APPLICATION OF THE $E-\varepsilon$ TURBULENCE MODEL TO THE ATMOSPHERIC BOUNDARY LAYER

H. W. DETERING and D. ETLING

Institut für Meteorologie und Klimatologie, Universität Hannover, Herrenhäuser-Str. 2, 3000 Hannover 21, F.R.G.

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Abstract. In the so called $E-\varepsilon$ turbulence model, an eddy viscosity is evaluated from turbulent kinetic energy E and energy dissipation ε. Although still a first-order closure method in its simpler form, the $E-\varepsilon$ model yields eddy viscosity for complex turbulent flows without a prior prescription of a length scale needed in so-called mixing-length models. The $E-\varepsilon$ model has been successfully applied to many flow problems in engineering applications for non-rotating boundary layers. In this paper, the $E-\varepsilon$ method is extended to the atmospheric boundary layer for which a modification of the dissipation equation is found to be necessary in order to give results comparable with observational data.

1. Introduction

In recent years modelling of mesoscale phenomena has received growing interest not only among research scientists but also from applications in meteorology (Pielke, 1984). Especially in the mesoscale β and γ ranges which cover atmospheric phenomena with horizontal extensions less than 200 km, mesoscale models may be more or less regarded as two- or three-dimensional boundary-layer models applied to irregular terrain or non-homogeneous surface conditions (Orlanski, 1975). Hence one problem common to all mesoscale models is the parameterization of turbulent fluxes of momentum, heat, moisture or air contaminants. This so-called closure problem has been treated intensively for horizontal homogeneous boundary layers leading to numerous proposals of closure approximations; see, e.g., Bodin (1980) or Wyngaard (1982) for examples.

With respect to mesoscale models, no specific closure has been proposed so far but parameterization methods found for homogeneous boundary layers have also been applied. For the sake of simplicity, most models make use of a simple gradient transfer hypothesis where only a turbulent exchange coefficient has to be defined. This coefficient is often evaluated by a mixing-length hypothesis, where the mixing length is taken as a height-dependent function as proposed, e.g., by Blackadar (1962). Especially for flows over highly irregular terrain, e.g., steep hills or valleys (Hunt, 1980), it is not always obvious how to apply a mixing-length with respect to a varying underlying surface.

A few models have tried to circumvent this problem by use of a second-order closure model (Lewellen *et al.*, 1980; Yamada, 1978). This is a rather time consuming approach due to the many additional equations needed for second-order turbulence modelling and is not practicable for most mesoscale modellers at present. A compromise has been proposed by Mellor and Yamada (1974, 1982), called a level-2.5-model, and has been applied to mesoscale modelling by Yamada (1983). It takes the simple gradient transfer

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approach but uses a prognostic equation for the mixing length in connection with the turbulent energy equation. This results in an eddy viscosity coefficient variable in time and space due to mesoscale flow modifications. With this approach Yamada obtained better results compared to observations than with the usual mixing-length method.

A similar approach makes use of a prognostic equation for the energy dissipation ε instead of a length-scale equation. From this and the turbulent kinetic energy E, an eddy viscosity may also be calculated, as will be explained shortly. This method, originally proposed among others by Hanjalic and Launder (1972), called the $E - \varepsilon$ model, has become very popular in the field of fluid engineering in recent years. Numerous examples using the $E - \varepsilon$ method for numerical simulation of free shear flows, recirculating and separating flows, hydraulic and channel flows etc. can be found in Durst *et al.* (1979) or Rodi (1980). Most of these flows are highly inhomogeneous and some of them have similarities with atmospheric flows over irregular terrain.

The popularity of the $E - \varepsilon$ closure method in engineering applications raises the question of whether it could also be used for mesoscale modelling in the atmospheric boundary layer, especially for flows over irregular terrain, where a simple definition of a mixing length like that of Blackadar may not always be justified. We are aware of only a few applications of the $E - \varepsilon$ method for atmospheric (Lee and Kao, 1979) and oceanic boundary layers (Marchuk *et al.*, 1977; Svennsson, 1979). But as is shown later, applying the standard $E - \varepsilon$ model used in engineering applications to atmospheric flows yields unrealistic results compared to other frequently used closure methods. Hence a modification of the $E - \varepsilon$ method is proposed in this paper, which can be applied to modelling of mesoscale flows or atmospheric boundary-layer problems. The modified $E - \varepsilon$ model is presented for a one-dimensional homogeneous and neutral boundary layer, but can be extended easily to two- or three-dimensional stratified flows.

2. Basic Equations

Let us for simplicity consider the well known equations of motion for a horizontally homogeneous turbulent boundary layer which may be written in standard notation:

$$\frac{\partial \bar{u}}{\partial t} = f(\bar{v} - v_g) - \frac{\partial \overline{w'u'}}{\partial z} \tag{1a}$$

$$\frac{\partial \bar{v}}{\partial t} = -f(\bar{u} - u_g) - \frac{\partial \overline{w'v'}}{\partial z}. \tag{1b}$$

Velocity components of the geostrophic wind u_g and v_g as well as coriolis parameter f are as usual.

For the Reynolds stress, a simple gradient transfer approach (first-order closure) is taken:

$$\overline{w'u'} = -K \frac{\partial \bar{u}}{\partial z} \tag{2a}$$

$$\overline{w'v'} = -K\frac{\partial \bar{v}}{\partial z}. \tag{2b}$$

The eddy viscosity coefficient for momentum K will be evaluated using the Prandtl–Kolmogorov hypothesis:

$$K = c_0 l E^{1/2}. \tag{3}$$

The turbulent kinetic energy is denoted as $E = (\overline{u'^2} + \overline{v'^2} + \overline{w'^2})/2$, l is a turbulence length scale or mixing length and c_0 is a constant. A prognostic equation for turbulent kinetic energy E can be derived easily (Busch, 1973) which is for a neutrally stratified boundary layer:

$$\frac{\partial E}{\partial t} = K\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] + \frac{\partial}{\partial z}\alpha_e K\frac{\partial E}{\partial z} - \varepsilon. \tag{4}$$

In (4) first-order closure has been used for the production term and for turbulent transport of kinetic energy and pressure. The constant α_e links eddy viscosity coefficients for energy and momentum by $K_e = \alpha_e K$.

The energy-dissipation ε is modelled by the generally accepted Kolmogorov relation:

$$\varepsilon = c_\varepsilon \frac{E^{3/2}}{l}. \tag{5}$$

As in (3) l is a mixing length or characteristic turbulent length scale and c_ε is a constant. In the following, we shall adopt the commonly made assumption that the length scales in (3) and (5) are equal, although for stratified boundary layers there are some indications (Therry and Lacarrère, 1983) that different length scales should be used for dissipation and mixing length.

