Public Choice **64**: 21-41, 1990.

Susceptibility to manipulation*

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Abstract. All positional voting procedures can be manipulated, so it is natural to question whether some of these systems are more susceptible to being manipulated than others. In this essay, this susceptibility factor is measured for strategic action involving small groups. It is shown that the system least susceptible to micro manipulations for n = 3 candidates is the Borda Count (BC). The optimal choice changes with n, but the analysis shows that the BC always fares fairly well. On the other hand, the plurality and anti-plurality vote as well as multiple voting systems, such as approval voting and cumulative voting, always fare quite poorly with respect to susceptibility. Finally, it is shown why it is possible to justify any voting method by choosing an appropriate measure of susceptibility and imposing the appropriate assumptions on the profiles of voters. This statement emphasizes the importance of the basic assumptions of neutrality used throughout this essay.

1. Introduction

Many, if not most formal group decisions are made with positional voting procedures. These are the methods where, for n candidates, n specified weights $\mathbf{W}^n = (w_1, \ldots w_n)$, $w_j \geq w_{j+1}$, $w_1 > w_n = 0$, are used to tally the ballots. In the tabulation w_j points are assigned to the jth ranked candidate on a ballot, $j = 1, \ldots, n$, and the group ranking for a candidate is determined by the total number of points she receives. For instance, $(1, 0, \ldots, 0)$ corresponds to the plurality vote,[1] $(1, \ldots, 1, 0)$ to the anti-plurality vote (which is equivalent to voting *against* one candidate), and $\mathbf{B}^n = (n-1, n-2, \ldots, 0)$ to the Borda Count (BC). Because these voting procedures are so widely used,[2] it is important to analyze their properties to determine when one should avoid certain systems because of their defects.

Already in the 1780s it was recognized that one of the positional voting systems, the BC, has a serious flaw: it can be manipulated. Namely, there are situations in which a voter can change the election outcome to a personally more favorable one by strategically misrepresenting his true preferences.[3] Indeed, this weakness of the BC remains as an argument against using it. (See, for example, Riker, 1982; Sawyer and MacRae, 1962.) We now know, however,

* This research was supported, in part, by NSF grants IRI-8415348, IRI-8803505, and a Fellowship from the Guggenheim Memorial Foundation. Also, I would like to thank P. Aranson for his several helpful comments.

that it is premature to disqualify the BC or any other election procedure just because it admits strategic voting. This is because the Gibbard (1973) – Satterthwaite (1975) Theorem asserts for $n \geq 3$ that all reasonable election procedures, including all positional voting methods, can be manipulated.

If "manipulability" is to play a role in the choice of a positional voting method, then it must be in terms of "degrees." Are some systems more easily manipulated than are others? To answer this question we must measure how susceptible a system is to being manipulated. Once a measure for manipulability is adopted, the analytic issue is to discover which positional voting methods minimize the likelihood of successful strategic actions. It turns out that the optimal system need not be one of the more commonly discussed methods, so the analysis must involve all possible positional voting methods. This is done here. Second, as I show in this essay (Theorem 3), the answer can change with the choice of a measure. Indeed, as I indicate, a corollary of Theorem 3 is that is possible to create a scenario with specially chosen profiles or a measure of manipulability to justify the use of *any* positional voting method. This assertion has many consequences. One consequence is that there is no universal solution to the manipulability problem; a given positional voting method can be the best selection with some electorates, but a poor one with others. A second conclusion is that because of the sensitivity of the answer to the choice of a measure, we must take care when we adopt a measure or when we make any restrictive assumptions about how the voters will act.

Two natural ways to measure manipulation involve whether there is a macro or a micro attempt to alter the relative election rankings of two particular candidates; will a large percentage or a small percentage of the voters try in a coordinated fashion to manipulate the outcome?[4] The choice of which one is appropriate depends on the nature of the electorate. For instance, there are noteworthy examples where one might expect macro manipulations of an election, such as when there is a single manipulating voter in a small group, or when a large interest group tries to orchestrate the members' strategic voting. On the other hand, quite often only individual voters or small coalitions try to coordinate attempts to manipulate the system. Examples include those organizations, such as professional societies, in which not many voters are likely either to attempt a strategic manipulation or to possess the necessary information to discover an appropriate strategy. Other examples include heterogeneous societies and other organizations without large, organized interest groups. My main emphasis is to understand which positional voting systems are the least susceptible to micro manipulation.

I must emphasize that considerations of whether there is a micro or a coordinated macro attempt to manipulate the system are critical. This point is underscored by comparing my results with the assertions in two interesting essays Chamberlin (1985) and Nitzan (1985). These two authors consider a similar strategic issue with assumptions closely related to mine. What differs here is the methodology[5] and the choice of measures. While these authors investigate a kind of coordinated macro manipulation, I concentrate on micro manipulations. It is instructive and surprising to learn how radically their conclusions differ from mine. For instance, one can extrapolate from their computer simulations that those systems relatively immune to micro manipulations appear to be quite prone to macro manipulations, and vice versa. There are interesting mathematical explanations for this difference in our conclusions, and I outline them in Section 1.2.

For any choice of a measure of manipulation, the subsequent analysis must involve two factors. The first is to find the positional voting method, $\mathbf{W}^n$, which offers a strategic voter the largest impact on changing the final outcome. After all, each $\mathbf{W}^n$ provides a different vote differential for the strategic voter to affect the final tally. With the techniques developed here, one can show for $n = 3$ that $\mathbf{B}^3 = (2, 1, 0)$, the BC, maximizes the expected strategic impact. (Consequently, one might expect, and it probably is true, that for carefully coordinated macro manipulations of a system, the BC fares quite poorly among the positional voting methods.) However, even if a voter attempts to be strategic, he need not be successful. So, the second factor involves "opportunity." How often will a voter (or a small coalition) be in a situation where the manipulative scheme is effective? Thus, an analysis of manipulating requires combining both the impact that a strategic voter has on the final outcome and an accounting of how often such a strategy succeeds.

This essay's conclusions are based on this approach: Start with the sincere way that the voters mark ballots. Next, count the number of profiles for which a small group of voters, by changing how they choose to mark their ballots, can convert the outcome into a personally more preferable one. The more profiles for which the voters successfully can manipulate the outcome in this manner, the more susceptible is the sytem. (Because the total number of profiles is fixed, this approach is equivalent to computing the proportion of possible profiles for which a micro manipulation is successful.) In a sense that I later make precise, the surprising conclusion is that for $n = 3$, the BC is the positional voting system that is the *least* susceptible to micro manipulations! This means that although it is easy to concoct examples to demonstrate certain strategic failings of the BC, if we admit all possible situations, the BC optimally avoids the consequences of micro manipulations.

