SENSITIVITY ANALYSIS OF THE EQUATIONS FOR A CONVECTIVE MIXED LAYER

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Abstract. Jump or slab models are frequently used to calculate the depth of the convectively mixed layer and its potential temperature during the course of a clear day. Much attention has been paid theoretically to the parameterization of the budget for turbulent kinetic energy that is required in these models. However, for practical applications the sensitivity of the solutions of the model equations to variations in the entrainment formulation and in the initial and boundary conditions is also very important. We analyzed this sensitivity on the basis of an analytical solution for the model which uses the well-known constant heat flux ratio. The initial conditions for the mixed-layer height (h) and potential temperature (Θ_m) quickly lose their influence. Only the initial temperature deficit is important. The mixed-layer temperature at noon on convective days is insensitive to the entrainment coefficient c. It is governed by the integral of the heat input and by the stable lapse rate. A change in c from 0.2 to 0.5 leads to a variation of 20% in h. This is not very much considering the accuracy in the determination of h from actual observations.

1. Introduction

Jump or slab models are frequently used to calculate the evolution of the depth of the convectively mixed atmospheric boundary layer and its potential temperature during the course of a clear day (Figure 1). Many theoretical discussions have been devoted to the parameterization of the turbulent kinetic energy budget which is required in order to relate dh/dt to the energetics of the turbulence in the mixed layer (Tennekes and Driedonks,

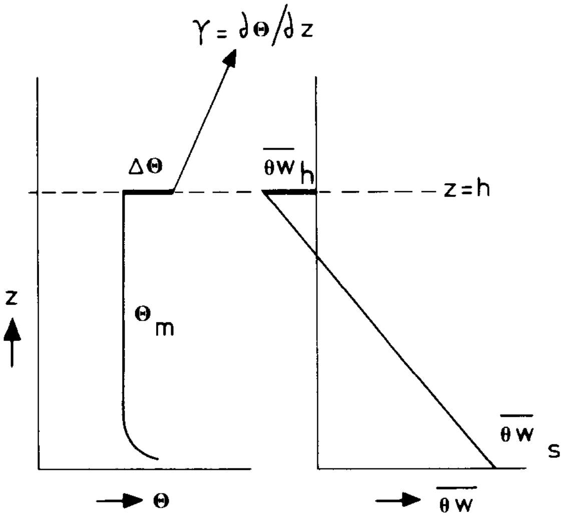

Fig. 1. The profile of potential temperature and the heat flux distribution in a jump model.

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1981). However, when we use these models for practical applications, we have to realize that the determination of the mixed-layer variables from atmospheric observations is difficult and that errors are inevitable, either due to the measured profiles not being as ideal as depicted in Figure 1 or due to statistical errors (Driedonks, 1981). Therefore it is equally important to assess the influence of various parameterizations on the solutions and to consider the effect of variations in the initial and boundary conditions. Here we analyze these effects for a widely used model.

2. Governing Equations

The equations that govern the dynamics of mixed-layer jump models are (Tennekes and Driedonks, 1981):

$$\frac{d\Theta_m}{dt} = (\overline{\theta w_s} - \theta w_h)/h \,, \tag{1}$$

$$\frac{d\Delta\Theta}{dt} = \gamma \frac{dh}{dt} - \frac{d\Theta_m}{dt} \,, \tag{2}$$

$$-\overline{\theta w_h} = \Delta\Theta \frac{dh}{dt} \,. \tag{3}$$

The notation is as usual and is depicted in Figure 1. Closure of this set of equations is achieved by a parameterized form of the budget for turbulent kinetic energy. For a convectively mixed layer, a well-known result of this parameterization leads to a constant heat flux ratio

$$-\overline{\theta w_h} = c\,\theta w_s \,, \tag{4}$$

where c is the entrainment constant. The values reported for c range between 0 and 1 with an average value of 0.2 (Stull, 1976). This variation may partly be caused by the fact that (4) was used in situations in which a more complicated parameterization was required. Another possibility is the difficulty in getting an estimate for c from actual observations in which errors in the determination of the variables in (1)–(4) are inevitable. Therefore we analyze the sensitivity of the solutions of (1)–(4) to the initial conditions, to variations in the entrainment coefficient c, to errors in the surface heat flux $\overline{\theta w_s}$ and to errors in the value of the stable gradient γ.

