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Randomized decision rules in voting games: a model for strict proportional power

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Abstract. The concept of strict proportional power is introduced, as a means of formalizing a desire to avoid discrepancy between the seat distribution in a voting body and the actual voting power in that body, as measured by power indices in common use. Proportionality is obtained through use of a randomized decision rule (majority rule). Some technical problems which arise are discussed in terms of simplex geometry. Practical implications and problems in connection with randomized decision rules are indicated.

1. Introduction

At the very heart of the principle of representational democracy lies the idea that voters' preferences should be faithfully reflected in the decision making of representational bodies. However, it is widely known that the distribution of actual voting power seldom reflects the distribution of voting weights in a voting body, e.g., the distribution of seats in a parliament.

In designing a voting system (committee, parliament, etc), a major problem is the size of the representational body; how to represent a huge electorate through a rather small decision making unit. Numerical problems involved in this "reduction process" have received considerable attention in political science literature (see, e.g., Rokkan (1968), Rae et al. (1971), Lijphart and Gibbard (1977), Laakso and Taagepera (1978), and Laakso (1979)). Published analyses unanimously show that common electoral formulas, such as d'Hondt, Imperial and St. Laguë, strongly influence the thresholds of representation, as well as the relationship between the proportion of votes received and the number of seats granted.

In this paper we are not primarily concerned with distortions caused by a lack of agreement between vote shares in an electorate and the number of seats in an assembly. Instead we are interested in the distorting effects on a priori voting power which are directly attributable to a fixed decision rule, such as simple majority rule.

The fact that the distribution of a priori voting power – as measured by power indices in common use – in representational bodies may deviate substantially from the distribution of seats, has been demonstrated by means of empirical data (see, e.g., Shapley and Shubik (1954), Frey (1968), Holler and Kellerman (1977), Dreyer and Schotter (1980)), as well as with the aid of mathematical arguments (see, e.g., Brams (1975), Brams and Affuso (1976), Fisher and Schotter (1978), and Holler (1982a)). The results thus obtained are a challenge to a basic tenet of representational democracy.

As a partial remedy for this shortcoming of voting systems having a fixed decision rule, Holler (1982b) recently proposed the adoption of randomized decision rules as a means of equating, on the average, the distribution of a priori voting power and the actual seat distribution within the voting body. The purpose of the present paper is to further discuss and formalize Holler's approach.

The paper is organized as follows. In section 2, notations and concepts used in modelling simple games are given, and the well-known Shapley-Shubik and Banzhaf indices are introduced as measures of a priori voting power. With a view to formalizing Holler's approach, the concept of a randomized voting game and that of strict proportional power are defined. Section 3 is devoted to the problem of obtaining strict proportional power in an n-member voting body. Starting from a numerical example given by Shapley and Shubik (1979), we examine certain problems in connection with strict proportional power, using arguments from simplex geometry. Finally, in section 4 we discuss certain practical implications of, and alternatives to randomized decision rules for voting bodies.

2. Simple games, power measures and strict proportional power

Assemblies or committees making decisions by means of voting and a majority rule – voting systems – are conveniently modelled using the concept of a simple game (cf. Shapley (1962), von Neumann and Morgenstern (1947)). Formally, a simple game is a set of players (members of the committee): $N = \{1, 2, \dots, n\}$, together with the set of winning coalitions: W. This set is a collection of subsets of N, or coalitions, with the following properties:

1. $\emptyset \notin W$
2. $N \in W$
3. if $S \in W$ and $S \subseteq T$, then $T \in W$.

A simple game described by the pair $[N, W]$ can also be given by its characteristic function ν, defined for all subsets S of N, and such that

$$\nu(S) = \begin{cases} 1 & \text{if } \quad S \in W \\ 0 & \text{otherwise.} \end{cases} \qquad [1]$$

For a weighted voting game, the set W is determined by

1. the distribution of voting weights $w_1, w_2, \dots, w_n$, $\Sigma w_i = 1$;
2. the decision rule, or quota, $d \in (0, 1)$.

