(x_1^I, y_1^I) \succ (x_2, y_2)$.

For $0 \leqslant k < I$, $(x_1^k, y_1^k) \in T_R$; hence, by Lemma 1 and the transitivity of $\succsim$ we have that for all $k \leqslant I$

$$(x_1^0, y_1^0) \sim (x_1^k, y_1^k); \; (x_1^i, y_1^i) \sim (x_1^0, y_1^0)$$

and $x_1^0 \; S_x \; x_1$ allows us to conclude that

$$((x_1)^*, y_1) \succ (x_2, y_2)$$

as desired.

Remark 5. Had we followed Rubinstein's [2, Proposition 2] proof, we could have shown an element $(x_1^I, y_1^I) \sim (x_1, y_1)$ such that $x_1^I \; S_x \; x_2$ and $y_1^I > y_2$. We could now pick $(y_2)_*$ to be sure that $y_1 \; \not{S}_y \; (y_2)_*$. But when $y_2 = 0$, or, more generally, when $(y_2)_* = 0$, $y_1^I > y_2$ does not guarantee that $y_1^I \; \not{S}_y \; (y_2)_*$ and ** cannot be applied to conclude that $(x_1^I, y_1^I) \succ (x_2, (y_2)_*)$.

Remark 6. Part (a) of Theorem 2 shows the existence of a monotone, continuous and complete preorder $\succsim$ satisfying ** relative to S_x and S_y. We can use this preorder $\succsim$ to construct a nonmonotone preference $\succsim'$ satisfying ** relative to the similarities S_x and S_y. An example will be the following: consider the set $U = \{(x, y) \in [0, 1]^2 : (x, y) \succsim (1_*, 1)\}$; now let $\succsim'$ restricted to $[0, 1]^2 \backslash U$ be identical to $\succsim$ and assume that $\succsim'$ makes the points in U indifferent. Then it is clear that $(S_x, S_y, \succsim') \in \hat{C}$ and that $\succsim'$ is not monotone.

The next theorem establishes a relationship between a given preference represented by $u(\,.\,)v(\,.\,)$ and the type of similarities which are consistent with it, when neither $u(0) = 0$ nor $v(0) = 0$ are required.

THEOREM 3. *Assume that $\succsim$ is represented by the utility function $v(y)u(x)$, where v and u are non-negative, continuous and strictly increasing functions. If $(S_x, S_y, \succsim) \in \hat{C}$ then there is a $\lambda > 1$ such that*

$$x_1 \; S_x \; x_2 \Leftrightarrow \frac{1}{\lambda} \leqslant \frac{u(x_2)}{u(x_1)} \leqslant \lambda$$

$$y_1 \; S_y \; y_2 \Leftrightarrow \frac{1}{\lambda} \leqslant \frac{v(y_2)}{v(y_1)} \leqslant \lambda$$

Proof. As pointed out by Rubinstein, the sets of real numbers $S_x[X] = \{u(x_1)/u(x_2) : x_1 \; S_x \; x_2, \; x_1 \neq x_2\}$ and $S_p[Y] = \{v(y_1)/v(y_2) : y_1 \; S_y \; y_2, \; y_1 \neq y_2\}$ are bounded because for any x_1, x_2 with $x_1 \; S_x \; x_2$ $(y_1, y_2, y_1 \; S_y \; y_2)$ $u(x_1)/u(x_2) < v(\bar{y}_1)/v(\bar{y}_2)$ $(v(y_1)/v(y_2) < u(\bar{x}_1)/u(\bar{x}_2))$, for any pair $\bar{y}_1, \bar{y}_2$ such that $\bar{y}_2 \not S_y \bar{y}_1, \bar{y}_2 > \bar{y}_1 (\bar{x}_1, \bar{x}_2, \bar{x}_1 \not S_x \bar{x}_2, \bar{x}_1 > \bar{x}_2)$. Let λ_x and λ_y denote the suprema of $S_x[X]$ and $S_y[Y]$, respectively. We show now that the equality $\lambda_x = \lambda_y = \lambda$ is maintained even in the case where $v(0) \neq 0$ or $u(0) \neq 0$. If, say, $\lambda_x > \lambda_y$, then we can find $\tilde{x}_1, \tilde{x}_2$, with $\tilde{x}_1 \; S_x \; \tilde{x}_2$ and $\tilde{y}_1, \tilde{y}_2$ with $\tilde{y}_1 \not S_y \tilde{y}_2$, such that

$$\lambda_x > \frac{u(\tilde{x}_1)}{u(\tilde{x}_2)} > \frac{v(\tilde{y}_2)}{v(\tilde{y}_2)} > \lambda_y$$

because $v(1)/v(0) > \lambda_y$ even if $v(0) > 0$ (note that if $v(1)/v(0) = \lambda_y$ it would imply that, for any $\varepsilon < 0$ small enough, $\varepsilon\ S_y\ 1-\varepsilon$ which in turn, by continuity of S_y, implies that $0\ S_y\ 1$ and this violates (S.5)).

The rest of the proof literally follows that of Rubinstein: property ** implies $(\tilde{x}_2, \tilde{y}_2) \succ (\tilde{x}_1, \tilde{y}_1)$ but $v(\tilde{y}_1)u(\tilde{x}_1) > v(\tilde{y}_1)u(\tilde{x}_1)$. Thus, $\lambda_x = \lambda_y = \lambda$.

To show that if $1/\lambda \leq u(x_2)/u(x_1) \leq \lambda$ $(1/\lambda \leq v(y_1)/v(y_2) \leq \lambda)$ then $x_1\ S_x\ x_2$ $(y_1\ S_y\ y_2)$, let $x_2 > x_1$, $x_2\ \not S_x\ x_1$ and nevertheless $u(x_2)/u(x_1) < \lambda$. Then there exists a pair y_1, y_2 such that $y_1\ S_y\ y_2$ and $v(y_2)/v(y_1) > u(x_2)/u(x_1)$. Thus $v(y_2)u(x_1) > v(y_1)u(x_2)$, which contradicts property **. Hence if $1/\lambda < u(x_2)/u(x_1) < \lambda$ then $x_1\ S_x\ x_2$. By the continuity of S_x, this must also be true when $u(x_2)/u(x_1) = \lambda$.

ACKNOWLEDGEMENTS

We are most grateful to Ariel Rubinstein for his encouragement, helpful comments and suggestions. The authors have benefited also from the comments by Tatsuro Ichiishi. Financial support from Eusko Jaurlaritza – Basque Government is gratefully acknowledged. We are solely responsible for any remaining errors.

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Jose Maria Aizpurua
Departamento de Economía,
Universidad Pública de Navarra-Nafarroako Unibertsitate Publikoa,
Carretera de Sadar s/n,
31006 Pamplona,
Basque Country, Spain

and

Institut d'Anàlisi Econòmica,
Universitat Autònoma de Barcelona,
08193 Bellaterra,
Barcelona, Catalonia, Spain.

Jorge Nieto
Departamento de Análisis Económico,
Universidad del País Vasco-Euskal Herriko Unibertsitatea,
Avenida del Ejército 83,
48015 Bilbao,
Basque Country, Spain.

Jose Ramon Uriarte
Departamento de Economía,
Universidad Pública de Navarra-Nafarroako Unibertsitate Publikoa.