First-order closure (2a, b) with eddy viscosity (3) in connection with turbulent energy equation (5) is frequently used in boundary-layer and mesoscale models (Bodin, 1980, Wyngaard, 1982). The main difference between the models is in the definition of the length scale l. Due to its simplicity, a simple functional form for $l(z)$ is adopted by most modellers. The well known mixing length introduced by Blackadar (1962) is widely used:

$$l(z) = \frac{\kappa z}{1 + \kappa z/\lambda}. \tag{6}$$

This mixing length yields a linear variation in the surface layer and approaches a constant value in the upper part of the boundary layer. Several methods for evaluating the maximum value of the mixing length λ have been proposed but will not be discussed here further.

Instead of (6), prognostic equations for a length scale have been developed in connection with second-order modelling of atmospheric boundary layers by Shir (1973), Lewellen and Teske (1973) and others or, for a combination of E and l, by Mellor and

Yamada (1982). Still another form of a length scale equation can be obtained by changing the Kolmogorov relation (5) to:

$$l = c_\varepsilon \frac{E^{3/2}}{\varepsilon} . \tag{7}$$

Using (7), an additional prognostic equation for the energy dissipation ε is needed; this is given below. If we insert (7) into the mixing-length relation for eddy viscosity (3), we obtain

$$K = c_k \frac{E^2}{\varepsilon} \tag{8}$$

with $c_k = c_0 c_\varepsilon$.

Eddy viscosity relation (8) is the central part of what is called the $E - \varepsilon$ closure method in the literature*. A prognostic equation for energy dissipation ε may be derived from the equations of motion or from an equation for the turbulent vortex intensity (Tennekes and Lumley, 1972). This equation can only be used for modelling purposes after some approximations and parameterizations; these will not be repeated here. Instead we refer to papers by Hanjalic and Launder (1972), Lumley (1978) or Marchuck *et al.* (1977) for a derivation. The resulting equation which may be referred to as the standard form can be written as:

$$\frac{\partial \varepsilon}{\partial t} = c_1 \frac{\varepsilon}{E} P - c_2 \frac{\varepsilon}{E} \varepsilon + \frac{\partial}{\partial z} \alpha_\varepsilon K \frac{\partial \varepsilon}{\partial z} . \tag{9}$$

In (9), the abbreviation p for the production term of the kinetic energy equation is used:

$$P = K\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] \tag{10}$$

where c_1, c_2, and α_ε are constants to be determined; α_ε links eddy viscosity for dissipation transport to that for momentum, i.e., $K_\varepsilon = \alpha_\varepsilon K$. A somewhat more complicated form of (9), which takes non-isotropic turbulence into account, has been given by Zeman and Lumley (1979), but we will restrict ourselves to the more widely used notation as in (9).

Eddy viscosity formulation (8) together with energy equation (4) and dissipation equation (9) constitute the boundary-layer approximation of the $E - \varepsilon$ closure method. It should be recalled that this is still a first-order closure method, because a simple gradient approach (2a, b) is taken for Reynolds stress. It differs from the more frequently

* It is quite standard to denote turbulent kinetic energy by q, q^2 or k in engineering science and elsewhere. Hence, in most cases one will find the $E - \varepsilon$ model referred to as $k - \varepsilon$ closure model. But since we have reserved K for the eddy viscosity coefficient, as is common practice in meteorology, the model will be referred to as the $E - \varepsilon$ model here.

used *l-E* closure, consisting of (2a, b), (3), (4) and a form of mixing-length approach like (6), in that the eddy viscosity is evaluated from two prognostic variables, E and ε, hence leaving one more degree of freedom for K being a property of turbulent flow structure. It may be noted that an equation for energy-dissipation ε is also needed in most second-order closure methods (Lumley and Khajeh-Nouri, 1974; Wyngaard *et al.*, 1974; Launder *et al.*, 1975), because in these models the triple correlations of turbulent quantities are usually parameterized using a turbulent timescale $\tau \sim E/\varepsilon$.

The $E - \varepsilon$ method is quite similar to the $q^2 - 1$ closure model ($E - 1$ in our notation) used by Yamada (1983) for simulation of mesoscale drainage flows. But as pointed out by Mellor and Herring (1973) or Rodi (1980), all methods using a prognostic equation for a length scale l, a combination of energy E and l, or a dissipation equation, are more or less equivalent and need some approximations for their derivation. It is sometimes argued that the dissipation equation is more physically based than some form of a length scale equation. However, the result is more or less a type of closure method for turbulent flows, which needs some justification anyway. Here the objective is simply to investigate whether the widely used $E - \varepsilon$ model in simulating engineering turbulent flows can also be applied to the atmospheric boundary layer.

3. Evaluation of Constants and Boundary Conditions

As in every closure method, there are some constants to be determined. In the equation for turbulent kinetic energy, only one constant is needed, $\alpha_e = K_e/K$, which is a kind of an inverse Prandtl-number, because it is often assumed that eddy diffusivities for turbulent kinetic energy K_e and heat K_h are the same. Hence a value of $\alpha_e = 1.3$ may be taken for neutrally stratified boundary layers.

The constants c_0, c_1, c_2 and c_ε will be evaluated from the consideration that relations (3), (5), and (8) should also be valid in the surface layer, i.e., in the lowest 10 m of the atmospheric boundary layer. Within the surface layer (constant-flux layer) instead of (3) and (5), we have respectively (Tennekes, 1973):

$$K = \kappa u_* z \tag{11}$$

$$\varepsilon = \frac{u_*^3}{\kappa z} \tag{12}$$

where κ is Karman's constant and u_* is the friction velocity. Combining eddy viscosities (3) with (11) one gets for the constant c_0:

$$c_0 = u_*/E^{1/2} \tag{13a}$$

which gives a relation between Reynolds stress and turbulent kinetic energy in the surface layer, i.e., $u_*^2 = c_0^2 E$. The dissipation forms (5) and (12) yield for constant c_ε:

$$c_\varepsilon = c_0^3. \tag{13b}$$

Consequently one has for the constants in the eddy viscosity form (8):

$$c_k = c_0 c_\varepsilon = c_0^4. \tag{13c}$$

Finally the constants in the dissipation equation have to be determined. This is usually done (Rodi, 1980) by assuming that (9) should also be valid in the equilibrium layer near the lower solid boundary (law of the wall or Prandtl-layer). Here one has approximately $P = \varepsilon$ since turbulent kinetic energy E is nearly constant and leads to a vanishing diffusion term in Equation (4). The diffusion term in the dissipation equation (9) does not vanish, but has to be evaluated from (11) and (12) in the surface layer. Since friction velocity u_* is assumed to be constant within the surface layer, one finally obtains from (9):

$$c_1 = c_2 - \kappa^2 \frac{\alpha_\varepsilon}{c_0^2}. \tag{14}$$

The constant c_2, which is connected with the dissipation term in (9), is the only constant which has been determined by experiments so far. For decaying grid turbulence, it was found to lie in the range 1.8–2.0 (Launder and Spalding, 1972). Once the fundamental constant c_0 relating Reynolds stress to turbulent kinetic energy in the surface layer has been obtained from measurements, constants c_1 and α_ε in (14) are usually found through 'computer optimizing' by comparing simulated flow profiles with data from laboratory measurements. The constant c_0 has been evaluated from channel, pipe or boundary-layer flows or from atmospheric field observations. A mean value collected from laboratory data (Mellor and Yamada, 1982) is given by $c_0 = 0.55$ ($u_*^2 = 0.3E$). This is widely used in application of the $E - \varepsilon$ model to engineering flow problems. For the atmospheric boundary layer, measurements indicate $c_0 = 0.40$ ($u_*^2 = 0.16E$) (Panofsky *et al.*, 1977). Most $E - \varepsilon$ models have been applied using some standard values for constants as given, e.g., by Launder and Spalding (1974). These may be written in our notation as: $c_0 = 0.55$, $c_1 = 1.44$, $c_2 = 1.92$, $\alpha_e = 1.0$, $\alpha_\varepsilon = 0.77$. The values for these constants to be used in atmospheric boundary-layer calculations will be given later.