The BC is not the optimal answer for $n > 3$. For $n = 4$, the unique method is $(2, 2, 0, 0)$ (or, equivalently, $(1, 1, 0, 0)$ – "vote for your two top ranked candidates"), and for $n = 5$ it is $(2, 2, 1, 0, 0)$. It may appear that a pattern is beginning to develop in which the answer for $n = 6$ should be $(2, 2, 2, 0, 0, 0)$. There is a pattern, but it is not the obvious one because for $n = 6$ the optimal choice is, essentially, $(6, 6, 5, 1, 0, 0)$. (The precise answer requires two of the w_j's to be irrational numbers. This result underscores my earlier comment that a careful examination of manipulability must involve all possible $\mathbf{W}^n$'s.) For all other values of n, the zeros of a set of algebraic equations coming from the gradient of the expression given in Eqs. 3.17 determines the optimal choice of $\mathbf{W}^n$. It appears from these equations that when n has a sufficiently large value, the optimal $\mathbf{W}^n$ must have the differences between the successive w_j's approach a common value. This is, of course, a definition for the BC. Thus, for n = 3 and for larger values of n, it appears that the BC optimally avoids the effects of micro manipulation of a pair of candidates. It also is true that even when the BC is not the best choice, it is not far from being so. I show this for n = 4, 5, by comparing the ratios of the susceptibility measures for the BC versus the optimal choice. This approach extends for all values of n. Finally, my results prove that for $n \geq 3$, (1, 0, ..., 0) (plurality voting) and (1, 1, ..., 1, 0) (anti-plurality voting) always share the dubious honor of being the most susceptible to binary manipulation.

In addition to positional voting, I analyze multiple voting systems. A multiple system (Saari and Van Newenhizen, 1988a) is a voting system equivalent to allowing a voter to first rank the candidates on his ballot and then select how the ballot is to be tallied from among several specified positional voting methods. Some of the better known multiple systems include cumulative voting, approval voting, and any positional voting procedure that tolerates "truncated" ballots. As one might suspect, the conclusion is that multiple systems are far more susceptible to manipulation than are any of the defining individual systems. The mathematical approach developed here, moreover, extends to run-off elections, to other elimination and social choice functions, as well as to certain other allocation and decision problems that do not admit standard incentive structures.[6]

1.1. *The properties of a measure*

A measure for manipulability should reflect the goal of encouraging a sincere vote. "Telling the truth" must play a central role; we do not want a system that penalizes honesty. But, what system satisfies these conditions? To understand the motivation for my assumptions, I briefly review some of the incentive literature.

The Gibbard-Satterthwaite theorem informs us that for any choice of a voting procedure, when $n \geq 3$, there always are situations where someone can replace a sincere vote with a strategic one to achieve a personally better outcome. In game theoretic terminology, this assertion means that "telling the truth" is not a dominant strategy. Because it is impossible to attain our goal of "honesty" for all possible profiles, the next step is to restrict attention to certain "natural" ones. In incentive theory, one approach is to seek those mechanisms where a person cannot do better than telling the truth when everyone else does so. That is, with such mechanisms, truth-telling is a Nash equilibrium. This approach does not mean that such a system cannot be manipulated, that there are not strategies equally as good as being sincere, or that one cannot try to manipulate the system. It only means that if everyone else is truthful, a voter cannot successfully manipulate the system to his advantage. Of course, carried to the extreme, one might question the value of truth-telling. Here a remarkable result called the *revelation principle* is invoked. This result, which Gibbard (1973) first recognized, and then several other authors extended to many other settings, essentially asserts that if an appropriately defined implementation can be done at all, then it can be done truthfully. See Meyerson (1988) for an excellent introduction.

Positional voting methods form an important class of mechanisms for which the incentive-manipulation problem is not resolved even with the Nash approach. This means that for any choice of a positional voting system, there are situations in which a manipulating voter can alter the outcome to a personally more favorable one even though all voters are sincere. So, the next step is to find out which $\mathbf{W}^n$'s minimize the *likelihood* that a voter or a small coalition can successfully manipulate the system if everyone else is sincere. This is the basic theme of this essay.

An important factor in studying manipulation is to determine: "who knows what" and "who is saying what to whom." For instance, if I know that there is a close contest between two candidates, then I may try to manipulate the outcome to favor my preferred candidate. But, without any such added information, my unguided efforts could be counterproductive. If this kind of extra information about the electorate is available, it would be useful to an architect of an "one time only" voting system in the selection of a method to encourage sincere voting. To illustrate, suppose the architect knows that of the candidates $\{c_1, c_2, c_3\}$, c_1 and c_2 will closely contest the top position. Suppose the architect also knows either that nobody ranks c_3 in top position, or that all voters with c_3 as top-ranked will vote sincerely. Here, the *plurality vote*, (1, 0, 0), encourages sincere voting, while the anti-plurality vote, (1, 1, 0), is the worst choice. Conversely, if the architect knows that nobody ranks c_3 in last place, then the answer is just the opposite: the plurality system is the worst system, while the anti-plurality system encourages a sincere vote. This theme extends. Theorem 3 of this essay asserts that for any choice of a positional voting system, $\mathbf{W}^n$, there are distributions of voter's preferences for which $\mathbf{W}^n$ minimizes the susceptibility to manipulation. In other words, *with the appropriate assumptions, with a correctly constructed scenario, any system can be justified as being strategically the best.* As a corollary, one must view with suspicion those assertions about the manipulability of a system that rely only on a finite number of examples or on restrictive assumptions. The answer may change when we admit more general situations.

While an architect of a system intended for a single usage can exploit information about the characteristics of the electorate, such knowledge need not exist when a voting system is being selected for an institution. Here, it is understood that the system is to be used for all votes on all issues independent of membership changes. As we cannot judge the future, we do not know which alternatives or candidates will be the target of an manipulation attempt; we do not know who is going to vote strategically; and we do not know how the preferences of the voters are distributed. This means that we require a neutral approach to select $\mathbf{W}^n$. Therefore, my basic assumptions are: for any given set of three or more candidates, it is equally likely for any pair of them to be the target of a manipulation attempt; any profile or distribution of voters' types is equally likely; and it is equally likely that a strategic voter or small coalition of strategic voters has any particular preference profile. It is in this sense that my results hold.

As stated, my emphasis is to discover which positional voting methods minimize the likelihood that the relative ranking of a pair of candidates is successfully (micro) manipulated. There are many other questions that one could raise, such as: which method minimizes the likelihood that the relative ranking of a pair, or a triplet, or any other subset can be manipulated? With each new question, the choice of an optimal system can change. However, each of these issues as well as the macro manipulation concerns can be analyzed by using the analytical mathematical techniques developed here. Based on these techniques and the symmetry derived from the neutrality assumptions, I offer the conjecture that if one is concerned about the micro manipulation of any number of the n candidates, then the Borda Count is the method that either minimizes, or comes close to minimizing, the likelihood of a successful manipulation.

1.2. *Why the Borda method?*

Why should the BC be either the optimal choice, or close to it? Intuitively, it is because the BC is the voting system that symmetrically splits the difference between the worst choices of the plurality and the anti-plurality vote. The mathematically more subtle explanation is related to the reasons that the BC is the unique positional voting method that significantly minimizes both the number and the kinds of (sincere) voting paradoxes that ever could occur. (See Saari 1989a, b). Essentially, the traits of the BC grow out of the symmetry property that there is a fixed value for the successive differences between the w_j's in $\mathbf{B}^n$. It is this symmetry that minimizes the number and kinds of election paradoxes; it is this symmetry that pushes the BC to the forefront in terms of its minimal susceptibility to micro manipulation; but it also is this symmetry that forces the BC to be vulnerable to carefully coordinated macro manipulations of the system.