For a weighted voting game, a subset $S \in W$ if and only if $\Sigma_{i \in S}\, w_i \geqslant d$.

There are two well-known and widely used indices for measuring a priori power in voting systems, the Banzhaf index and the Shapley-Shubik index (for a discussion and also modifications of these measures, see, e.g., Coleman (1971) and Packel and Deegan (1980)). Both indices make use of the difference

$$\Delta_i(S) = \nu(S) - \nu(S - i) \qquad [2]$$

where S is an arbitrary coalition and i indexes the player whose power we wish to measure. If the removal of player i turns a winning coalition S into a losing one, then the difference [2] is equal to 1, otherwise it will be equal to 0. the two indices differ with respect to how the differences [2] are weighted (see, e.g., Stenlund and Lane (1984)).

The formula for the Shapley-Shubik index is

$$\varphi_i = \sum_{S \subset N} \frac{(n-s)!(s-1)!}{n!} \Delta_i(S), \quad i = 1,2,\ldots,n \qquad [3]$$

where s is the cardinal number of the subset S. The vector of players' indices will be written $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$. By its definition, the Shapley-Shubik index is normalized, i.e. $\Sigma\varphi_i = 1$. The corresponding formula for the Banzhaf index is

$$\beta_i = \frac{1}{c} \sum_{S \subset N} \Delta_i(S), \quad i = 1, 2, \ldots, n \qquad [4]$$

where the constant c is chosen so that the index is normalized, i.e. $\Sigma\beta_i = 1$. The vector of Banzhaf indices will be denoted $\beta = (\beta_1, \ldots, \beta_n)$.

Although the indices [3] and [4] have a similar algebraic structure, they may be deduced using seemingly quite different arguments. To illustrate this, consider first a permutation of the set of players N: $\{\pi(1), \ldots, \pi(n)\}$. Now if coalitions are formed by starting with $\{\pi(1)\}$, and then successively adding new numbers in turn: $\{\pi(1), \pi(2)\}$ and so forth, eventually a winning coalition will arise. The player who accomplishes this is called pivotal. It is readily seen that the Shapley-Shubik index [3] is precisely equal to the number of permutations in which player i is pivotal, divided by the total number of permutations. The Banzhaf index [4], on the other hand, is based on the concept of a swing. When the difference [2] is equal to 1, we speak of a swing for player i. The sum in formula [4] is thus the number of swings for player i, and the normalizing constant c is the total number of swings.

We note that for a given voting game, there is absolutely no need for a vector of power indices to coincide with, or even be roughly equal to the distribution of voting weights. This is clearly brought out by the following numerical example. Consider the voting game $[d = 0.55;\ w = (0.5, 0.4, 0.1)]$. The Shapley-Shubik and Banzhaf index vectors are $\varphi = (2/3, 1/6, 1/6)$ and $\beta = (3/5, 1/5, 1/5)$, respectively.

When applied to actual voting bodies, the Shapley-Shubik and Banzhaf indices often agree reasonably well. They tend to differ considerably, however, when applied to what is called nearly oceanic games, i.e. games with one or a few powerful players and a great number of weak players.

Consider now a set of players $N = \{1, 2, \ldots, n\}$ with fixed weights $w = (w_1, w_2, \ldots, w_n)$. Let D be a set of decision rules, $D \subseteq (0.5, 1)$, and Q a probability measure over D. Instead of assuming a fixed decision rule, we now let the decision rule vary at random over the interval (0.5, 1), or a subset thereof, according to the probability measure Q. We will refer to the triplet $[d \in D;\ w;\ Q]$ as a randomized voting game.