With regard to boundary conditions, it is assumed that an undisturbed geostrophic flow is reached above a certain height z_g which should be at some distance from the actual boundary-layer height. Hence we have:

Upper boundary conditions:

$$z = z_g: u = u_g,\ v = v_g \tag{15a}$$

$$E = 0,\ \varepsilon = 0. \tag{15b}$$

The lower boundary conditions are applied at some height z_p within the surface (constant stress) layer. This is necessary for practical applications because due to computer storage and execution time it is not always possible to resolve the lower part of the boundary layer with a fine resolution grid. Therefore, nearly all boundary-layer or mesoscale models apply a law of the wall, where the height of the surface layer z_p is taken as between 10 and 50 m. This is particularly necessary for the $E - \varepsilon$ model due to the rapid variation of dissipation ε in the lower part of the boundary layer. Hence, well-known surface-layer laws will be applied at $z = z_p$ as lower boundary conditions:

$$u(z_p) = \frac{u_*}{\kappa} \ln (z_p/z_0) \cos \alpha_0 \tag{16a}$$

$$v(z_p) = \frac{u_*}{\kappa} \ln (z_p/z_0) \sin \alpha_0 \tag{16b}$$

$$E(z_p) = u_*^2/c_0^2 \tag{16c}$$

$$\varepsilon(z_p) = u_*^3/\kappa z_p \tag{16d}$$

$\alpha_0 = \arctan (v/u)$ is the cross-isobar angle at $z = z_p$ and z_0 is the surface roughness length.

With these boundary conditions, it follows from (8) and (13c) that $K(z_p) = \kappa u_* z_p$ for the neutral surface layer.

4. Standard $E - \varepsilon$ Model Applied to the Atmospheric Boundary Layer

Closure method (8) with energy and dissipation equations in the form of (4) and (9) may be called the 'standard $E - \varepsilon$ model'. This method, as applied to atmospheric boundary-layer equations (1a, b), has been used so far, to our knowledge, only by Mason and Sykes (1980) and in a somewhat different form by Lee and Kao (1979). We will refer to these papers shortly.

Regarding the constants as given in Section 3, the constant c_0, relating turbulent energy to surface stress, will be taken from atmospheric surface-layer observations (Panofsky *et al.*, 1977) for neutral stratification. A value of 0.4 for von Karman's constant is used. All other constants are as used in engineering applications (Rodi, 1980). Relation (14) finally yields the following set:

$$c_0 = 0.40, \quad c_1 = 1.13, \quad c_2 = 1.90, \quad \alpha_e = 1.35, \quad \alpha_\varepsilon = 0.77.$$

For comparison of model results with atmospheric data, boundary equations, closed with the $E - \varepsilon$ method, have been solved numerically using a finite difference scheme with a vertically stretched grid to yield better resolution in the lower part of the boundary layer. A time-dependent problem was run from an initial state to a final steady state, and was compared to the well known Leipzig wind profile (Lettau, 1962). Since numerical methods are now quite standard, we only note that because we used centered time differencing (leap-frog), diffusion and dissipation terms in Equations (1a, b), (4), and (9) have to be evaluated at times $t - \Delta t$ and that all other terms are taken at the usual centre time t in order to yield stable results.

For comparison with a more common mixing-length model, all cases were also run with the eddy viscosity hypotheses (3) and with energy equation (4) but dissipation evaluated from the Kolmogorov relation (5). The mixing length was calculated using (6) where the maximum value λ was evaluated according to

$$\lambda = bu_*/f \tag{17}$$

where $b = 0.0063$ is Blackadar's constant.

In order to run the boundary-layer model, two external parameters – geostrophic wind $\mathbf{v}_g$ and roughness length z_0 – have to be prescribed. For the Leipzig wind profile, Lettau gives $|\mathbf{v}_g| = 17.5 \text{ m s}^{-1}$ ($u_g = 17.5 \text{ m s}^{-1}$, $v_g = 0 \text{ m s}^{-1}$ for the coordinate system chosen here) and $z_0 = 0.3$ m.

Model results and observations are shown for the velocity components in Figure 1.

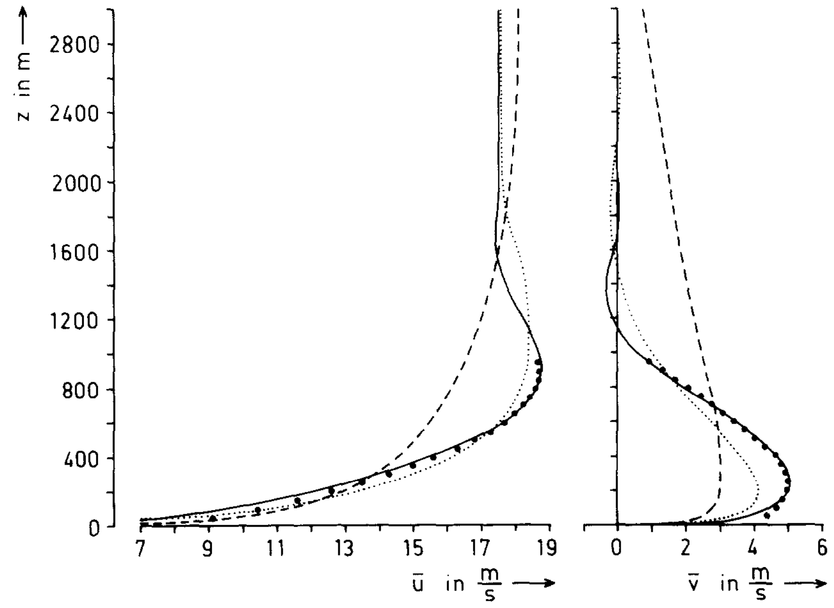

Fig. 1. Observed and simulated wind components $\bar{u}(z)$ and $\bar{v}(z)$. (• • • •) Leipzig data; (——) mixing-length model; (– – – –) $E - \varepsilon$ model (standard); (· · · ·) $E - \varepsilon$ model (modified constants).