The success of the BC rests on the geometry of the set of all profiles identified with an election outcome. Call this set of profiles a ranking set. Each $\mathbf{W}^n$ determines the geometry of the ranking sets, and the fixed differences in the weights of $\mathbf{B}^n$ impose a symmetry on the geometry of each of its ranking sets. In a micro manipulation, the susceptible profiles must be sufficiently close to the appropriate boundaries of a ranking set. The reason these profiles must be close to the boundary is because a small strategic change in the profile must cross the boundary to change the election ranking. Therefore, the "opportunity factor" which constitutes a major part of the measure of susceptibility must be related to the higher dimensional surface area of the boundary of this set. As it is commonly known, the more symmetrical a set (with a fixed volume), the smaller the surface area. (As an illustration, recall that of all rectangles with a fixed area, the square has the minimum perimeter.) Because of the symmetry of the BC ranking sets, the surface area of the boundary of the ranking sets is comparatively small. Indeed, this boundary is sufficiently small to compensate for the larger vote differential that $\mathbf{B}^n$ provides each strategic voter. It is for this reason that the BC fares well with respect to micro manipulations.

In a coordinated macro manipulation of an election, one can carefully orchestrate the strategic choices of *all* of the voters. In such settings the most important factor becomes the differential that each $\mathbf{W}^n$ provides each of the large group of strategic voters. Again, it is the symmetry properties of $\mathbf{B}^n$ that make this value larger for the BC than for even the plurality and anti-plurality vote. Consequently, one should not expect the BC to fare as well as the plurality method for highly coordinated macro manipulations of elections.

2. Framework and main results

For the n candidates, $C^n = \{c_1, \ldots, c_n\}$, there are n! different linear rankings. Each ranking defines a *voter type*. Let p_j denote the fraction of all voters that are of the jth type, $j = 1, \ldots, n!$, and let $\mathbf{p} = (p_1, \ldots, p_{n!})$. As $p_j \geq 0$, and $\Sigma_j p_j = 1$, each profile $\mathbf{p}$ is a point in the unit simplex, Si(n!), of the positive orthant of $R^{n!}$. In what follows, the profile $\mathbf{p}$ always represents the voters' truthful preferences while $\mathbf{p}'$ includes attempts at manipulation. For a specified voting vector, $\mathbf{W}^n$, let $F(\mathbf{p}; \mathbf{W}^n)$ be the function that gives the tally of the ballot. More precisely, compute the vote tally for c_j as determined by $\mathbf{p}$ and $\mathbf{W}^n$. Let this value be the jth component of a vector in R^n. In this manner, $F(\mathbf{p}; \mathbf{W}^n)$ defines a vector in $R_+{}^n$;

$$F(-; \mathbf{p}) : Si(n!) \to R_+{}^n, \quad 2.1$$

where $R_+{}^n$ is the positive orthant of R^n.

The values of the coordinates of $F(\mathbf{p}; \mathbf{W}^n)$ give the election ranking of the candidates. So, if the vector $F(\mathbf{p}; \mathbf{W}^n)$ is closer to the jth axis than the kth (i.e., c_j received a larger vote than c_k), then the relative election ranking is $c_j > c_k$. If the coordinates of R^n are given by $(x_1, \ldots, x_n)$, then the plane $x_j = x_k$ corresponds to the set of all election outcomes where there is a tie vote between c_j and c_k. Call this plane the *indifference surface between c_j and c_k.*

The truthful profile is $\mathbf{p}$, so the sincere outcome is $F(\mathbf{p}; \mathbf{W}^n)$. If a voter (or a small group) votes strategically, then the actual election tally is $F(\mathbf{p}'; \mathbf{W}^n)$ where $\mathbf{p}' = \mathbf{p} + \mathbf{v}$ and $\mathbf{v}$ represent the strategic change in voting. The manipulation $\mathbf{v}$ affects the election outcome if and only if $F(\mathbf{p}; \mathbf{W}^n)$ and $F(\mathbf{p} + \mathbf{v}; \mathbf{W}^n)$ are on different sides of an indifference surface. For such a change to be "successful," the new outcome must be personally more favorable for the manipulators.

To start the analysis, I formalize my assumptions. Assume that

1. *any $p \in Si(n!)$ is equally likely, and that*
2. *each pair of alternatives is equally likely to be the target of an attempted manipulation.*

To measure the "success" of any attempted manipulation, I follow the lead of the Nash equilibrium by analyzing the situation in which a strategic voter, or a small group of strategic voters, tries to reverse the relative ranking of a particular pair of candidates, while all other voters vote sincerely. Without loss of generality, assume *that the strategic voter(s) attempts to influence the relative election ranking of c_1 and c_2 in the direction $c_1 > c_2$.* These assumptions define $\mathbf{v}$.

To motivate the definition of $\mathbf{v}$, let $n = 3$ and let the voter types be labelled as

Type	Ranking	Type	Ranking
1	$c_1 > c_2 > c_3$	4	$c_3 > c_2 > c_1$
2	$c_1 > c_3 > c_2$	5	$c_2 > c_3 > c_1$
3	$c_3 > c_1 > c_2$	6	$c_2 > c_1 > c_3$

By assumption, a strategic voter is of type 1, 2, or 3. The most successful way that such a voter can manipulate the system is to pretend to be a type 2 voter. Any other choice either is not strategically maximal or is counterproductive. When a type-1 voter assumes the characteristics of a type-2 voter, the profile changes. For instance, with twenty voters where $\mathbf{p} = 20^{-1}(6, 3, 2, 2, 3, 4)$, a single "type-1" strategic voter changes the profile to $\mathbf{p}' = 20^{-1}(5, 4, 2, 2, 3, 4)$, so the vector $\mathbf{v}$ characterizing the attempted manipulation is $20^{-1}(-1, 1, 0, 0, 0, 0)$. Likewise, a strategic type-3 voter defines the vector $\mathbf{v} = \mathbf{m}^{-1}(0, 1, -1, 0, 0, 0)$, where $\mathbf{m}$ is the total number of voters. The theme of neutrality leads to the assumption that

3. *it is equally likely for a manipulating voter to be of any strategic type. Such a voter assumes a strategy (voter type) to maximize the effect of the manipulation. If there are several maximal strategies, the voter selects the one most consistent with his actual type.* (Here we understand "consistency" to mean that a maximal number of the relative rankings of the pairs is preserved.)

The second part of Assumption 3 has meaning only for $n \geq 4$.

Example. If a manipulating voter is of type $c_1 > c_2 > c_3 > c_4$, then the two maximal strategies are to assume type $c_1 > c_3 > c_4 > c_2$ or $c_1 > c_4 > c_3 > c_2$. Of these two strategies, the first is more nearly consistent with the voter's true type.

According to the third assumption, if $n = 3$, then it is equally likely for the strategic voter to be of type 1 or 2. Consequently, the expected manipulation is a scalar multiple of $(-1/2, 1, -1/2, 0, 0, 0)$. In general, $\mathbf{v}$, the *expected manipulation vector* (**EMV**), is determined by averaging the effects of strategic changes over all strategic types. The scalar multiple of the **EMV** depends upon the small fraction of voters that are strategic. For instance, with $\mathbf{m}$ voters and q manipulating voters, the multiple is $q/\mathbf{m}$. Everything that follows holds should the magnitude of the **EMV**, $|\mathbf{v}|$, be sufficiently small.[7] Consequently, these results hold whenever the manipulating voters form a small fraction of the total electorate.

Definition. Let $\mathbf{m} > 2$, and an **EMV** $\mathbf{v}$ be given. Let $\mu(\mathbf{W}^n; \mathbf{m})$, the $\mathbf{m}$ *voter measure of* binary *susceptibility* of $\mathbf{W}^n$, be the number of $\mathbf{p}$'s in Si(n!) with a common denominator $\mathbf{m}$, so that $F(\mathbf{p} + \mathbf{v}; \mathbf{W}^n)$ has the relative ranking $c_1 > c_2$ while $F(\mathbf{p}; \mathbf{W}^n)$ has the relative ranking $c_2 \geq c_1$.

