Public Choice 42: 235-246 (1984).
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# Arrow's theorem with social quasi-orderings

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## Abstract

The collective rationality requirement in Arrow's theorem is weakened to demanding a social quasi-ordering (a reflexive and transitive but not necessarily complete binary relation). This weakening leads to the existence of a group such that (a) whenever all members of the group strictly prefer one alternative to another then so does society and (b) whenever two members of the group have opposite strict preferences over a pair of alternatives then the pair is socially not ranked. This theorem is then used to provide an axiomatization of the strong Pareto rule. These results are compared and contrasted to Gibbard's oligarchy theorem and Sen's axiomatization of the Pareto extension rule.

## 1. Introduction

A social welfare function is a mapping from the admissible profiles of individual orderings (reflexive, complete, and transitive binary relations) into a social ordering of the alternatives. Arrow (1963) demonstrated that the only social welfare functions that satisfy unlimited domain, independence of irrelevant alternatives, and the weak Pareto principle must be dictatorial.

Our understanding of Arrow's theorem has been greatly enhanced by the many studies that have considered modifications in the axioms and structure of Arrow's problem. One particularly active area of research has been to consider the implications of relaxing the assumption that the social

* I would like to thank David Donaldson, Dieter Sondermann, and the referees for their comments. Thanks are also due to Peter Aranson for his editorial remarks. My work on this topic was initiated while I was a visitor to the Center for Operations Research and Econometrics. My stay at CORE was financed by the generous support of the Institut des Sciences Economiques of the Université Catholique de Louvain.

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preference relation satisfies all of the properties of an ordering, while maintaining the other features of the problem. Weakening transitivity has resulted in the development of decision rules that satisfy Arrow's other axioms. For the most part, these rules embody distributions of decision-making authority that, while not as extreme as dictatorial rule, are not very satisfactory either.[1]

Alternatively, we can maintain the full strength of transitivity while dropping the requirement that the social preference relation be complete. In other words, the social preference relation must be a quasi-ordering. This article determines the implications of this modification of Arrow's problem.

The results of this endeavour are most compactly expressed in terms of Gibbard's (1969) definition of an oligarchy. A set of individuals is an oligarchy if and only if (i) the unanimous strict preference of all group members for x over y implies that x is socially preferred to y and (ii) the strict preference of any member of the group for x over y implies that y is not socially preferred to x. Property (i) says that oligarchies are decisive, while property (ii) says that members of an oligarchy have some veto power. Gibbard only considered complete social preference relations, in which case the conclusion in (ii) may be equivalently stated as 'x is socially at least as good as y'. This statement implies that if two members of an oligarchy have opposite strict preferences for a pair of alternatives, then socially the pair is ranked indifferent. Here we refer to the oligarchies Gibbard considered as $\alpha$-oligarchies. We also introduce a new kind of oligarchy, a $\beta$-oligarchy. For a $\beta$-oligarchy the conclusion in (ii) is strengthened to state that 'x is either socially preferred to y or x and y are socially non-ranked', or, more compactly, 'y is not socially weakly preferred to x'. If two members of a $\beta$-oligarchy have opposite strict preferences for a pair of alternatives, then that pair is socially non-ranked.

In this article, we show that modifying Arrow's theorem to require social preferences that are quasi-orderings leads to the existence of a $\beta$-oligarchy. This proposition is the counterpart to Gibbard's (1969) theorem on the existence of $\alpha$-oligarchies if Arrow's theorem is modified to require reflexive, complete, and quasi-transitive social preference relations.[2]

In principle, any group of individuals could be an oligarchy. Demanding that the decision procedure treat individuals symmetrically implies that the whole society forms the oligarchy. If this is the case, by strengthening the Pareto principle used in the $\beta$-oligarchy theorem we obtain an axiomatization of the strong Pareto rule. An analogous modification of the axioms in the $\alpha$-oligarchy theorem yields Sen's (1970a, Theorem 5*3, p. 76) axiomatization of the Pareto extension rule.