(The coordinate is chosen with the u-component in the direction of the surface geostrophic wind.) Whereas the mixing-length model simulates the observed wind profile quite reasonably, the standard $E - \varepsilon$ model exhibits a very different behaviour. The u-component does not reach the free-stream value of u_g within the lowest 3 km of the boundary layer and also the cross-wind component v becomes zero only near the top of the model atmosphere, i.e., at about 12 km. The flattened v-profile in the lower boundary layer also yields a rather small cross-isobar angle $\alpha_0 = 14{,}2°$ compared to 26.1° as obtained from the observations. Friction velocity u_* was overestimated by the $E - \varepsilon$ model with $u_* = 0.80 \text{ m s}^{-1}$ compared to observed 0.65 m s^{-1}. The reason for this discrepancy becomes clear if the results for the turbulent kinetic energy and eddy viscosity coefficient in Figure 2a, b are compared. The eddy viscosity from the mixing-

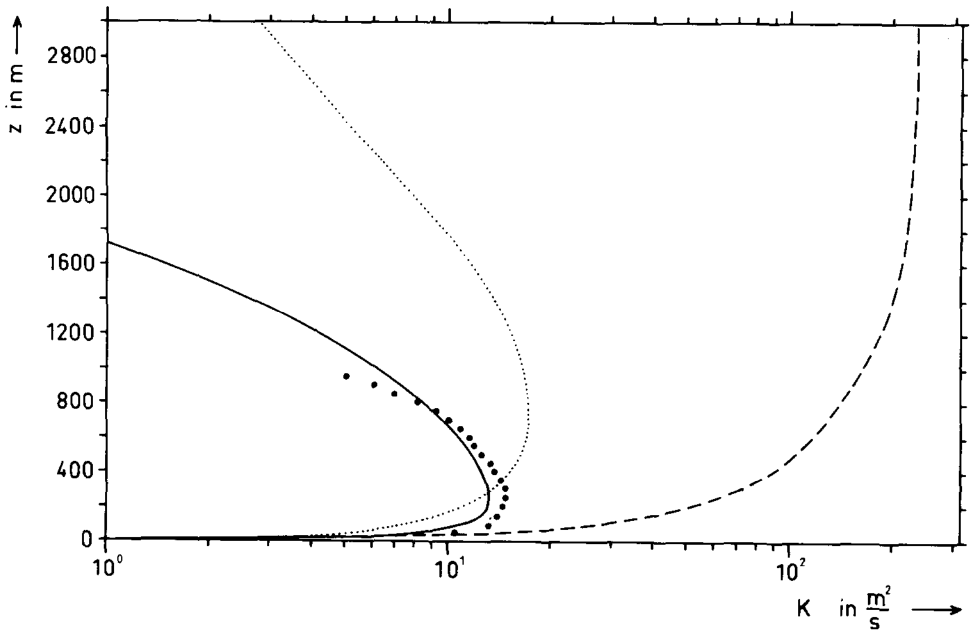

Fig. 2a. Profile of eddy viscosity coefficient $K(z)$ as obtained from model calculations and observations. Designation of curves as in Figure 1.

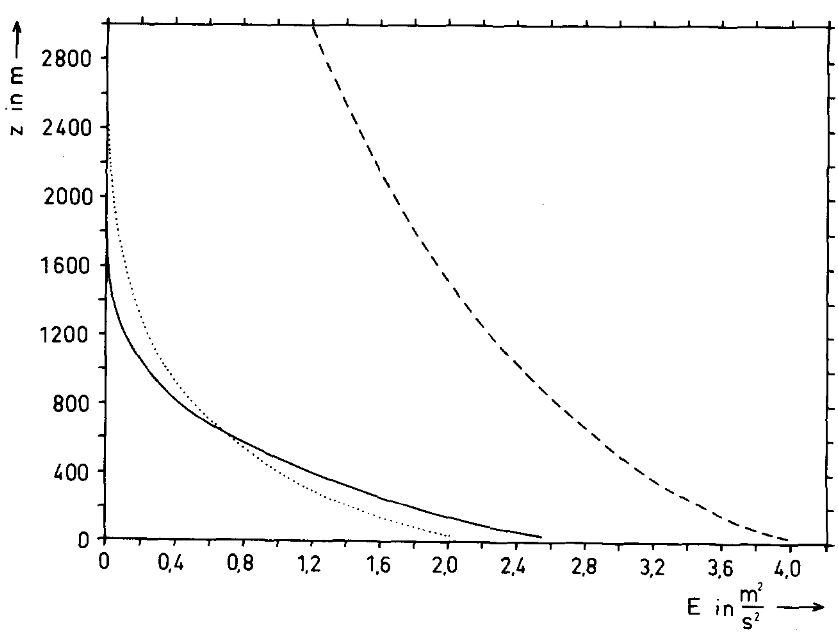

Fig. 2b. Same as Fig. 2a but for the turbulent kinetic energy $E(z)$.

length model and from observations show a variation with height typical for atmospheric boundary layers, a more or less linear increase near the surface, a maximum value in the lower boundary layer and decreasing magnitudes above. The $E - \varepsilon$ model yields an eddy viscosity K increasing to very large values (note the logarithmic axis) even well above the observed boundary-layer height. This large eddy viscosity causes a very deep boundary layer to develop as can be also seen from the velocity-profiles in Figure 1. This effect is also observed in the calculated profile of the turbulent kinetic energy E, shown in Figure 2b. In the mixing-length model, kinetic energy E vanishes at a height of about 1600 m, whereas in the $E - \varepsilon$ model, E slowly decreases with height with non-zero values even in the upper part of the model atmosphere. In addition, the surface magnitudes of E are larger from the $E - \varepsilon$ model due to the larger friction velocity u_* ($E = u_*^2/c_0^2$).

In summary, the standard $E - \varepsilon$ model, when applied to the atmospheric boundary layer, yields a very deep boundary layer, large eddy viscosity, large friction velocity and small cross-isobar angle compared to observations or mixing-length models. The same effect was recognized by Mason and Sykes (1980) in a two-dimensional model for simulating vortex roll development due to inflection point instability. Using the standard $E - \varepsilon$ method for parameterisation, they found the boundary layer to be completely stable with respect to two-dimensional perturbations, whereas vortex roll developed for a mixing-length type of parameterisation. The reason for this behaviour may be explained from the eddy viscosity coefficients obtained by $E - \varepsilon$ closure models as shown in Figure 2a. The large values of K result in a dynamical interpretation of a low (turbulent) Reynolds-number flow, which is stable for an inflection point instability mode; see Brown (1970).

Similar results (too large u_* and K, too small α_0) were also obtained by Lee and Kao (1979) for a related $E - \varepsilon$ model and by Shir (1973) using a length-scale equation. Deviations in these models were not as great as those presented here, presumably due to a rather limited model height ($z_g = 0.45u_*/f$) used in their calculations.

A possible reason for this behaviour of the $E - \varepsilon$ model, when applied to the atmospheric boundary layer, may be deduced by comparing a mixing length reevaluated after (7) with the Blackadar mixing length (6). This is shown in Figure 3 where scales have been normalized with respect to a boundary-layer height H evaluated from mean wind profiles. For the Blackadar mixing length, one gets the typical behaviour found for the atmospheric boundary layer, a linear increase with height near the surface and a rapid transition to a near constant value with a maximum mixing length l_m of approximately $l_m = 0.03H$.