This definition considers all possible profiles with $\mathbf{m}$ voters (the common denominator of the $\mathbf{p}$'s), while the magnitude of the **EMV**, $\mathbf{v}$, determines the fraction of strategic voters. The measure determines the number of profiles for which a micro manipulation is successful with $\mathbf{W}^n$, where smaller values mean that the voting method is more immune to the strategic action. Again, because the number of profiles is fixed, a smaller value for $\mu(\mathbf{W}^n; \mathbf{m})$ also means that a smaller percentage of all profiles can be successfully manipulated. The idea is to discover which $\mathbf{W}^n$'s minimize the value of this $\mathbf{m}$-voter measure of susceptibility.

Unfortunately, each value of **m** admits an uncountable number of $\mathbf{W}^n$'s to minimize $\mu(\mathbf{W}^n; \mathbf{m})$. This is because for each **m** and $\mathbf{W}^n$, there is a continuum of positional voting methods for which an election outcome is indistinguishable from that of $\mathbf{W}^n$. For instance, an election tallied with (1, 0, 0) or one tallied with $(1, \beta, 0)$, $0 \leq \beta < \mathbf{m}^{-1}$, always agree because the difference in the weights is not reflected by such a small electorate. This means that the answer to the susceptibility problem changes with the value of **m**. So, to remove this annoying dependency on **m**, I retain only those $\mathbf{W}^n$'s that are optimal for all sufficiently large values of **m**. This means that there are values of $\mathbf{W}^n$'s that are "optimal" for smaller values of **m**, but they are dropped from consideration once **m** becomes larger.

Definition. A voting system $\mathbf{W}^{n\prime}$ is *susceptibility efficient* if

$$\mu(\mathbf{W}^{n\prime}; \mathbf{m}) \leq \mu(\mathbf{W}^n; \mathbf{m}) \qquad 2.2.$$

for all choices of $\mathbf{W}^n$ and for all sufficiently large values of **m**. The voting system $\mathbf{W}^{n\prime}$ is *susceptibility inefficient* if Inequality 2.2 is reversed.

A straightforward argument proves that $\mathbf{W}^n$ is susceptibility efficient if $\mathbf{W}^n$ minimizes the value of the next measure, the measure of (binary) susceptibility of $\mathbf{W}^n$.

Definition. Let **EMV v** have a sufficiently small value. Define $\mu(\mathbf{W}^n)$, the *measure of* (binary) *susceptibility* of $\mathbf{W}^n$, to be the volume of $\{\mathbf{p} \in Si(n!) \mid F(\mathbf{p} + \mathbf{v}; \mathbf{W}^n)$ has the relative ranking $c_1 > c_2$ while $F(\mathbf{p}; \mathbf{W}^n)$ has the relative ranking $c_2 \geq c_1\}$.

Theorem 1. a. For all $n \geq 3$, the plurality vote, $(1, 0, \ldots, 0)$, and the antiplurality vote, $(1, 1, \ldots, 1, 0)$, are susceptibility inefficient, and $\mu((1, 0, \ldots, 0)) = \mu((1, 1, \ldots, 1, 0))$.

b. For $n = 3$, the unique susceptibility efficient system (USE) is the BC. For $n = 4, 5$, the USE are, respectively, (2, 2, 0, 0) and (2, 2, 1, 0, 0).

c. The following ratios compare choices of $\mathbf{W}^n$ with the USE:

$$\begin{array}{ll} \mu((1,0,0))/\mu(\mathbf{B}^3) = 1.027 & \mu((1,0,0,0))/\mu(2,2,0,0) = 4.6064 \\ \mu((1,0,0,0,0))/\mu((2,2,1,0,0)) = 2254.003 & \mu((2,1,0,0))/\mu((2,2,0,0)) = 3.358 \\ \mu(\mathbf{B}^5)/\mu((2,2,1,0,0)) = 13.914 & \mu(\mathbf{B}^4)/\mu((2,2,0,0)) = 2.014 \end{array} \qquad 2.3$$

Theorem 1 extends to all $n \geq 3$ by use of Eq. 3.17; this equation gives the measure of susceptibility for all choices of $\mathbf{W}^n$.

The ratios in Eq. 2.3 compare the relative vulnerability of the different systems. I find it surprising that the ratios are so close to unity for $n = 3$. (Because this ratio compares the worst situation with the USE, this value bounds all other ratios for $n = 3$.) On the other hand, these ratios prove for $n \geq 4$ that the plurality and the anti-plurality systems are very susceptible to micro manipulations. Notice that while the BC is not the USE for $n = 4, 5$, the ratio is relatively small. The same conclusion holds for $n \geq 4$. This result indicates that the BC might be adopted over the USE should it satisfy other criteria; for example, see Saari (1989a, b).

Theorem 1 concerns the social welfare ranking of some two alternatives, but they need not be the two top ranked ones. In many elections the purpose is to find a "winner," the top ranked candidate. So it is interesting to find out whether the conclusions of Theorem 1 change should we restrict attention only to those profiles for which c_1 and c_2 are contesting for the top position. There are certain symmetries on the space of profiles that permit the answer to remain the same for all values of n.

Theorem 2. The systems that are least susceptible to a binary manipulation of the two top ranked candidates are $\mathbf{B}^3$, (2, 2, 0, 0), and (2, 2, 1, 0, 0).

The conclusions of Theorems 1 and 2 rely on the neutrality assumptions. The next issue is to discern what happens if the neutrality assumption on who is a strategic voter does not hold. To be more specific, consider $n = 3$ and suppose with probability c a type-one voter is strategic, and with probability $(1-c)$ that a type-three voter is strategic. The resulting **EMV**, $\mathbf{v}_c$, is a multiple of $(-c, 1, -(1-c), 0, 0, 0)$, rather than $(-1/2, 1, -1/2, 0, 0, 0)$. The choice of an optimal voting method requires finding $\mathbf{W}^n$ to minimize the volume of $\{\mathbf{p} \in Si(n!) | F(\mathbf{p} + \mathbf{v}_c; \mathbf{W}^n)$ has the relative ranking $c_1 > c_2$ while $F(\mathbf{p}; \mathbf{W}^n)$ has the relative ranking $c_2 \geq c_1\}$. The surprising conclusion is that the answer changes with c, the *electorate's manipulation characteristic.* Part of the surprise is that this conclusion puts in doubt those standard arguments about the strategic properties of certain methods that rely on specialized examples or restrictive assumptions on strategic action. This conclusion, moreover, highlights the importance of the neutrality assumptions.

Theorem 3. Let $\mathbf{W}^3$ be given. There is a value for c, the electorate's manipulation characteristic, so that $\mathbf{W}^3$ is the USE.

This assertion holds for all values of $n \geq 3$ as well as for all multiple voting systems such as approval or cumulative voting.

Definition. A *simple* voting system for n candidates is one in which all ballots are tallied with a specified $\mathbf{W}^n$. A *multiple voting system* for n candidates is one in which (i) there is a specified set of at least two simple voting methods $\{\mathbf{W}^n_j\}$, where the difference between any two of them is not a scalar multiple of (1, . . . , 1), and (ii) each voter can select any one of the voting methods to tally his ballot.