For a randomized voting game, the appropriate power measure is the average, or expected (a priori) voting power for player i. Thus using the Shapley-Shubik index [3], we would have

$$\bar{\varphi}_i = \int_D \varphi_i(d)\, dQ, \quad i = 1, 2, \ldots, n \qquad [5]$$

as expected power for player i, where $\varphi_i\ (d)$ is the voting power of player i in the game $[d;\ w]$. For a discrete set of decision rules, $D = (d_1, d_2, \ldots, d_k)$, the expected power is simply the weighted average of Shapley-Shubik or Banzhaf indices:

$$\bar{\varphi}_i = \sum_{\nu=1}^{k} \varphi(d_\nu) q_\nu, \quad i = 1, 2, \ldots, n$$

where $q_1, \ldots, q_k$ are the probabilities with which the games $[d_\nu;\ w]$, $\nu = 1, \ldots, k$, are played.

We are now in a position to give a precise definition of the concept of strict proportional power. Consider the randomized voting game $[d \in D;\ w;\ Q]$. If the set D and the probability measure Q are such that the expected a priori power [5] of each player is exactly equal to his voting weight, i.e. if $\bar{\varphi}_i = w_i$ for each $i \in N$, then we have a case of strict proportional (Shapley-Shubik) power. For a given pair $[d \in D;\ w]$, it may be possible to find a measure Q to accomplish strict proportional power. This problem will be further discussed in the following section.

Two further definitions will be useful in discussing some implications of strict proportional power in voting games. A decision rule d is a dictator rule if, for a given set of voting weights, it assigns power index = 1 to one of the players, while the others are given index = 0. A decision rule d, finally, is a veto rule if $w_i \geqslant d$ for all $i \in N$. This implies power index $= 1/n$ for all n players.

3. Methods for obtaining strict proportional power

Shapley (1962) proved that what we call strict proportional power for a voting game is obtained if the decision rule d is a random variable, uniformly

Table 1.

d =	Voting weights			
	1	2	3	4
	β'_1	β'_2	β'_3	β'_4
1, 10	1	1	1	1
2, 9	0	2	2	2
3, 8	1	1	3	3
4, 7	1	1	3	5
5, 6	1	3	3	5
Σ	4	8	12	16

distributed over the interval (0, 1), and if power is measured by the Shapley-Shubik index. In other words, the randomized voting game $[d \in (0, 1);\ w;\ Q]$, with Q uniform, is characterized by strict proportional power, as conceived here. More recently, Dubey and Shapley (1979) have shown that this property also holds when power is measured by the Banzhaf index.

The following is a slightly simplified version of a numerical example discussed by Dubey and Shapley (1979). Consider the class of voting games $[d;\ (1, 2, 3, 4)]$, where $d = 1, 2, \ldots, 10$. Table 1 shows the raw swing scores required for calculating the Banzhaf index.

Following Dubey and Shapley, we note that choosing the games listed in Table 1 with equal probability leads to an average power for the players equal to relative voting weight. More specifically, if we play one of the voting games $[d;\ (1, 2, 3, 4)]$, where $d = 6, 7, 8, 9$ and 10, chosen with uniform probability = 1/5, then the expected Banzhaf indices will be 1/10, 2/10, 3/10 and 4/10, i.e., we have strict proportional power for the randomized voting game.

It is interesting to note, however, that in order to obtain expected voting power equal to relative voting weight, it suffices in this case to alternate between the games [6; (1, 2, 3, 4)] and [8; (1, 2, 3, 4)] with probability 1/2. The point we wish to make here is that for a voting game with n players we do not need as many as $n(n+1)/2$ games to achieve strict proportional power.

Let us now briefly examine what occurs if we instead measure voting power by the Shapley-Shubik index. In Table 2 we have listed the same set of games as before, now with entries proportional to Shapley-Shubik scores. Again, as the table shows, we can play the games listed with uniform probability and have expected Shapley-Shubik power indices equal to relative voting weight.