The mixing length reevaluated from the $E - \varepsilon$ model departs from the linear profile only slowly, reaching very large values with a maximum mixing length $l_m = 0.16H$. It resembles closely the well known Nikuradse formula for mixing length in pipe and channel flows (see e.g., Launder and Spalding, 1972) which is also plotted in Figure 3. This fact may be understood by recalling that the constants used in the standard $E - \varepsilon$ model have been tuned using (14) by comparing model results with observations from non-rotating turbulent boundary layers, such as wall or channel flows, for which Nikuradse's formula or similar functional forms of a mixing length hold. Hence one

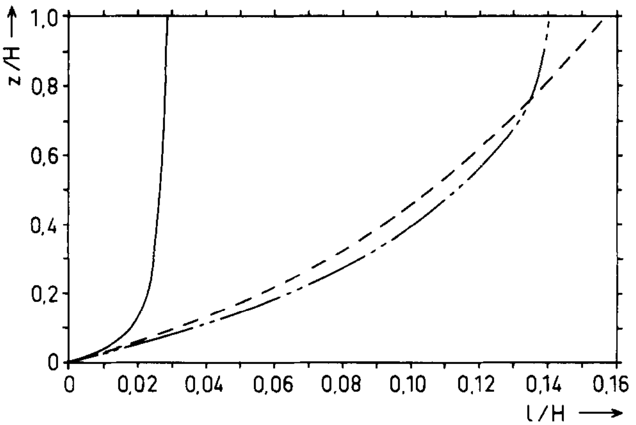

Fig. 3. Normalized mixing length profiles. (——) after Blackadar (Equation (6)); (- - - -) $E - \varepsilon$ model (Equation (7)); (— - — -) Nikuradse's mixing length.

cannot expect that rotating turbulent boundary layers, as found in atmosphere and ocean, can be modelled appropriately with a turbulence model adjusted to observations in non-rotating boundary layers.

This experience might have led Marchuk *et al.* (1977) to a simple modification of a standard $E - \varepsilon$ model for use in simulation of the oceanic boundary layer. Although they developed a relation for the constants in the dissipation equation (9) similar to (14) leading to a constants $c_1 = 1.38$ and $c_2 = 1.9$, using our notation, the actual calculations were carried out with $c_2 = 1.4$, leading to a difference $c_2 - c_1 = 0.02$ instead of 0.52 according to (14).

Some experimentation with varying constants for the standard $E - \varepsilon$ model used in our calculations led to the following conclusions: If relation (14) is used for fixing constants in the dissipation equation (9), results are only very slightly dependent on the fundamental constant c_0 for velocity and viscosity profiles and also for friction velocity u_* and cross-isobar angle α_0. Only profiles of turbulent kinetic energy E are different in magnitude due to relation (13a) used in boundary condition (16c). However, if all constants were kept fixed except c_1 or c_2 (as has been done by Marchuk *et al.* 1977), results for all variables were very sensitive with respect to variations of either c_1 or c_2. (Increasing c_1 and hence the production term in (9), and leaving c_2, hence the dissipation term, fixed has the same effect as fixing c_1 and varying c_2.)

Having recognized this behaviour of the $E - \varepsilon$ method with respect to constants to be used, we decided to keep c_2 fixed because this is the only constant evaluated from measurements of decaying turbulence. We then varied c_1 in such a way that the resulting profile for the eddy viscosity K was on the order of observed magnitudes and thus approximately in agreement with mixing-length model results (Figure 2a).

The resulting constant c_1 keeping all other constants as given before, was $c_1 = 1.83$, yielding $c_2 - c_1 = 0.07$. This model version will be called 'modified $E - \varepsilon$ model'. The results obtained are also shown in Figures 1 and 2a, b. The velocity profiles in Figure 1 still did not agree with those obtained by a mixing-length model, but were far closer to observations than those from the standard $E - \varepsilon$ model. The resulting eddy viscosity profile was near observed values, but the maximum was found at a height of 700 m compared to about 240 m from observations and mixing-length model. Also there was not such a rapid decrease above that maximum as in the other profiles, indicating a still larger boundary-layer height. This can be estimated from the turbulent kinetic energy profile (Figure 2b) as $z_g \approx 2600$ m compared to 1600 m in the mixing-length model. Values for friction velocity u_* and cross-isobar anàgle α_0 were 0.58 m s^{-1} and 22.2°, respectively, and hence closer to observations as can be seen from Table I.

TABLE I

Friction velocity u_* and cross-isobar angle α_0 for the Leipzig data (Obs.) and different model results

	Obs.	1	2	3	4
u_* (m s^{-1})	0.65	0.64	0.66	0.58	0.80
α_0 (°)	26.1	26.3	26.6	22.2	14.2

1 mixing length model.
2 E–ε model using Equation (20).
3 $E - \varepsilon$ model, standard with modified constants.
4 $E - \varepsilon$ model, standard, Equation (9).

5. Modified Dissipation Equation

The failure of a standard $E - \varepsilon$ model to predict atmospheric boundary-layer profiles correctly is not surprising if one looks into the literature concerning application of the $E - \varepsilon$ method to real flow problems. As Rodi (1980) pointed out, universality of constants used in the $E - \varepsilon$ model could not be expected for all kinds of turbulent flows. Indeed there are many examples (see also Rodi's paper) where the $E - \varepsilon$ method had been successfully applied only after modification of constants involved in the dissipation equation. It has turned out that the standard $E - \varepsilon$ model predicts well for strong shear flows near solid boundaries, but gives poor results for weak shear flows as in the outer part of a free jet. In the latter case, for example, Rodi (1980) proposed the constant c_k in eddy viscosity form (8) to be a function of P/ε. Since $c_k = c_0^4$ also enters relation (14), it has the effect of increasing the constant c_1 in the dissipation equation in weak shear regions, where dissipation is larger than production in the turbulent energy equation. Predictions of weak shear flows have been improved using this correction method (see also Gibson and Launder, 1976).

In the atmospheric boundary layer, a region of strong shear can be found near the lower boundary (surface layer), whereas the middle and upper parts of the boundary layer are dominated by weak shear. Hence it seemed to be appropriate to modify model constants for regions above the surface layer. This modification was not done with respect to the basic constant c_0 or c_k as proposed by Rodi because mixing-length models with eddy viscosity according to (3) have been applied successfully in many models of the atmospheric boundary layer with fixed c_0. Also the constant c_2 in dissipation equation (9) was not changed since its value is generally derived from measurements in decaying turbulence.