A justification for considering social quasi-orderings is provided by the fact that any finite quasi-ordered set has at least one maximal element in the set, that is, an element which is not dominated by any other element in the set with respect of the binary relation being considered. Furthermore, if the set in question has a non-empty choice set, that is, the collection of elements in the set that are at least as good according to the binary relation as each of the other elements in the set, then the choice set is identical to the set of maximal elements.[3]

A further justification for studying social quasi-orderings is the resulting deeper understanding of existing results concerning social preferences that are reflexive, complete, and quasi-transitive. To emphasize this justification, we present new proofs of Gibbard's $\alpha$-oligarchy theorem and Sen's axiomatization of the Pareto extension rule which parallel the development of our $\beta$-oligarchy theorem and our axiomatization of the strong Pareto rule. Social indifference plays a role in Gibbard's and Sen's theorems analogous to the role social non-comparability plays in our results.

Section 2 presents the notation and definitions used in the rest of the article. Section 3 presents and discusses our results. After developing a few preliminary lemmas, we establish a general theorem on the existence of oligarchies, which yields the $\alpha$-oligarchy and $\beta$-oligarchy theorems as simple corollaries. We then use these theorems to axiomatize the Pareto extension rule and the strong Pareto rule. Section 4 discusses the relationship between the propositions established in Section 3 and the theory of filters. Section 5 considers possible modifications and extensions of the $\beta$-oligarchy theorem.

## 2. Notation and definitions

For the most part, our notation follows Sen (1983). X denotes the set of alternatives, which may be finite or infinite but if it is finite, we assume that X contains at least three elements. H denotes the set of individuals, who are finite in number.

The binary relation, R, 'at least as preferred as,' serves as the primitive concept. Strict preference, P, is defined as $\forall x, y \in X$: $xPy \leftrightarrow [xRy$ and $\neg(yRx)]$ while indifference, I, is defined as $\forall x, y \in X$: $xIy \leftrightarrow [xRy$ and $yRx]$. In addition, N denotes that two alternatives are not ranked; N is defined as $\forall x, y \in X$: $xNy \leftrightarrow [\neg(xRy)$ and $\neg(yRx)]$.

A binary relation, R, is an *ordering* if and only if R is reflexive, complete, and transitive. A binary relation, R, is a *quasi-ordering* if and only if R is reflexive and transitive. A binary relation R is *quasi-transitive* if and only if P is transitive. $B$ is a set of binary relations of X, while $R$ is the set of orderings of X. $R^{|H|}$, the Cartesian product of $R$ $|H|$-times, is the set of logically possible *profiles* of individual orderings.

A *collective choice rule*, f, is a function from the admissible set of profiles $D \subset R^{|H|}$ to a set of binary relations of X. Social preferences remain unsubscripted while subscripts distinguish individuals' preferences. A profile of preferences is denoted $<R_i>$.

A collective choice rule satisfies *unlimited domain* if and only if $D = R^{|H|}$.

A collective choice rule satisfies the *weak Pareto principle* if and only if $\forall x, y \in X, \forall <R_i> \in D$: $xP_iy, \forall i \in H \rightarrow xPy$.

A collective choice rule satisfies the *strong Pareto principle* if and only if $\forall x, y \in X, \forall <R_i> \in D$: (i) $xR_iy, \forall i \in H \rightarrow xRy$ and (ii) $[xR_iy, \forall i \in H$ and $\exists i \in H: xP_iy] \rightarrow xPy$.

A collective choice rule, f, satisfies (binary) *independence of irrelevant alternatives* if and only if $\forall x, y \in X, \forall <R_i> \in D$: $[xR_iy \leftrightarrow xR_i'y, \forall i \in H] \rightarrow [xRy \leftrightarrow xR'y]$ where $R = f(<R_i>)$ and $R' = f(<R_i'>)$.