Example. Cumulative voting, as used for half century in Illinois (Sawyer and MacRae, 1962) is defined by the set ⟨(3, 0, . . . , 0), (2, 1, 0, . . . , 0), (3/2, 3/2, 0, . . . , 0), (1, 1, 1, 0, . . . , 0)}. *Approval voting* is defined by {(1, 0, . . . , 0), (1, 1, 0, . . . , 0), . . . , (1, 1, . . . , 1, 0)}. The simple system, given by $\mathbf{W}^n$, defines a multiple system, called the *truncated ballot*, if it includes provisions to tally those ballots in which the required number of candidates are not ranked. A multiple system that has an uncountable number of ballots is one in which each voter can split ten points in any desired manner among the candidates.

The next theorem asserts that multiple systems are more susceptible to manipulation than are any of the defining simple systems. This conclusion is reasonable; after all, each of the constituent simple systems offers voters certain strategies and opportunities to manipulate the outcome. We should suspect that a multiple system offers the union of these opportunities and strategies to the strategic voter, so the added opportunities should make a multiple system more susceptible to manipulation. The actual argument is more complicated, because even more strategies are admitted, but this intuition serves us well.

Theorem 4. Let $n \geq 3$. If $\mathbf{W}^n$ is one of the simple voting methods in a specified multiple voting system, then the multiple system is more susceptible to binary manipulation than is the simple system $\mathbf{W}^n$.

Corollary. For $n \geq 3$, both cumulative and approval voting are more susceptible to binary manipulation than is any simple system, including the susceptibility inefficient system of plurality voting.

Niemi (1984) contends that if one relaxes certain of the basic assumptions supporting approval voting, then this method begs to be manipulated. Theorem 4 and its corollary rely on different arguments and a different approach, but they support Niemi's assertions. Theorem 3 provides a partial way to understand the conflict in assertions between the various positive assertions supporting the strategic action associated with approval voting and the conclusion of the corollary just stated. Much of the theory of approval voting rests on assumptions about what is an "admissible" strategic action. If one adopts these assumptions, then approval voting appears to fare quite well. This is to be expected because, as Theorem 3 asserts, any system can be justified, or be judged to be strategically optimal, just by imposing the appropriate restrictive assumptions. But when a strategic voter tries to manipulate the election, he may not wish to restrict the strategic options only to those satisfying certain theoretical assumptions. If there are other strategic actions that achieve his desired objectives, he may wish to use the successful options. Thus, a more realistic approach is to relax these restrictive restrictions to admit all possible strategic actions admitted by the neutrality assumptions. Once this is done, the conclusions differ significantly.

The approach and argument supporting Theorem 4 only admit those strategic actions that require a misrepresentation of truthful preferences. Consequently, this theorem does not admit any of those manipulative actions for which a voter remains true to his preferences, but where he uses the choice a tallying method as a strategic variable.[8] Thus a complete analysis would require including this second kind of strategic action. By introducing this second kind of strategy, however, it becomes clear that multiple systems are even more manipulable than suggested by the above results. Thus it is not necessary to complete this analysis here because Theorem 4 already makes the point.

While the emphasis in this essay is on manipulating the relative ranking of a pair of candidates, there are many other interesting kinds of manipulations. For instance, an ambitious, strategic voter may wish to manipulate the relative rankings of more than just two of the n candidates. The analysis of this problem remains open, but the symmetry properties of the BC lead me to suspect that the optimal choice either is the BC or it is a system that is very close to the BC. For instance, notice that the USE for $n = 3, 4, 5$ definitely reflects the fact I am analyzing only binary manipulations. It is reasonable to suspect that when one analyzes the effects of a manipulation of larger numbers of candidates, the optimal $\mathbf{W}^n$ will reflect higher order symmetries. This identifies the BC.

3. Proofs

Part of the proof relies on an interesting symmetry property based on reversing mappings. Let R_r be an involution map that reverses a ranking. For instance, $R_r(c_2 > c_1 > c_3) = c_3 > c_1 > c_2$. As such, R_r reverses the ranking of a profile. For example, if $\mathbf{p}$ consists of four type-1 voters and one type-2 voter, then $R_r(\mathbf{p})$ consists of four type-4 voters and one type-5 voter. It is obvious that $F(R_r(\mathbf{p}); \mathbf{W}^n) = R_r(F(\mathbf{p}; \mathbf{W}^n))$.

As it is well known, the election rankings associated with $\mathbf{p}$ are the same if we tally the ballots with $\mathbf{W}^n$ or with $a\mathbf{W}^n + b\mathbf{E}_n$ for $a > 0$ and $\mathbf{E}_n = (1, \ldots, 1)$. We use this fact to define a "reversing mapping," R_w, for voting methods. Namely, $R_w(\mathbf{W}^n)$ is the class of voting methods given by $-a\mathbf{W}^{n\prime} + b\mathbf{E}_n$, where $\mathbf{W}^{n\prime} = (w_n, w_{n-1}, \ldots, w_1)$. For example, $R_w((1, 0, 0)) = -\{(1, 0, 0)\}' +$

$(1,1,1) = -(0,0,1) + (1,1,1) = (1,1,0)$. Thus, the involution of the plurality vote is the anti-plurality vote. These involution operators are related in this manner:

Proposition. For $n \geq 3$, $R_r(F(\mathbf{p}; \mathbf{W}^n)) = F(R_r(\mathbf{p}); R_w(\mathbf{W}^n))$.

Proof. The ranking $R_r(F(\mathbf{p}; \mathbf{W}^n))$ is given by the ranking obtained from $b\mathbf{E}_n - F(\mathbf{p}; \mathbf{W}^n) = b\mathbf{E}^n + F(\mathbf{p}; -\mathbf{W}^n) = F(R_r(\mathbf{p}); R_w(\mathbf{W}^n))$.

This proposition proves there is a relationship between the strategic properties of $\mathbf{W}^n$ and $R_w(\mathbf{W}^n)$. It shows that whatever happens for profile $\mathbf{p}$ with $\mathbf{W}^n$ is mimicked by the profile $R_r(\mathbf{p})$ with $R_w(\mathbf{W}^n)$. Therefore, it is straightforward to show that both systems embrace the same kinds of paradoxes and strategic action. This is why the susceptibility measure for both the plurality and anti-plurality votes agree. (As a different proof, the conclusion follows from the symmetry properties of the indices in the susceptibility measure given by Eq. 3.17.)

Corollary. For $n \geq 3$, $\mu(\mathbf{W}^n) = \mu(R_w(\mathbf{W}^n))$.

The number of subscripts needed for the proofs of the theorems grows rapidly with the value of n, and there is a danger that this proliferation of notation will obscure the basic ideas. Therefore, I first prove the theorems for $n = 3$.

Proof of Theorem 1; $n = 3$. We use $\mathbf{W}^3$ to tally an election for the candidates $\{c_1, c_2, c_3\}$. With the notation of Section 2 and for profile $\mathbf{p}$, the tally of c_1 is $p_1w_1 + p_2w_1 + p_3w_2 + p_4w_3 + p_5w_3 + p_6w_2$, while that for c_2 is $p_1w_2 + p_2w_3 + p_3w_3 + p_4w_2 + p_5w_1 + p_6w_1$. Thus, $H_3 = \{\mathbf{p} \in Si(3!) |$ the election ranking has c_1 and c_2 tied$\}$ is given by

$$\langle \mathbf{E}_6, \mathbf{p} \rangle = 1 \qquad 3.1$$
$$\langle \mathbf{N}, \mathbf{p} \rangle = 0, \qquad 3.2$$

where $\mathbf{N} = (w_1-w_2, w_1-w_3, w_2-w_3, w_3-w_2, w_3-w_1, w_2-w_1)$ and $\langle -,- \rangle$ is the standard Euclidean scalar product.