Experience made with changing the constant c_1 in the modified $E-\varepsilon$ model, as described before, led to the conclusion that increasing c_1, hence decreasing the difference c_2-c_1, with height, should improve the results for the $E-\varepsilon$ model with respect to the atmospheric boundary layer. Some trial and error calculations indeed showed very good agreement for a linear increase of c_1 with height. But as the $E-\varepsilon$ method is proposed also for flows over highly irregular terrain, a purely geometric correction term was not easy to justify in general. Here it should be pointed out that Yamada (1983) also used a correction term in his length-scale equation, but for the constant c_2 (in our notation) connected with a dissipation term. The correction was correlated to the relation of $l/\kappa z$, where l is the length scale and z the vertical coordinate. Because l approaches a limited value in the upper boundary layer (with z increasing), the correction term effectively reduces dissipation in comparison to production (constant c_1) and hence the difference c_2-c_1 with height. Although Yamada's length scale equation is not exactly comparable to the dissipation equation (9) used here, his correction method is equivalent to our choice of increasing constant c_1 as the upper boundary layer is approached.

The correction method proposed here is based on the assumption that the constant c_1 related to the production term in the dissipation equation (9), should depend on characteristic length scales of the turbulent flow considered. Hence it is proposed to modify c_1 into c_1' as follows:

$$c_1' = c_1 l/h \tag{17}$$

where l is a turbulent length scale and h is a characteristic scale for the atmospheric boundary layer, which is taken as:

$$h = c_h u_*/f. \tag{18}$$

The length h is proportional to the depth of the atmospheric boundary layer which may be defined as $H = au_*/f$ for near neutral stratification, where the constant a is accepted to range from 0.2 to 0.45.

If the length scale l is evaluated after the Kolmogorov relation (7), the constant c_1' becomes:

$$c_1' = c_1 \frac{c_\varepsilon E^{3/2}}{c_h u_* \varepsilon}. \tag{19}$$

Replacing the constant c_1 in (9) by the constant c_1' as given by (19) a modified dissipation equation is obtained as:

$$\frac{\partial \varepsilon}{\partial t} = c_1 \frac{c_\varepsilon f}{c_h u_*} E^{1/2} P - c_2 \frac{\varepsilon}{E} \varepsilon + \frac{\partial}{\partial z} \alpha_\varepsilon K \frac{\partial \varepsilon}{\partial z}. \tag{20}$$

Although there is no convincing argument for choosing (17) as a correction term, it yields the desired result: with l increasing with distance from the earth's surface (as suggested by observations and other mixing-length models proposed), the constant c_1' connected with the production term on the right-hand side of (20) increases too. There is no geometric dependence of this constant with respect to a lower boundary because the length scale l is evaluated from local values of E and ε.

It might be added in this context that a recent numerical simulation of isotropic turbulence in rotating fluids (Aupoix *et al.*, 1983) has led to a correction of the constant c_2 in the dissipation equation (9) of a form $c_2 \sim fE/\varepsilon$, which is similar in structure to (17) if use of (7) and (18) is made.

The additional constant c_h in (18) and (20) still has to be determined. This has been done by leaving all other constants fixed for the standard $E-\varepsilon$ model given in the previous section and running the boundary-layer model for different values of c_h. The results were compared to observations (Leipzig wind profile) as already done for the standard $E-\varepsilon$ model. The 'optimum' value of c_h chosen as best fit between model results and observations, was found as $c_h = 0.0015$.

Given a typical value for the friction velocity $u_* = 0.4$ m s^{-1}, one obtains for the characteristic length $h \approx 6$ m which is of the order of a surface-layer thickness. Hence the proposed relation (17) modifies the constant c_1, related to the production term of the dissipation equation, with the ratio between a turbulence length scale and a surface-

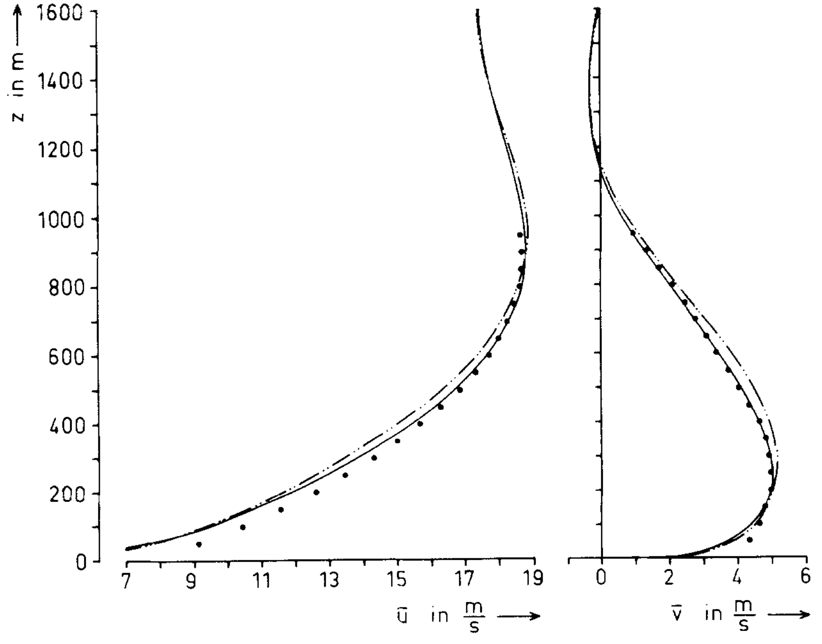

Fig. 4. Observed and simulated wind components $\bar{u}(z)$ and $\bar{v}(z)$. (• • • • •) Leipzig Data; (——) mixing-length model; (– · · –) $E-\varepsilon$ model with Equation (20).

layer scale. This is, in principle, similar to the modification of the constant c_2 in the length-scale equation by Mellor and Yamada (1982).

Results of the model simulations using the modified dissipation equation (20) but leaving all other equations as before, are shown for the mean wind profiles in Figure 4. Results obtained from both the modified ε-equation and the $1 - E$ mixing-length model agree quite well with observations. Boundary-layer height, as indicated by the vanishing cross-isobar component v, was approximately 1600 m, corresponding to $0.38\ u_*/f$.

Also the friction velocity u_* and the cross-isobar angle α_0 were predicted well compared to observations by the modified $E - \varepsilon$ closure model, as can be seen from Table I. The eddy viscosity coefficient obtained according to (8) is shown in Figure 5a together with results from the usual Kolmogorov relation (3). Both methods give values in fair agreement with the eddy viscosity derived from observed wind profiles. A mixing length evaluated for the $E - \varepsilon$ model from (7) is shown in Figure 5b. It shows similar behaviour to a mixing length obtained from (6) near the surface and the middle part of the boundary layer. The decrease of mixing length above a certain height has also been found in the models of Shir (1973) and Lee and Kao (1979). But is is not clear whether the mixing length should approach a constant value or decrease to zero when geostrophic equilibrium is reached.