A collective choice rule, f, satisfies *anonymity* if and only if $\forall <R_i>, <R_i'> \in D$: if $<R_i'>$ is a reordering of $<R_i>$, then $f(<R_i>) = f(<R_i'>)$.

A set of individuals, $G \subset H$, is *almost decisive over the ordered pair* (x, y), if and only if $\forall <R_i> \in D$: $[xP_iy, \forall i \in G$ and $yP_ix, \forall i \notin G] \rightarrow xPy$.

A set of individuals, $G \subset H$, is *decisive over the ordered pair* (x, y) if and only if $\forall <R_i> \in D$: $xP_iy, \forall i \in G \rightarrow xPy$.

A set of individuals, $G \subset H$, is *decisive* if and only if $\forall x, y \in X$: G is decisive for (x, y).

A set of individuals, $G \subset H$, is an *oligarchy* if and only if (a) G is decisive and (b) $\forall x, y \in X, \forall <R_i> \in D$: $[\exists i \in G: xP_iy] \rightarrow \neg(yPx)$.

A set of individuals, $G \subset H$, is an *$\alpha$-oligarchy* if and only if (a) G is an oligarchy and (b) $\forall x, y \in X, \forall <R_i> \in D$: $[\exists i \in G: xP_iy] \rightarrow xRy$.

A set of individuals, $G \subset H$, is a *$\beta$-oligarchy* if and only if (a) G is an oligarchy and (b) $\forall x, y \in X, \forall <R_i> \in D$: $[\exists i \in G: xP_iy] \rightarrow \neg(yRx)$.

An individual $i \in H$ is a *dictator* if and only if $\{i\}$ is decisive.

A collective choice rule, f, is the *Pareto extension rule* if and only if $\forall x, y \in X, \forall <R_i> \in D$: $xRy \leftrightarrow [\exists i \in H: xP_iy$ or $\forall i \in H: xR_iy]$ where $R = f(<R_i>)$.

A collective choice rule, f, is the *strong Pareto rule* if and only if $\forall x, y \in X, \forall <R_i> \in D$: $xRy \leftrightarrow xR_iy, \forall i \in H$ where $R = f(<R_i>)$.

A collection F of subsets of H is a *filter* if and only if (a) $H \in F$, (b) $\phi \notin F$, (c) $[G^1$ and $G^2 \in F] \rightarrow G^1 \cap G^2 \in F$, and (d) $[G^1 \in F$ and $G^1 \subset G^2 \subset H] \rightarrow G^2 \in F$.

The strong Pareto *principle* is a condition on a collective choice rule while the strong Pareto *rule* is a particular collective choice rule. If two individuals have opposite strict preferences over a pair of alternatives, then the alternatives remain unranked by the strong Pareto rule but are considered to be indifferent by the Pareto extension rule.

## 3. Theorems

We proceed by first establishing four lemmas. A major component of the proof of Arrow's theorem is what Sen (1983) refers to as the 'field expansion lemma'. This lemma demonstrates that if a group is ever almost decisive over a pair of alternatives, then it is decisive. This lemma is valid for much weaker social rationality assumptions than the orderings that Arrow considered; we only require reflexivity and quasi-transitivity.[4]

*Lemma 1*: For any collective choice rule with range *B* contained in the set of reflexive and quasi-transitive relations of X satisfying unlimited domain, independence of irrelevant alternatives, and the weak Pareto principle, if a group is almost decisive over any pair of alternatives, then it is decisive.

Oligarchies are decisive, consequently it is not possible to have more than one oligarchy.

*Lemma 2*: For any collective choice rule satisfying unlimited domain and independence of irrelevant alternatives, there can be at most one oligarchy.