If we look for a solution with fewer than 5 games, then we see that there is no probability combination of just two voting games giving strict proportional power. Let us therefore try the four games $[d;\ (1, 2, 3, 4)]$, $d = 6, 7, 8$, and 9, played with probabilities given by a vector $q = (q_1, q_2, q_3, q_4)$, $\Sigma q_i = 1$. Although the corresponding four vectors of Shapley-Shubik indices are linearly independent, there is no solution in the form of a probability vector, i.e. a vector with non-negative coordinates. If we instead consider $[d;\ (1, 2, 3, 4)]$, with $d = 7, 8, 9$ and 10, then it is easily confirmed that $q =$

Table 2.

	Voting weights			
	1	2	3	4
d = 12x	φ_1	φ_2	φ_3	φ_4
1, 10	3	3	3	3
2, 9	0	4	4	4
3, 8	1	1	5	5
4, 7	1	1	3	7
5, 6	1	3	3	5
Σ	6	12	18	24

(0.30, 0.15, 0.30, 0.25) provides a solution, i.e. a randomized voting game with strict proportional expected voting power.

Certain problems encountered in connection with voting games and strict proportional power are conveniently discussed using geometrical arguments. In particular, n-dimensional simplex arguments are useful. First, let S denote the 3-dimensional, ordered simplex

$$S = \{x = (x_1, x_2, x_3): x_1 \geqslant x_2 \geqslant x_3 \geqslant 0,\ x_1 + x_2 + x_3 = 1\} \qquad [5]$$

Geometrically, the simplex [5] corresponds to the triangle ABC in Fig. 1 below, where the coordinates of a point are the distances to the opposite sides of the larger triangle. Any voting game with weights $w = (w_1, w_2, w_3)$ ordered and summing to 1, can be represented by a point in S or, equivalently, by a point in the triangle ABC. Furthermore, players' power indices, when normalized, can be represented as points in the simplex. Thus in Fig. 1, the points on the boundary A, B, C and D represent the four possible Banzhaf index vectors, namely (1/3, 1/3, 1/3), (1/2, 1/2, 0), (1, 0, 0), and (3/5, 1/5, 1/5). A and C correspond to the veto rule and the dictator rule, respectively.

Consider a weighted voting game $[d; w]$ with w in BCD. If we let the decision rule d vary from 0.5 to 1, the Banzhaf index points are A, B, C, and D. If equality, on the average, between relative weights and voting power is desired, then there are several possibilities. Thus, for example, we can choose three d-levels, d_1, d_2, d_3, such that the players' Banzhaf indices correspond to the points B, C, and D. The probabilities $q = (q_1, q_2, q_3)$ with which to play the voting games $[d_i; w]$, $i = 1, 2, 3$, will be given by

$$q_1 = 2(w_2 - w_3)$$
$$q_2 = w_1 - w_2 - 2w_3$$
$$q_3 = 5w_3.$$

Alternatively, we can take three d-levels corresponding to Banzhaf index points A, B, and C. However, there is no solution if we take d-levels generating Banzhaf points A, B, and D, since the point w we consider is exterior to the set of points forming the triangle ABD. This is what happened above in connection with Table 2: we had four different d-values giving four

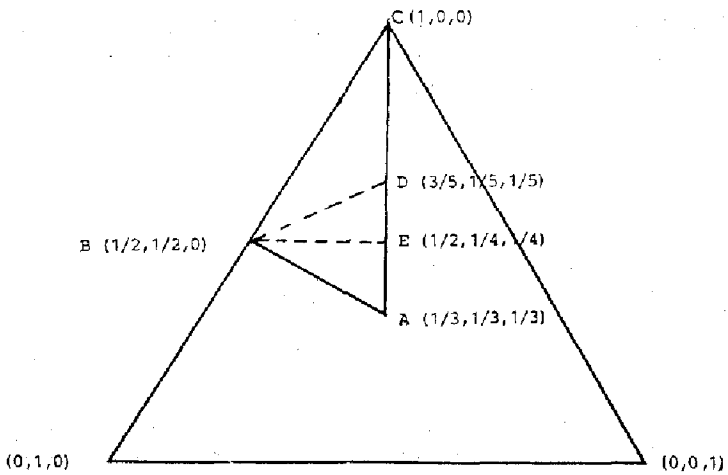

Figure 1.

distinct Banzhaf vectors. However, the vector of voting weights w was exterior to the simplex generated by the Banzhaf vectors.