Another possible test for boundary-layer models is the use of Rossby-number similarity and resistance laws for boundary-layer parameterisation (Blackadar and Tennekes, 1968; Wippermann, 1972). According to those theories, the geostrophic drag coefficient $c_g = u_*/|\mathbf{v}_g|$ and the cross-isobar angle α_0 should be functions of the surface Rossby number only (for neutral stratification), defined by $\mathrm{Ro}_0 = |\mathbf{v}_g|/fz_0$ as usual. Results for model calculations of c_g and α_0 for different values of Ro_0 are shown in Figure 6a, b and compared to the resistance law and data from the Leipzig measurements. Results for the modified $E - \varepsilon$ model and $1 - E$ closure are quite similar to the resistence law as given by Wippermann (1972), but the standard $E - \varepsilon$ model (Chapter 4) yields too low a cross-isobar angle and higher drag coefficients as compared to other results.

Results for varying surface Rossby number have been obtained for fixed geographic latitude (51° for the Leipzig data) and hence fixed coriolis parameter f. The Rossby-number similarity should yield the same results regardless of whether Ro_0 is varied with changing $|\mathbf{v}_g|$, z_0 or f. To test this, we made two additional calculations for the $E - \varepsilon$ model with fixed geostrophic wind and surface roughness for latitudes 70° N and 20° N. The resulting values of c_g and α_0 fell well onto the line in Figure 6a, b valid for the $E - \varepsilon$ model. If we varied the coriolis parameter in the equation of motion but not in the modified constant c_1' (19) of the dissipation equation (20) (simply taking a value for 51° as fixed in c_1'), we obtained the results marked by open crosses in Figure 6a, b. It is obvious that in this case the model does not give results as required by Rossby-number similarity. Thus the inclusion of a characteristic length scale $h = c_h u_*/f$ in the dissipation equation of modified form (20) seems to be necessary for application of the $E - \varepsilon$ method to rotating boundary layers, such as found in atmosphere and oceans. Therefore the standard $E - \varepsilon$ model using (9) instead of (20) would also fail with respect to variation of coriolis parameter in Rossby-number similarity.

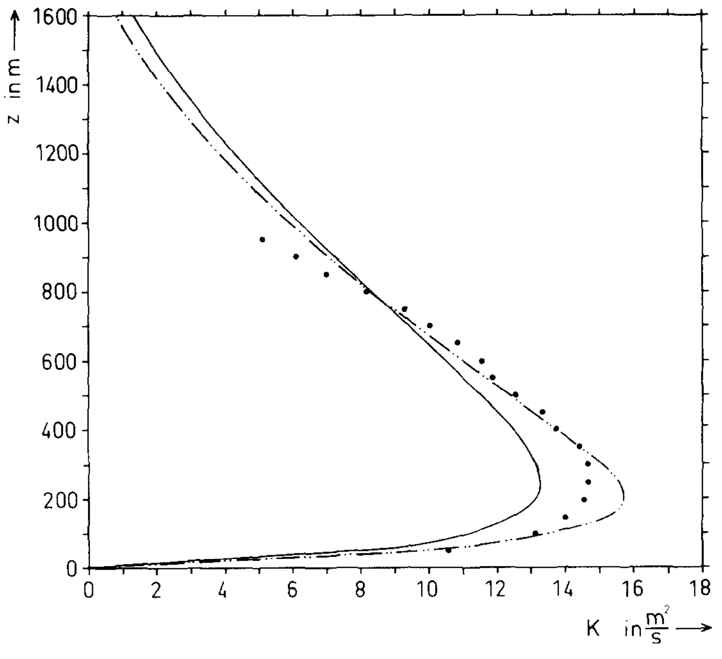

Fig. 5a. Same as Figure 4 but for the eddy viscosity $K(z)$.

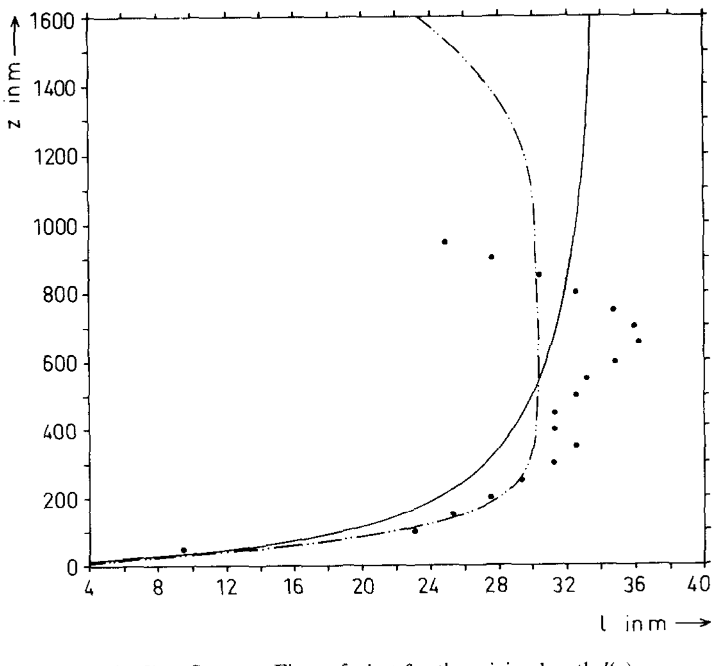

Fig. 5b. Same as Figure 5a but for the mixing length $l(z)$.