3. Solutions of the Model

The set of Equations (1)–(4) can be reduced to

$$c\overline{\theta w_s} = \Delta\Theta \frac{dh}{dt} \,, \tag{5}$$

$$\frac{d\Delta\Theta}{dt} = \gamma \frac{dh}{dt} - \frac{\overline{\theta w_s}}{h}(1 + c) . \tag{6}$$

With the initial conditions: $h(t = 0) = h_0$, $\Delta\Theta(t = 0) = \Delta\Theta_0$, these equations have an analytic solution of the form:

$$\Delta\Theta = \gamma \frac{c}{1 + 2c} h + \left(\frac{h_0}{h}\right)^{(1 + c)/c} \left(\Delta\Theta_0 - \gamma \frac{c}{1 + 2c} h_0\right) . \tag{7}$$

Substitution of (7) in (5) and integration gives the following implicit solution for h ($\equiv h(t)$):

$$\tfrac{1}{2}\gamma \frac{c}{1 + 2c} h^2 - \left(\frac{h_0}{h}\right)^{1/c} \left(h_0\Delta\Theta_0 - \gamma \frac{c}{1 + 2c} h_0^2\right) = c\,\{I(t) + \tfrac{1}{2}\gamma h_0^2 - h_0\Delta\Theta_0\} , \tag{8}$$

where $I(t)$ is the integral of the heat input:

$$I(t) = \int_0^t \overline{\theta w_s}(t')\, dt' . \tag{9}$$

Equation (8) gives $h(t)$ as a function of the initial and boundary conditions (through h_0, $\Delta\Theta_0$, γ, and $I(t)$), and of the entrainment coefficient c.

The solution for the temperature Θ_m is related to (7) and (8) by

$$\begin{aligned} \Theta_m(t) - \Theta_{m0} &= \gamma(h - h_0) + \Delta\Theta_0 - \Delta\Theta \\ &= \gamma \frac{1 + c}{1 + 2c} h - \left(\frac{h_0}{h}\right)^{(1 + c)/c} \left(\Delta\Theta_0 - \gamma \frac{c}{1 + 2c} h_0\right) + \Delta\Theta_0 - \gamma h_0 . \end{aligned} \tag{10}$$

4. The Initial Conditions

A typical value for the entrainment coefficient c is 0.2. As a consequence the terms in (7), (8), and (10) in which $(h_0/h)^{1/c}$ is involved, decay very quickly when h starts growing. For example, when $c = 0.2$ and $h = 3h_0$, these terms are only of the order of 0.1% of their initial value. Thus, when a short time has elapsed, we may neglect these terms and approximate (7)–(10) by

$$\Delta\Theta = \gamma \frac{c}{1 + 2c} h , \tag{11}$$

$$\tfrac{1}{2}\gamma h^2 = (1 + 2c)(I(t) - D_0) , \tag{12}$$

$$\Theta_m(t) = \Theta_{00} + \gamma \frac{1 + c}{1 + 2c} h , \tag{13}$$

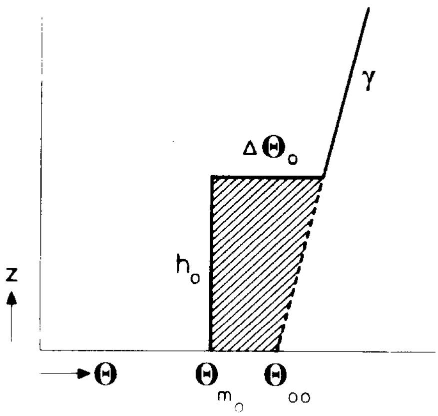

Fig. 2. Initial temperature profile. The shaded area represents the initial temperature deficit D_0.

where $D_0 = h_0\Delta\Theta_0 - \frac{1}{2}\gamma h_0^2$ and $\Theta_{00} = \Theta_{m_0} + \Delta\Theta_0 - \gamma h_0$ (Figure 2). These results are a generalization of those obtained by Manton (1980), who used a constant heat flux and simplified initial conditions.

For $t \gg 0$ (and $h \gg h_0$), the effect of the initial conditions on the solutions as given in (11)–(13) manifests itself only in (12) through the initial temperature deficit $D_0 = h_0\Delta\Theta_0 - \frac{1}{2}\gamma h_0^2$. The initial conditions thus play a role only in this combination; the individual values of Θ_{m_0} and h_0 are not important. This is a fortunate result for the prediction of the mixed-layer depth and temperature at noon.