Replace $\mathbf{W}^3$ with the equivalent representation $(1, u, -1)$, where $|u| \leq 1$. With this notation, the BC, the plurality vote, and the anti-plurality vote are given, respectively, by the u values of 0, -1, and 1. Also

$$\mathbf{N} = (1-u, 2, u+1, -(1+u), -2, -(1-u)). \qquad 3.3$$

The goal is to find the volume of the $\mathbf{p}$'s that are close enough to H_3 so that $\mathbf{p} + \mathbf{v}$ crosses H_3. (Recall that $\mathbf{v}$ is a small positive multiple of $(-1/2, 1, -1/2,$

0, 0, 0). Because I only compute the ratios of measures, I suppress this and all other common multiples.) Thus, we compute $\mu(\mathbf{W}^n)$ by using the higher dimensional surface area of H_3, $S(H_3)$, and the length of the component of $\mathbf{v}$ orthogonal to H_3. So, for any u, the measure is given by

$$\mu(\mathbf{W}^3) = \mu(u) = b \langle \mathbf{N}/|\mathbf{N}|, \mathbf{v}\rangle S(H_3) \qquad 3.4a$$

where b is a common multiple and $|\mathbf{N}|$ is the length of $\mathbf{N}$. It is an elementary computation to show that

$$\langle \mathbf{N}/|\mathbf{N}|, \mathbf{v}\rangle = 3/[2\{3+u^2\}^{1/2}]. \qquad 3.4b$$

Notice that H_3 is a four-dimensional surface in R^6. To compute $S(H_3)$, I use two changes of variables to reduce the problem to an integration problem over a region in R^4. We obtain the first change by solving Eq. 3.1 for p_5.

$$y_1 = p_1, y_2 = p_2, y_3 = p_3, y_4 = p_4, y_5 = p_6, p_5 = (1-\Sigma_j y_j). \qquad 3.5$$

The integrating factor for this change of variables is $6^{1/2}$, so it is suppressed. In these variables the domain is

$$y_j \geq 0, \langle \mathbf{E}_5, \mathbf{y}\rangle \leq 1, \qquad 3.6$$

while the equation $\langle \mathbf{N}, \mathbf{p}\rangle = 0$ becomes

$$\langle \mathbf{N}^*, \mathbf{y}\rangle = 2, \qquad 3.7$$

where $\mathbf{N}^* = (3-u, 4, u+3, 1-u, 1+u)$.

We obtain the second change of variables by solving Eq. 3.7 for y_2, the only other variable with a coefficient independent of u. This defines the new x variables as $x_1 = y_1, x_2 = y_3, x_3 = y_4, x_4 = y_5$, where we find the value of y_2 from the x's through Eq. 3.7. The integrating factor for this change of variables is $\{4^2 + (3-u)^2 + (3+u)^2 + (1-u)^2 + (1+u)^2\}^{1/2}$, which is a scalar multiple of

$$\{9+u^2\}^{1/2}. \qquad 3.8$$

The geometry of the domain in the $\mathbf{x} = (x_1, x_2, x_3, x_4)$ variables is given by

$$\langle \mathbf{N}_j, \mathbf{x}\rangle \leq 2, \text{ and } x_j \geq 0, \qquad 3.9$$

where $\mathbf{N}_1 = (1+u, 1-u, 3+u, 3-u)$ and $\mathbf{N}_2 = (3-u, 3+u, 1-u, 1+u)$. The obvious symmetry between $\mathbf{N}_1$ and $\mathbf{N}_2$ results from the symmetry of $\{\mathbf{p} \in Si(3!) | c_1 > c_2\}$ and $\{\mathbf{p} \in Si(3!) | c_2 > c_1\}$.

The volume of the domain defined by Eq. 3.9 can be determined by elementary arguments. Let $\mathbf{e}_j$ be the unit vector with unity in the jth component. The region defined by the equation $\langle \mathbf{N}_1, \mathbf{x}\rangle \leq 2$ is the convex region defined by $\mathbf{0}$ and the four vertices; $2\mathbf{e}_1(1+u)^{-1}$, $2\mathbf{e}_2(1-u)^{-1}$, $2\mathbf{e}_3(3+u)^{-1}$, and $2\mathbf{e}_4(3-u)^{-1}$. Correspondingly, the region defined by $\langle \mathbf{N}_2, \mathbf{x}\rangle \leq 2$ is convex region defined by $\mathbf{0}$ and the four vertices; $2\mathbf{e}_4(1+u)^{-1}$, $2\mathbf{e}_3(1-u)^{-1}$, $2\mathbf{e}_2(3+u)^{-1}$, and $2\mathbf{e}_1$ $(3-u)^{-1}$. The domain defined by Eq. 3.9 is the intersection of these two regions, so it is the union of two congruent regions, where one of them is the convex region defined by the five points $\mathbf{0}$, $2\mathbf{e}_1(3-u)^{-1}$, $2\mathbf{e}_2(3+u)^{-1}$, (1/2, 0, 0, 1/2), and (0, 1/2, 1/2, 0). The four dimensional volume of this object is a scalar multiple of $(9-u^2)^{-1}$.

From these results and the integrating factors given by Eqs. 3.8, 3,4, it follows that

$$\mu(u) = \mu(-u) = D\,(3+u^2)^{1/2}/[(9-u^2)(9+u^2)^{1/2}], \tag{3.10}$$

where D is a scalar factor determined by the suppressed constants. The minimum value is at $u = 0$ – the BC – while the maxima are at $u^2 = 1$ – the plurality and anti-plurality votes. This proves the first part of Theorem 1 (for $n = 3$); the last part is a simple computation.

Proof of Theorem 2; $n = 3$. Here, we integrate Eq. 3.4a over the region corresponding to $c_1 = c_2 > c_3$. By the symmetry of R_r, the surface volume of $c_1 = c_2 > c_3$ equals that of $c_1 = c_2 < c_3$. Thus, the value of the new measure is one-half that given in Eq. 3.10. The conclusion follows.

Proof of Theorem 3; $n = 3$. Here we replace $\mathbf{v}$ with $\mathbf{v}_c = (-c, 1, -(1-c), 0, 0, 0)$, so $\langle \mathbf{N}, \mathbf{v}_c\rangle = 3+u(1-2c)$. Therefore, $\mu_c(u)$ is a scalar multiple of $(3+u(1-2c))(3+u^2)^{1/2}/[(9-u^2)(9+u^2)^{1/2}]$. For the c values of 1, 1/2, and 0, the minimum values for μ_c are attained, respectively, for the u values of 1 (anti-plurality), 0 (BC), and -1 (plurality). Because the minimum point is a continuous function of c, it follows from the intermediate value theorem that any choice of u is the optimal choice for some value of $c \in [0, 1]$.