*Proof*: Suppose G and G′ are both oligarchies. Consider any profile with $xP_iy$, $\forall i \in G$ and $yP_ix$, $\forall i \in G' \setminus G$. G being decisive implies that xPy, while G′ being an oligarchy implies that $\neg(xPy)$, a contradiction. □

Besides being decisive, members of an oligarchy have veto powers. $\beta$-oligarchies differ from $\alpha$-oligarchies in only one respect, the nature of this veto. With an $\alpha$-oligarchy a strict preference for x over y on the part of any oligarchy member guarantees that xPy or xIy. For a $\beta$-oligarchy the latter possibility is replaced by xNy. The significance of this distinction manifests itself most clearly in the comparison of the social preferences that result if two members of an oligarchy strictly rank a pair of alternatives in the opposite way. For an $\alpha$-oligarchy this disagreement leads to social indifference, while for a $\beta$-oligarchy the alternatives are socially noncomparable.

*Lemma 3*: For any $\alpha$-oligarchy, G, if any two members of G have opposite strict preferences over a pair of alternatives, then the pair of alternatives is ranked indifferent: $\forall x, y \in X$, $\forall \langle R_i \rangle \in D$: $[\exists i \in G\text{: } xP_iy \text{ and } \exists j \in G\text{: } yP_jx] \rightarrow xIy$.

*Lemma 4*: For any $\beta$-oligarchy, G, if any two members of G have opposite strict preferences over a pair of alternatives, then the pair of alternatives is not ranked: $\forall x, y \in X$, $\forall \langle R_i \rangle \in D$: $[\exists i \in G\text{: } xP_iy \text{ and } \exists j \in G\text{: } yP_ix] \rightarrow xNy$.

By maintaining the axioms of Arrow's theorem but modifying the requirement that social preferences be orderings it is possible to obtain a general oligarchy theorem. To do so, social preferences are assumed to be reflexive and quasi-transitive; that is, transitivity is weakened to quasi-transitivity and completeness is dropped.

*Theorem 1*: For any collective choice rule with range *B* contained in the set of reflexive and quasi-transitive relations of X satisfying unlimited domain, independence of irrelevant alternatives, and the weak Pareto principle, there exists a unique oligarchy.

*Proof*: By the weak Pareto principle, H is decisive, and since $|H|$ is finite, there must exist a smallest decisive group, G. We shall demonstrate that G is an oligarchy, which by Lemma 2 implies that it is unique.

If G contains a single member, then G is trivially an oligarchy. If $|G| \geq 2$, for a pair of alternatives x, y $\epsilon$ X, we then define the sets $A = \{i \epsilon G \mid xP_iy\}$, $B = \{i \epsilon G \mid yP_ix\}$, and $C = \{i \epsilon G \mid xI_iy\}$. By unlimited domain we may consider any profile that contains the rankings:

| | | | |
|---|---|---|---|
| A | B | C | H\G |
| x | y | (x, y) | w |
| w | x | w | y |
| y | w | | |

where (x, y) denotes that x and y are indifferent. Suppose $A \neq \phi$ and $B \cup C \neq \phi$. The case of $B \cup C = \phi$ is trivial.

Since G is decisive, xPw. Suppose yPx. Transitivity of P would then imply that yPw. But if yPw, Lemma 1 implies that B∪C is decisive, which contradicts the assumption that G is the smallest decisive group. Thus $\neg$(yPx). Since G is decisive, this establishes that G is an oligarchy.□

It is now straightforward to establish Gibbard's $\alpha$-oligarchy theorem and our $\beta$-oligarchy theorem with the aid of Theorem 1. To obtain an $\alpha$-oligarchy, the social rationality assumption in Theorem 1 is strengthened to demand completeness. If instead of demanding completeness, quasi-transitivity is strengthened to transitivity, we obtain a $\beta$-oligarchy.

*Corollary 1* [Gibbard (1969)]: For any collective choice rule with range *B* equal to the set of complete, reflexive, and quasi-transitive relations of X satisfying unlimited domain, independence of irrelevant alternatives, and the weak Pareto principle, there exists a unique $\alpha$-oligarchy.