Suppose next we have a voting game with weights w in ABD, and strict proportional power is desired. We can then make use of the fact that the point w is interior to either of the two triangles ABC and ABD. We can thus achieve the desired proportionality without having to resort to the dictator rule. It is worth noticing that for any weight vector w in BED, we need not make the strongest player a dictator. Although, for points in BEC, w_1 is greater than 1/2 and simple majority voting thus would make player 1 dictator, we can still attain strict proportional power through a linear (probability) combination of voting games, having the points A, B, and D as Banzhaf vectors. However, we would have to include the veto rule, since the point A must be used in the linear combination.

More generally, if we extend the idea behind the triangle ABC in Fig. 1 to the n-dimensional case, we have a closed, ordered simplex:

$$S^{(n)} = \left\{ x = (x_1, \ldots, x_n) : x_1 \geq \ldots \geq x_n \geq 0, \; \sum x_i = 1 \right\} \qquad [6]$$

All linear combinations of points in the simplex [6] remain in the simplex, provided the coefficients are non-negative and sum to unity. Any vote distribution $w = (w_1, \ldots, w_n)$ of a voting game, with weights ordered and normalized, corresponds to a point in $S^{(n)}$, as do the associated power index vectors.

The extreme points, or corner points, of the simplex (6) are given by the vectors: $\alpha_1 = (1, 0, \ldots, 0)$, $\alpha_2 = (1/2, 1/2, \ldots, 0), \ldots, \alpha_n = (1/n, 1/n, \ldots, 1/n)$. This set of vectors spans the simplex (6) and forms a convenient basis for it. Hence for any point $\underset{\sim}{x} \in S^{(n)}$, there is a unique representation:

$$x = q_1\alpha_1 + q_2\alpha_2 + \ldots + q_n\alpha_n \qquad [7]$$

In specific applications, the coefficients q_i of [7] can be interpreted as probabilities, being non-negative and summing to unity. The coefficients are best determined recursively using the formulas:

$$q_i = \begin{cases} i(x_i - x_{i+1}), & \text{if } i = 1, 2, \ldots, n-1 \\ nx_n, \text{ if } i = n. \end{cases} \qquad [8]$$

Consider now the sequence of simple games on $N = \{1, 2, \ldots, n\}$ for which we have the following sequence of minimal winning coalitions

$$W_1^m = \{1\}, \quad W_2^m = \{1, 2\}, \ldots, \quad W_n^m = \{1, 2, \ldots, n\} \qquad [9]$$

The reader will recognize what is called unanimity games, or pure bargaining games, with dummy players added. The corresponding power vectors, Banzhaf or Shapley-Shubik, are precisely the corner points of the simplex [6], i.e. the vectors $\alpha_1, \ldots, \alpha_n$. We note that it is always possible to obtain strict proportional power by resorting to a random mixture of the pure bargaining games defined by [9], with the mixing probabilities given by [8]. The number of simple games on a fixed set of n players, $N = \{1, 2, \ldots, n\}$, increases rapidly with increasing n, since we are dealing with sets of sets. Thus there are numerous other ways of replacing a given weighted voting game by a mixture of simple games, such that proportionality between voting weights and expected voting power is obtained.

If a solution is desired in the form of a randomized voting game, we must find n d-levels: $D = \{d_1, d_2, \ldots, d_n\}$, with corresponding power vectors, say $\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(n)}$, which are linearly independent, and such that the vector of voting weights, $w = (w_1, w_2, \ldots, w_n)$, is interior with respect to the simplex generated by the n power vectors. The voting weight vector being interior, guarantees the existence of a probability vector $q = (q_1, q_2, \ldots, q_n)$, where q_i, $i = 1, 2, \ldots, n$, is the probability with which to play the voting game $[d_i; w]$.