Proof of Theorem 1; $n \geq 3$. Normalize the voting vector so that $\mathbf{W}^n = (1, u_1, \ldots, u_{n-2}, -1)$, where $u_j \geq u_{j+1}$ and $|u_j| \leq 1$. As in the proof for $n = 3$, the first condition is that $\langle \mathbf{E}_{n!}, \mathbf{p}\rangle = 1$. We determine the replacement for Eq. 3.2 by setting the tallies for c_1 and c_2 equal to each other. Thus, if p_j is the fraction of voters of the jth type, the coefficient for p_j depends on how these voters rank c_1 and c_2. More precisely, this coefficient is the difference between the weights assigned to c_1 and c_2. The vector $\mathbf{M} = (1-u_1, 1-u_2, \ldots, 1-(-1); u_1-u_2, u_1-u_3, \ldots; u_{n-2}-(-1))$ lists all of the possible combinations for $c_1 > c_2$ where the first series has c_1 top ranked, the second has c_1 second ranked, etc. Notice that $-\mathbf{M}$ captures all of the possibilities for $c_2 > c_1$. Moreover, for each of the fixed relative rankings of c_1 and c_2, there are $(n-2)!$ ways to rank the remaining candidates. Thus, with an appropriate labeling of the types of voters, we replace Eq. 3.2 with

$$\langle \mathbf{N}, \mathbf{p} \rangle = 0 \text{ where } \mathbf{N} = (\mathbf{M}, \ldots, \mathbf{M}, -\mathbf{M}, \ldots, -\mathbf{M}). \quad 3.11$$

(Both **M** and $-\mathbf{M}$ are repeated $(n-2)!$ times.) These two constraints define H_n.

To compute **v**, note that for each of the $n(n-1)/2$ ways there are to rank c_1 and c_2, there are $(n-2)!$ ways to rank the remaining candidates. The assumptions on a strategic voter mean that he selects a ranking where c_1 is top-ranked, c_2 is bottom-ranked, and the relative rankings of $\{c_3, \ldots, c_n\}$ remain truthful. Thus, where the relative ranking of $\{c_3, \ldots, c_n\}$ is fixed, the expected change from **p** is a scalar multiple of $(-1, \ldots, -1, [n(n-1)/2]-1, \ldots, -1)$, where the positive value is in the same component as 2 in **M**. So, by my neutrality assumptions, the EMV is a scalar multiple of $\mathbf{V} = (\mathbf{v}, \mathbf{v}, \ldots, \mathbf{v}, 0, \ldots, 0)$ where the factor **v** is repeated $(n-2)!$ times. It now is a simple computation to show that $\langle \mathbf{N}/|\mathbf{N}|, \mathbf{V} \rangle$ is a scalar multiple of

$$[(n-2)(n-1)-\Sigma_j u_j(n-2j-1)]/[4+2\{n-2+\Sigma_j u_j^2\} + \Sigma_{j<k}(u_j-u_k)^2]^{1/2}. \quad 3.12$$

Following the lead of the proof for $n = 3$, we use a change of variables to calculate $S(H_n)$, the $n!-2$ dimensional surface volume of H_n. We find the **y** variables by solving for a p_j that has a coefficient -2, where a truthful ranking has c_2 top-ranked and c_1 bottom-ranked. The domain of the **y** variables is

$$\langle \mathbf{E}_{n!-1}, \mathbf{y} \rangle \leq 1, \text{ all } y_j \geq 0, \quad 3.13$$

and I suppress the integrating factor because it is a scalar that depends only on n. In these new variables Eq. 3.11 becomes $\langle \mathbf{N}^*, \mathbf{p} \rangle = 2$ where $\mathbf{N}^* = 2\mathbf{E}_{n!-1} + \mathbf{N}'$ and $\mathbf{N}'$ is the projection of **N** defined by dropping the coordinate direction corresponding to p_j. (These equations follow by solving for p_j from $\langle \mathbf{N}, \mathbf{p} \rangle = 0$. Namely, $p_j = 1-\langle \mathbf{N}_{n!-1}, \mathbf{p} \rangle$.)

As true for $n = 3$, the change to the x variables uses $\langle \mathbf{N}^*, \mathbf{p} \rangle = 2$. I derive this equation by solving for a y_k term corresponding to a ranking where c_1 and c_2, respectively, are top- and bottom-ranked. the integrating factor for this change of variables is a scalar multiple of

$$[2(n-2)\{4(n-3)! + n+2\} + 2\Sigma_j u_j^2 + \Sigma_{j<k}(u_j-u_k)^2]^{1/2}. \quad 3.14$$

The constraints $y_k \geq 0$ lead to

$$\langle \mathbf{N}_2, \mathbf{x} \rangle \leq 2, \ x_i \geq 0. \quad 3.15$$

The first equation follows from $\langle \mathbf{N}^*, \mathbf{y} \rangle = 2$, so it is immediate that $\mathbf{N}_2$ is obtained from $\mathbf{N}^*$ with the same kind of projection argument where the projection is defined by the coordinate direction y_k. Now, either by using direct algebraic substitution, or by using symmetry arguments based on interchanging the order in which these two change of variables are obtained, it follows that Eq. 3.13 becomes $\langle \mathbf{N}_1, \mathbf{x} \rangle \leq 2$ where $\mathbf{N}_1$ is obtained from $\mathbf{N}_2$ by the same symmetry condition, caused by the symmetry of $\{\mathbf{p} \in \mathrm{Si}(n!) \mid c_1 > c_2\}$ and $\{\mathbf{p} \in \mathrm{Si}(n!) \mid c_2 > c_1\}$, that held for $n = 3$.

To compute $S(H_n)$, first we compute the volume of the region defined by $\langle \mathbf{N}_j, \mathbf{x} \rangle \leq 2$, $j = 1, 2$. This object is the union of two geometrically congruent convex regions defined by $\mathbf{0}$ and $n!-2$ vertices. One-half of these vertices are the $\mathbf{e}_i$ points determined by the equation $\langle \mathbf{N}_j, \mathbf{x} \rangle = 2$ for one value of j, while the remaining vertices are determined by the intersection of the two surfaces $\langle \mathbf{N}_j, \mathbf{x} \rangle = 2$. By symmetry, these last vertices are points where the symmetrical coordinates equal 1/2 and all others are zero. (The symmetrical coordinates are the ones transferred into each other in the construction of $\mathbf{N}_1$ and $\mathbf{N}_2$.) These intersection points do not depend on $\mathbf{W}^n$ or on any of the u_j's. Thus, the volume of each congruent region is a scalar multiple of the volume defined by the vertices on the coordinate axes. This is a scalar multiple of

$$1/\{\Pi_j(9-u_j^2)\}\{\Pi_{j<k}(u_j-u_k+3)^{(n-2)!}\}. \qquad 3.16$$

By using Eq. 3.16, the integrating factors, and $\mathbf{V}$, $\mu(\mathbf{W}^n)$ is a multiple of

$$\frac{[(n-2)(n-1)-\Sigma_j u_j(n-2j-1)][(2n-4)(4(n-3)! + n+2) + 2\Sigma_j u_j^2 + \Sigma_{j<k}(u_j-u_k)^2]^{1/2}}{\{\Pi_j(9-u_j^2)\}\{\Pi_{j<k}(u_j-u_k+3)^{(n-2)!}\}\{2[n+\Sigma_{j<k}(u_j-u_k)^2\}^{1/2}}. \qquad 3.17$$

Theorem 1 now follows by applying elementary, but messy, calculus techniques to $\mu(\mathbf{W}^n)$. It follows from the symmetry considerations that at a critical point, $u_j = -u_{n-(j+1)}$. Therefore, if n is odd, then $u_{(n+1)/2} = 0$. For $n = 4, 5$, the minimum point is on the boundary of the region. For $n \geq 6$, some of the values of u_j differ from 1, 0, or -1. In particular, for $n = 6$, u_2 is an irrational number. The USE specified in the introductory section is a rational approximation for this value.