*Proof*: The result follows trivially from the observation that in the presence of completeness $\neg$(yPx) is equivalent to xRy.□

*Corollary 2*: For any collective choice rule with range *B* equal to the set of quasi-orderings of X satisfying unlimited domain, independence of irrelevant alternatives, and the weak Pareto principle, there exists a unique $\beta$-oligarchy.

*Proof*: In the proof of Theorem 1, instead of assuming yPx, suppose that yRx. Transitivity of R would then imply that yPw. By the reasoning in the proof of Theorem 1, we conclude that $\neg$(yRx), which establishes that G is a $\beta$-oligarchy.□

As noted previously, the differences between the two kinds of oligarchies exhibit themselves in situations in which two oligarchy members have opposite strict preferences over a pair of alternatives. Demanding completeness of the social preference relation results in an $\alpha$-oligarchy and, by Lemma 3, forces the ranking of the alternatives to be indifference. This indifference violates transitivity of social preferences (but not quasi-transitivity). Demanding transitivity of the social preference relation instead of completeness results in a $\beta$-oligarchy and, by Lemma 4, leaves the alternatives unranked.

Demanding both completeness and transitivity (in addition to reflexivity) eliminates from consideration all oligarchies containing two or more individuals. We are left with Arrow's (1963) theorem, as a dictator is simply an oligarchy consisting of a single member. A feature of the proof of Arrow's theorem is what Sen (1983) calls the "group contraction lemma". This lemma establishes that, for the Arrow problem, if any group is decisive and contains at least two persons, then this group contains a smaller decisive group. Completeness and transitivity of the social preference relation are used in an essential fashion in this lemma.

By adding anonymity to the list of axioms in Theorem 1, we obtain the polar case to Arrow's dictator, the oligarchy must consist of the whole society. If we also strengthen the Pareto assumption, then we can use Corollary 1 to establish Sen's axiomatization of the Pareto extension rule,[5] while we can use Corollary 2 to axiomatize the strong Pareto rule.

*Theorem 2* [Sen (1970a, Theorem 5*3, p. 76)]: If a collective choice rule, f, has a range *B* equal to the set of complete, reflexive, and quasi-transitive relations of X, then (a) it satisfies unlimited domain, independence of irrelevant alternatives, the strong Pareto principle, and anonymity if and only if (b) f is the Pareto extension rule.

*Proof*: That (b) implies (a) is easy to verify. To show that (a) implies (b) we must demonstrate that:

$$xRy \leftrightarrow [\exists i \in H: xP_iy \text{ or } \forall i \in H: xR_iy]. \tag{1}$$

By Corollary 1 and anonymity H is the $\alpha$-oligarchy. If there exists an $i \in H$ such that $xP_iy$, H being an $\alpha$-oligarchy implies xRy. If for all $i \in H$ $xR_iy$, the strong Pareto principle implies xRy. Hence,

$$[\exists i \in H: xP_iy \text{ or } \forall i \in H: xR_iy] \rightarrow xRy. \tag{2}$$

The antecedent in (2) is false if and only if

$$\forall i \in H: yR_ix \text{ and } \exists i \in H: yP_ix.$$

By the strong Pareto principle,

$$[\forall i \in H: yR_ix \text{ and } \exists i \in H: yP_ix] \rightarrow yPx. \tag{3}$$

Since R is complete, (2) and (3) establish (1).□

*Theorem 3*: If a collective choice rule, f, has a range ***B*** equal to the set of quasi-orderings of X, then (a) it satisfies unlimited domain, independence of irrelevant alternatives, the strong Pareto principle, and anonymity if and only if (b) f is the strong Pareto rule.