4. Discussion

The Banzhaf index has been applied to the representation of legislative districts in the State of New York (see Grofman and Scarrow (1979)). The New York State courts have in fact explicitly endorsed the use of the Banzhaf index as a measure of legislative power, whereas the US Supreme court has rejected this index as unsuitable for measuring voting power in the case of single and multimember districts. However, let us for the moment put aside the question of choice of an appropriate power index. Then, as our analysis shows, given a power measure there is always a way of accomplishing strict proportionality between a priori power and voting weight in a voting body, provided the majority rule is allowed to vary. However, objections to the adoption of a randomized decision rule may be raised on technical or practical grounds, as well as on psychological grounds. Let us therefore take a closer look at some of the possible objections.

It might be argued that simple majority voting possesses several appealing features, which we forego by admitting alternative decision rules in a probability mixture. If, for example, a two thirds majority is required, then a minority can block a decision. But the blocking of a motion could be regarded as a decision or a vote for *status quo*, or a vote for the continuation of a voting bargaining process. Furthermore, simple majority voting minimizes the expected total disappointment of the voters, an appealing property which was shown by Curtis (1972). On the other hand, Buchanan and Tullock (1962, p. 81) in their analysis of decision making costs in voting bodies find that:

> On a priori grounds there is nothing in the analysis that points to any uniqueness in the rule that requires simple majority to be decisive. The $(N+1)/2$ point seems, a priori, to represent nothing more than one among the many possible decision rules, and it would seem very improbable that this rule should be "ideally" chosen for more than a very limited set of collective activities.

Buchanan and Tullock take into account the external costs for the individual and the fact that the voting body may vote against his preferences. However, in accordance with Curtis, they also show that under specific assumptions, simple majority voting minimizes the decision costs for the individual.

Lastly, we may share the outlook of Grotius, the Father of Law (see Gough, 1957, p. 81), who endeavored to make a natural law out of the right of the majority. According to this view, the majority should remain a majority and decisions must be made by majorities, regardless of what representational procedure we use; minorities should thus be prevented from having an influence on the outcome of law-making and governing. Of course, this view is incompatible with the use of decision rules not based on simple majority, and therefore also with the application of randomized decision rules, which do lend influence to minorities. The rationale behind the concept of a "sacred majority" is the idea that a larger number of voters is more likely to choose the best of a limited set of alternatives than a smaller number of voters. However, to invoke a statistical likelihood principle to justify simple majority voting presupposes the existence of a truly best political decision which can be identified, or at least approximated by a majority decision.

Political reality indicates that we do not always content ourselves with the "natural law" of simple majority; especially not when it involves amending a constitution. Indeed, some European parliaments make use of one, or even two qualified majority rules, in addition to simple majority rule. In general, the adoption of alternative decision rules is motivated by the importance of the decision to be taken. Thus, to cite an example, in the Finnish parliament a bill on a constitutional law may be declared as urgent by a 5/6 majority, and then be approved by a two thirds majority in the same parliament. Or, alternatively, the bill may be approved by a simple majority for being held. Then, after elections for a new parliament, the bill must be approved by a two thirds majority. Although in this example several different decision rules are used, one might hesitate to accept this as support for the idea of a randomized decision rule. In the Finnish example, this particular mixture of decision rules is based upon legal and institutional determinism, and no stochastic element is involved.

A practical difficulty with the adoption of a randomized decision procedure concerns the stability and continuity of the decision making process. Decisions made by a voting body often form part of an integrated whole, a policy. Such continuity of policy is difficult to obtain if the coalition structure varies in an unpredictable fashion from one vote to the next. In theory, it would suffice to make the random choice of a decision rule at the beginning of an election period, when the seat distribution is determined. In practice, however, it might be preferable to change the decision rule at shorter intervals. Over a period of time, the voting power actually obtained by the players is then likely to be close to the voting weights or seat distribution. Normally, as our analysis shows, it would be possible to avoid including decision rules, which might endanger the functioning of a political system, such as the veto rule.

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