With Equations (11)–(13) we now can estimate by partial differentiation the effect of the entrainment coefficient c and of the forcing terms $I(t)$ and γ on the resulting values of h and Θ_m.

5. The Sensitivity of *h*

We first consider the influence of c on h through (12). Partial differentiation of this equation with respect to c leads to the result

$$\frac{\delta h}{h} = \frac{\delta c}{1 + 2c}, \tag{14}$$

where δh is the variation of h when we vary c by an amount δc. From this relation we see that a change in c from $c = 0$ to $c = 0.2$ will cause only a change in h of about 20% at noon. The same change in h occurs when c is changed from 0.2 to 0.5.

The variation of h due to a change in the integral heat input I is given by

$$\frac{\delta h}{h} = 0.5 \frac{\delta I}{I - D_0}. \tag{15}$$

When we use $D_0 \approx 0.3I$ as a typical value on clear days (Driedonks, 1981), we see that a change of 30% in the integral heat input causes a change of 20% in h.

A change in the stable lapse rate γ will lead to a change in h according to

$$\frac{\delta h}{h} = -0.5\,\frac{\delta\gamma}{\gamma}\,. \tag{16}$$

A typical stable lapse rate is 0.005 K m^{-1}. An inaccuracy of 0.001 K m^{-1} will then lead to a change in h of about 10%.

From these considerations, we see that inaccuracies in $I(t)$ and γ may lead to errors in h which are of the same order of magnitude as those caused by a large variation in the entrainment coefficient c. Therefore it will be difficult to get accurate estimates of c from atmospheric observations.

6. The Sensitivity of Θ_m

The influence of a variation in c on the mixed-layer temperature $\Theta_m(t)$ is small. It may be estimated by partial differentiation of (13):

$$\delta\Theta_m = 2\,\frac{(I - D_0)c}{h(1 + 2c)}\,\delta c \quad (^\circ\mathrm{C})\,. \tag{17}$$

Typical values at noon are: $I - D_0 \approx 1000$ K m, $h \approx 1000$ m. We then see that the change in Θ_m caused by a change in c from 0.2 to 0.5 is only very small, less than 0.1 °C. This difference can usually be neglected.

The influence of a change in I on Θ_m is somewhat larger. From (13) and (15) we estimate

$$\delta\Theta_m = 0.5\,\frac{1 + c}{1 + 2c}\,\gamma h\,\frac{\delta I}{I - D_0} \quad (^\circ\mathrm{C})\,. \tag{18}$$

At noon, γh is typically 5 °C. A change of 20% in I will then change the temperature of the mixed layer at noon by 0.5 °C. This is much larger than could be caused by a change in c. The mixed-layer temperature is thus more sensitive to a change in the integral heat input than to a change in the entrainment coefficient.

A change in γ will lead to a change in Θ_m whch may be estimated from (13) and (16) as

$$\delta\Theta_m = \frac{1 + c}{1 + 2c}\cdot\tfrac{1}{2}h(\delta\gamma) \quad (^\circ\mathrm{C}). \tag{19}$$

For typical values $c = 0.2$, $h \approx 1000$ m, we see that a change of $\delta\gamma = 0.001$ K m^{-1} causes a change of $\delta\Theta_m$ of about 0.5 °C.

7. Conclusions

On convective days the initial conditions for h_0 and Θ_{m0} quickly lose their influence on the solutions for $h(t)$ and $\Theta_m(t)$. Only the initial temperature deficit $D_0 = h_0\Delta\Theta_0 - \frac{1}{2}\gamma h_0^2$ is important.

The mixed-layer temperature Θ_m at noon on convective days can be calculated without paying much attention to the value of the entrainment coefficient c. It is mainly influenced by the integral heat input I and by the stable lapse rate γ.

Although the mixed-layer height h is somewhat more sensitive to variations in the entrainment coefficient than Θ_m, it will be difficult to estimate c accurately from atmospheric observations. We saw that a change in c from 0.2 to 0.5 leads to a variation of 20% in h. This is still not very much and the inaccuracy in the determination of h from actual observations will tend to obscure partly the effect of c.

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