*Proof*: That (b) implies (a) is easy to verify. To show that (a) implies (b) we must demonstrate that:

$$xRy \leftrightarrow xR_iy, \forall i \in H. \tag{4}$$

By Corollary 2 and anonymity, H is the $\beta$-oligarchy. If there exists an $i \in H$ such that $yP_ix$, H being a $\beta$-oligarchy implies $\neg(xRy)$. Taking the contrapositive, $xRy \rightarrow \neg(\exists i \in H: yP_ix)$ or, since individual preferences are complete,

$$xRy \rightarrow \forall i \in H: xR_iy. \tag{5}$$

If for all $i \in H$ $xR_iy$, the strong Pareto principle implies xRy, that is,

$$\forall i \in H: xR_iy \rightarrow xRy. \tag{6}$$

Together, (5) and (6) establish (4).□

With the Pareto extension rule, Pareto non-comparable alternatives are ranked indifferent while for the strong Pareto rule they are left unranked.

Thus, the main difference between these two rules arises because the Pareto extension rule makes society an $\alpha$-oligarchy while the strong Pareto rule makes society a $\beta$-oligarchy. In terms of the axiomatizations, these collective choice rules differ only in the maintained social rationality condition; the other axioms are common to both rules.

For $\beta$-oligarchies the polar cases of Arrow's dictator and the strong Pareto rule highlights that there is a tradeoff involved between spreading the decision-making authority among a large group of people and the desire to have the social relation as complete as possible. At the extreme of dictatorshop, there is little representation but there is near-completeness of the social preference relation (if the dictator is indifferent, alternatives need not be ranked). At the other extreme there is complete representation, but Pareto non-comparable alternatives are left unranked. $\beta$-oligarchies between these extremes adopt a middle ground. Similarly, a tradeoff between the equality of the distribution of decision-making power and the proportion of the ordered pairs that are socially indifferent occurs with $\alpha$-oligarchies, a tradeoff that Blair and Pollak (1982) have observed more generally in their study of social preference relations which are reflexive, complete, and acyclic.

This discussion of the differences between $\alpha$- and $\beta$-oligarchies should not obscure their similarities. In particular, suppose that for any feasible set of alternatives we identify the maximal elements generated by a social preference relation. If the profile of individual preferences contains no individual indifference, then both $\alpha$- and $\beta$-oligarchic rule by the group G lead to precisely the same maximal elements. In constructing maximal elements it is of no consequence whether x and y are indifferent or whether they are non-comparable.

## 4. Filters

With oligarchies it is particularly easy to characterize all of the decisive groups in society. If G is an oligarchy, then the set of decisive groups consists of all coalitions of individuals that contain G as a subset. It is straightforward to check that such a collection of coalitions forms a filter.

Kirman and Sondermann (1972) develop the relationship between Arrow's theorem and the theory of filters. Subsequently, Hansson (1976) and Brown (1975) used the theory of filters to generate oligarchy theorems. Brown's article also contains interpretations of the properties of filters in terms of social choice axioms. These articles have related their results to the theory of $\alpha$-oligarchies; this section comments on the relationship of this work to the theory of $\beta$-oligarchies.

Hansson's (1976) terminology might suggest that he never used completeness of R as an axiom; but his 'transitivity' condition requires R to be an ordering. As already noted earlier, completeness is used in an essential way in both Arrow's theorem and Gibbard's theorem; the same is true with Hansson's theorems. However, it is easy to discern the consequences for Hansson's results of relaxing completeness. For example, observing that completeness of R is not used in parts (a) – (d) of Hansson (1976, Theorem 1, pp. 91–92), it is possible to conclude that, with the assumptions of our Theorem 1, the set of decisive groups forms a filter. With this result in hand, it is not too difficult to provide an alternative derivation of our $\beta$-oligarchy theorem.

Brown (1975) does not require R to be complete. Using quite weak rationality conditions, Brown (1975: 462–463) demonstrates that the social choice procedure will be oligarchic if the set of decisive coalitions forms a filter. If the social preference relation is complete, reflexive, and quasi-transitive, then Gibbard's $\alpha$-oligarchy theorem becomes an immediate corollary of Brown's general theorem. However, Brown uses the strict preference relation as his primitive, which means that it is impossible to distinguish noncomparability from indifference. Consequently, it is also not possible to obtain the $\beta$-oligarchy theorem from his proposition if R is required to be a quasi-ordering.