It is a straightforward asymptotic analysis argument to show that for large values of n, the important factor in determining the optimal value is

$$[(n-2)(n-1)-\Sigma_j u_j(n-2j-1)]/\{\Pi_j(9-u_j^2)\}\{\Pi_{j<k}(u_j-u_k+3)^{(n-2)!}\}.$$

It follows from the symmetry of the terms that, modulo some boundary considerations, for all j, u_j-u_{j+1} tends to a fixed value as $n \to \infty$. This, of course, is the BC.

Proof of Theorem 2. The integration is over the subset of H_n, where $c_1 = c_2$ are the two top-ranked candidates. On H_n there are (n−1)! different rankings; by symmetry, each ranking defines the same surface volume. Of these (n−1)! regions, (n−2)! have $c_1 = c_2$ top-ranked. Thus, the measure for proving this assertion is $\mu(\mathbf{W}^n)/(n-1)$. The conclusion now follows.

Proof of Theorem 4. There are several ways to prove this theorem. The most obvious is to follow the lead of the intuitive comments describing Theorem 4 by showing that the set of **p**'s that can be manipulated with each choice of $\mathbf{W}^n{}_k$ is a proper subset of the set of **p**'s that can be manipulated with the multiple voting system. I provide an alternative proof that introduces the space of profiles and choices of tallying methods described in footnote 8.

Assume that the multiple system is defined by the voting vectors $\{\mathbf{W}^n{}_1, \ldots, \mathbf{W}^n{}_k\}$. (If the multiple system admits more than a finite number of simple systems, then let this be a proper subset of the admissible choices.) Each voter has k ways to select how to have his ballot tallied. Let $q_{j,i}$, $i = 1, \ldots, k$, be the fraction of the voters of the jth type that use $\mathbf{W}^n{}_i$ to tally the ballots. This defines a point $\mathbf{q}_j \in Si(k)$. Thus, the true domain for a multiple system is $(\mathbf{p}; \mathbf{q}_1, \ldots, \mathbf{q}_{n!}) \in Si(n!)x(Si(k))^{n!}$. This can be viewed as a fiber space over Si(n!) where a point in each fiber indicates how the voters choose to split their ballots.

The $c_1 = c_2$ indifference surface in $Si(n!)x(Si(k))^{n!}$, $H_{n,k}$, is a hyperplane in this fiber space. In describing this equation, the coefficient for the $\mathbf{p}_j\, q_{j,i}$ variable comes from the difference between the weights assigned to c_1 and c_2 as defined by $\mathbf{W}^n{}_i$ for the jth type of voter. It is simple to show (see Saari and Van Newenhizen, 1988a, b) that for any $\mathbf{q} \in (Si(k))^{n!}$, there is a $\mathbf{p} \in Si(n!)$ so that the pair is on $H_{n,k}$. By the condition on the $\mathbf{W}^n{}_i$'s, which leads to the indeterminacy of the election outcomes, this hyperplane creates an angle that differs from 90° with Si(n!). The measure of susceptibility is the number of profiles close enough to $H_{n,k}$ so that the outcome can be changed with a small scalar multiple of **v**. To compare this measure with a component simple system $\mathbf{W}^n{}_1$, use the same space $Si(n!)x(Si(k))^{n!}$ where there is no restriction on the **q** vectors. For this setting, $H_{n,1}$ is above H_n and orthogonal to the base space Si(n!). The conclusion now follows from this geometry. The surface area for $H_{n,k}$ must exceed that of H_n. (This is equivalent to comparing the length of all segments in a rectangle starting at a common vertex and ending on one of the sides. The shortest segment is one of the "orthogonal" edges.)

Notes

1. As the election rankings for $\mathbf{W}^n$ and $a\mathbf{W}^n$, $a > 0$, always are the same, all such multiples are identified with each other. The plurality vote, for example, corresponds to (a, 0, ..., 0) for all values of $a > 0$.

2. It is difficult to avoid positional voting. Just by invoking some seemingly innocucous assumptions about the class of choice functions, one ends up with only the positional voting methods. See Young (1973) and Saari (1989a).
3. Because $\mathbf{B}^3 = (2, 1, 0)$, a strategic voter with a true ranking $c_1 > c_2 > c_3$ who wants to alter the c_1, c_2 outcome should mark his ballot as $c_1 > c_3 > c_2$ to provide c_1 with a two point, not just a single point, differential over c_2.
4. I emphasize percentages, not numbers of the voters, because, quite obviously, one out of three voters has a much stronger impact on the election outcome than 1000 out of three million voters. So, a key factor is the percentge size of the "coordinated" manipulation attempt. While I do not carefully analyze the issue, my results include those situations of uncoordinated macro manipulations. This involves heterogeneous electorates in which many voters do try to manipulate the system, but they do so in uncoordinated, small groups. With the heterogeneity assumption, there is a cancellation of the strategic efforts of the different groups, and this can reduce the analysis to a micro manipulation problem.
5. The conclusions of these two essays rely on computer experiments; mine rely on analytic arguments that permit precise statements. Incidentally, Nitzan (1985) points out that "an analytic derivation of the various ... measures seems to be a hopelessly complex task." In light of his comment, one of the contributions of my essay is the development of the mathematical structures that now permits such an analysis. While my emphasis is on micro manipulation, the same techniques can be used to analyze macro manipulation of the systems.
6. This is by design. I view positional voting as being an important prototype to understand the properties of the kind of incentive issue posed here.
7. The smaller the value of $|\mathbf{v}|$, the more precise the conclusion. While I have not carried out any careful computations, it appears that if $|\mathbf{v}|$ is smaller than 1/20, then the conclusions hold.
8. That this choice of a tallying procedure is a strategic variable is the theme of Brams and Fishburn (1984) and it is discussed in Saari and Van Newenhizen (1988a, b). There is a debate over when the voter's choice of a $\mathbf{W}^n_j$ is sincere and when it is a strategic variable. We can completely resolve this modelling problem by extending the techniques developed here. Start with the space that consists of all possible truthful rankings of the candidates as well as their choices of a tallying method. (See the last section in this essay and in Saari and Van Newenhizen, 1988a). Each truthful point is of the form $(\mathbf{p}, \mathbf{q})$ where $\mathbf{q}$ indicates how the voters sincerely select the $\mathbf{W}^n$'s. A function $G(\mathbf{p}, \mathbf{q}; \{\mathbf{W}^n_j\})$, defined in the obvious manner, gives the sincere election outcome. Now suppose that a small group of voters strategically selects a tallying method. The point $(\mathbf{p}, \mathbf{q}')$ represents the strategic behavior where $\mathbf{q}' = \mathbf{q} + \delta$ (δ represents the strategic action) and $G(\mathbf{p}, \mathbf{q}'; \{\mathbf{W}^n_j\})$ gives the actual election outcome. With this modelling, the manipulation problem assumes a form similar to the one I use at the beginning of Section 2. The analysis continues in much the same way with similar conclusions. For instance, for mathematical reasons similar to those that explain why plurality voting does not fare well with respect to manipulations, it follows that approval voting does not fare well in a class of comparable multiple voting systems.

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