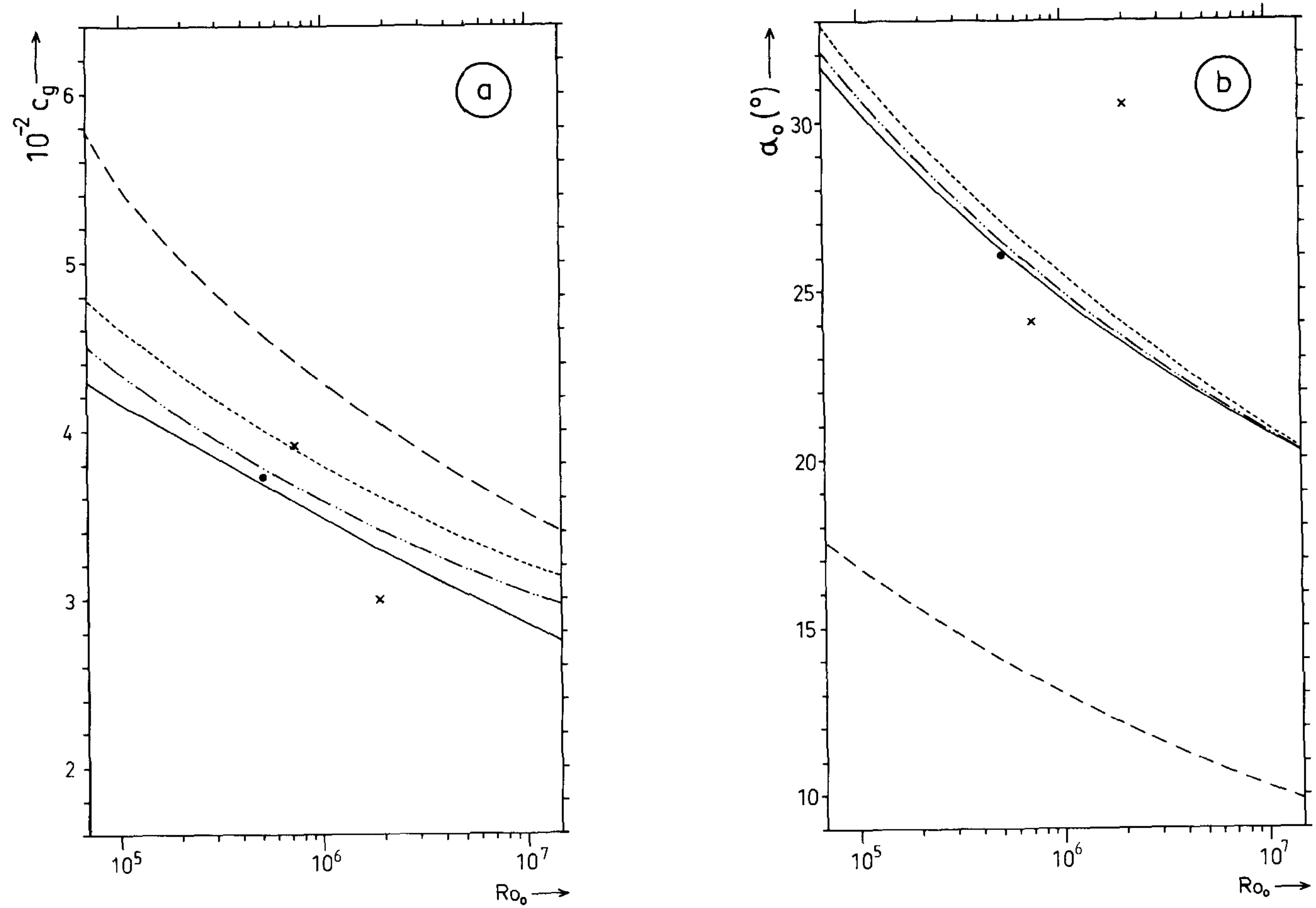

Fig. 6. Geostrophic drag coefficient c_g (a) and cross-isobar angle α_0 (b) as functions of surface Rossby number Ro_0. (———) mixing-length model; (–···–···–) $E-\varepsilon$ model (modified, Equation (20)); (— — —) $E-\varepsilon$ model (standard, Equation (9)); (-----) Resistance law (Wipperman, 1972); (●) Leipzig Data; (×) $E-\varepsilon$ model with fixed Coriolis parameter.

6. Conclusions

A modified $E - \varepsilon$ closure method, replacing the standard dissipation equation of form (9) by a modified version (20), has been applied for simulation of atmospheric profiles in a neutrally stratified boundary layer. The results compare well with observations and results obtained with a more generally used mixing-length closure. Although the modification of the production term cannot be justified from first principles, it yields the desired results, including Rossby-number similarity. The modification as proposed in (20) is no less arbitrary than the standard dissipation equation (9) where the production term has been derived only after considerable manipulations of the full dissipation equation as mentioned before. From the results presented here, one can conclude that the $E - \varepsilon$ closure model can be used for rotating boundary layers only after modification of production or dissipation terms in the dissipation equation (9), which has been applied to many problems of non-rotating turbulent flows with considerable success. The modification as proposed in (17)–(20) is one possible way of adjusting the $E - \varepsilon$ model to the atmospheric boundary layer.

As already pointed out, the standard $E - \varepsilon$ model has been successfully applied to complex engineering flow problems such as flow around obstacles, recirculating flows, etc. To make use of the modified $E - \varepsilon$ model, as proposed here, for complex atmospheric flows, the energy and dissipation equations, (4) and (20), have to be extended to two- or three-dimensional forms.

This can be done easily by adding an advection term to these equations and expanding the production term to a standard two- or three-dimensional form. The resulting eddy viscosity (8) will then be calculated locally at any point of the flow under consideration leading, for example, to a two- or three-dimensional field of eddy diffusivities.

A two-dimensional version of the (standard) $E - \varepsilon$ model has been applied for flow simulations over hills with regard to air quality problems by Lee (1979). An eddy diffusivity evaluated after (8) was used in a diffusion equation for air contaminants. Recently, there has been some application of the $E - \varepsilon$ model to dispersion of heavy gases in the vicinity of surface obstacles by Deaves (1984). With respect to atmospheric dispersion problems from ground-level or elevated sources, the modified $E - \varepsilon$ model might provide some useful estimates for diffusion coefficients especially if applied to flow over complex terrain.

For application of the $E - \varepsilon$ model to real atmospheric flow problems, one has, of course, to consider buoyancy effects too, which have been neglected in our analysis so far. Extension of the standard $E - \varepsilon$ model to stratified flows may be found, among others, in the papers by Gibson and Launder (1976) or Svennson (1980). Although the appropriate form of a stratified dissipation equation of form (9) is still a matter of discussion, an extension to stratified flows has been proposed simply by adding a thermal production term to energy and dissipation equations (4) and (9) respectively. This term results from the Boussinesque-approximation in the well known form:

$$P_t = \frac{g}{\theta_0}\overline{w'\theta'} = -\frac{g}{\theta_0}\alpha_h K \frac{\partial\bar{\theta}}{\partial z}$$

where $\alpha_h = K_h/K$ is an inverse turbulent Prandtl number and θ is the potential temperature.

This thermal production term P_t is added to the right-hand side of equation (4) for turbulent kinetic energy and also to the dissipation equation (9) or (20). Hence the production term P is replaced in (9) or (20) by $P = P_m + c_3 P_t$, where P_m is the mechanical production term as defined in (10). The additional constant c_3 is not as standard in the literature as are the others for non-stratified flows. Hence for turbulence modelling of stratified flows with the aid of the $E - \varepsilon$ closure model, some adjustment to the thermal production term $c_3 P_t$ has to be made.

With respect to the atmospheric boundary layer, many methods have been proposed for taking into account stratification into turbulence parameterisation even for simple mixing-length models. Although well known Monin-Obukov similarity laws exist for surface-layer turbulence, the influence of stratification on turbulence in the middle and upper part of the atmospheric boundary layer is not as easy to determine. The same problem exists for extension of the modified $E - \varepsilon$ model to stratified boundary layers. Work is currently in progress to extend this model to stably stratified boundary layers, where even less observational data are available. Here again modifications of the $E - \varepsilon$ model with respect to thermal stratification have to be verified against observational data, as has been done for neutral boundary layers in the paper presented here.

Finally a critical remark on the use of a dissipation equation like (9) for turbulence parameterisation by Hasse (1978) should also be mentioned. He questions whether the dissipation equation is a separate equation at all, and not dependent on the energy equation. Indeed the structure of both equations is quite similar, but this is also true for other length-scale equations, which have been used for turbulence modelling so far. As pointed out by Mellor and Yamada (1982), the weakest point in even higher-order closure methods is perhaps the length-scale equation, independant of the different forms chosen by different authors. This is also true for the use of a dissipation equation instead of a length-scale equation. Hence one might regard the modified $E - \varepsilon$ model as a closure method which gives similar results to other methods in use, but can be easily applied to complex atmospheric flow problems.

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