## 5. Modifications and extensions

There are many ways to modify and extend the basic $\beta$-oligarchy theorem. These concluding remarks address some of these possibilities.

For any oligarchy, if between two alternatives all individuals have strict rankings, then the social ranking (or non-ranking) of the pair must be based entirely on these rankings ignoring any descriptive features of the alternatives involved. Sen (1979) calls this neutrality property 'strict-ranking welfarism'.[6] Whenever there is a person in an oligarchy who is indifferent between two alternatives, neutrality does not necessarily hold. For example, if a dictator expresses indifference between alternatives, non-welfare information can be used to provide the ranking. However, even this tiny foothold for non-welfare considerations can be eliminated by using the strong Pareto principle, as is apparent in Theorem 3.

The ordinality of preferences is not essential for Arrow's theorem. Instead, if each persons's preferences are cardinal (representable by a function unique up to a positive affine transformation) but interpersonally non-comparable, Sen (1970a, Theorem 8*2, p. 129) has shown that Arrow's theorem remains valid if the definitions have been suitably changed to deal explicitly with utility functions. The same extension is possible with the results presented here. Intuitively, the social preference between a pair of alternatives must be based on the rankings of at least three alternatives for cardinality to be effective and independence of irrelevant alternatives only permits the information gained from binary comparisons to be used.

Fishburn (1974; 1976) also considers relaxing the completeness axiom but does not require that R be a quasi-ordering. In our study the pairs of alternatives that the social preference relation leaves unranked are endogenous and will vary from profile to profile, given a fixed group as the $\beta$-oligarchy. In Fishburn's work the pairs of alternatives that must be ranked are exogenous. In this framework he specifies a set of axioms leading to a dictatorial rule. His articles suggest that there is a close connection between having restricted domain assumptions and dropping completeness; it would be useful to have the precise relationship between these axioms explored more systematically.

There has been surprisingly little work done explicitly on social quasi-orderings or any other collective rationality requirement that does not demand completeness. In the literature that does exist, we can discern two themes. The first of these themes was initiated by Sen's (1970a: Chapter 7*; 1970b; 1972) study of social choice with partial interpersonal comparisons of utility, a theme that Blackorby (1975) has explored, among others. In this literature *sums of individual utilities* are used to generate social quasi-orderings. The theorems presented here do not assume that the social preference relation is based on comparing sums of utility gains and losses. The second theme, explored extensively by Sen (1970a: Chapter 12*) and Blackorby and Donaldson (1977), analyzes properties of various quasi-orderings, such as the Suppes (1966) grading principle. However, except in the case of utilities that are fully interpersonally comparable, this literature does not investigate the properties required for a social choice procedure to determine uniquely the class of social quasi-orderings under consideration. Thus very few of the implications of letting social preferences be incomplete have yet been determined.[7]

## NOTES

1. Sen (1983) has recently provided a characteristically lucid survey of these results, to which the reader is referred for further references.
2. This theorem was subsequently, and independently, established by a number of other individuals. See Sen (1983) for a discussion of these contributions.
3. Formal statements of these propositions appear in Sen (1970a, Chapter 1*) which is also a convenient reference for properties of quasi-orderings.
4. A proof of this proposition may be found in Sen (1983).
5. Sen (1970a) offered a self-contained proof of his theorem and did not relate his results to Gibbard's $\alpha$-oligarchy theorem.
6. It is easy to check that Sen's (1979, Theorem 1, p. 541) strict-ranking welfarism theorem is valid for social quasi-orderings.

7. We can obtain a few partial results from some of the theorems in Blair and Pollak (1979; 1982) and Brown (1975).

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