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Optimal Production Control of a Dynamic Two-Product Manufacturing System with Setup Costs and Setup Times

MOHSEN ELHAFSI[1] and SHERMAN X. BAI[2]
[1]*The A. Gary Anderson Graduate School of Management, University of California at Riverside, Riverside CA 92521, U.S.A.;* [2]*Department of Industrial and Systems Engineering, University of Florida, Gainesville, FL 32611, U.S.A.*

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Abstract. This paper deals with the optimal control of a one-machine two-product manufacturing system with setup changes, operating in a continuous time dynamic environment. The system is deterministic. When production is switched from one product to the other, a known constant setup time and a setup cost are incurred. Each product has specified constant processing time and constant demand rate, as well as an infinite supply of raw material. The problem is formulated as a feedback control problem. The objective is to minimize the total backlog, inventory and setup costs incurred over a finite horizon. The optimal solution provides the optimal production rate and setup switching epochs as a function of the state of the system (backlog and inventory levels). For the steady state, the optimal cyclic schedule is determined. To solve the transient case, the system's state space is partitioned into mutually exclusive regions such that with each region, the optimal control policy is determined analytically.

Key words: Dynamic setups, production and setup control, optimal control.

1. Introduction

The goal of the Production and Setup Scheduling Problem (PSSP) is to determine the optimal production rates and setup epochs of several products on a single machine. The latter is assumed to have controllable production rates. Each product has known constant demand rate and processing time. When production is switched from one product to the next, a constant setup time as well as a fixed setup cost are incurred. Backlog is allowed and the system is not necessarily in steady state. The objective is to control the production rate of each product as well as to control the setup change epochs so as to minimize the total setup, inventory and backlog costs over a finite or infinite planning horizon. The problem reduces to the Economic Lot Scheduling Problem (ELSP), when the planning horizon is infinite, the system is in steady state, the machine has fixed production rates, no backlog is allowed, and the objective is to determine lot sizes that minimize the average setup and inventory holding costs per unit time. Despite a large amount of research work on the ELSP, an optimal solution approach has not been proposed yet. Rather, good (some times excellent) heuristics have been suggested. A comprehensive review of the ELSP through 1976 is given in Elmaghraby (1978). Recent work on the ELSP includes the work of Goyal (1984), Roundy (1985), Dobson (1987), Gallego (1989), Jones and Inman (1989), and Carreno (1990). Recently, researchers have dropped the assumption of fixed production rates at the machine and have improved the ELSP model using controllable production rates. Recent work along this new direction includes the work of Buzacott and Ozkarahan (1983), Silver (1990), Moon *et al.* (1991) and Elhafsi and Bai (1996a).

The PSSP is concerned with the study of the system during its transient and steady state periods, throughout the planning horizon. In this paper, we consider a finite planning horizon and assume that the steady state is finite time achievable. This model may have many applications in practice. For instance, consider the following situation: We would like to plan the production of several products, periodically (say quarterly), on a bottleneck machine. The demand rate for each period of time is forecasted at the beginning of that period and should be satisfied during the planning period. The machine has enough capacity to meet the forecasted demand for each product and is fast enough to bring the inventory/backlog of all products to a steady state (which constitutes the most economical way of operating the system). Now, at the end of the planning horizon, the inventory and backlog of the products is checked and possibly different forecasted demand rates are to be used in the next planning horizon. It is clear that the output (inventory/backlog) of the previous period constitutes the input for the current period and hence new initial conditions as well as new parameters are to be used. As can be seen, at the beginning of each planning period we are faced with the problem of optimally driving the system to its steady state production cycle. Many other factors can change the conditions of the system and might drive it away from its steady state.

The PSSP is formulated as a feedback control problem. The control must respond to certain initial disruptions so as to minimize a certain criterion. This kind of formulation is usually classified under *production flow control* models (see Olsder and Suri (1980) and Kimemia and Gershwin (1983)). Using the production flow control formalism, Sharifnia *et al.* (1991) investigated a single machine setup scheduling problem. They proposed a feedback setup scheduling policy which uses corridors in the surplus/backlog space to determine the epochs of setup changes. The corridors are chosen so as to guide the surplus trajectory to a target cycle which is referred to as the *Limit Cycle*. Srivatsan and Gershwin (1990) extended the ideas of Sharifnia *et al.* and developed methods for choosing the parameters of the corridors when the setup frequencies are not all the same. Caramanis *et al.* (1991) derived the optimality conditions for setup changes and solved them numerically for a two-part type system using a quadratic cost criterion. Hu and Caramanis (1992, 1995) solved the three-part type setup problem numerically and deduced structural properties of the optimal policies. Based on the numerical results, they proposed near-optimal policies. Perkins and Kumar (1989) and Kumar and Seidman (1990) studied the performance of distributed real-time setup scheduling policies and investigated the conditions under which the system remains stable. Connolly (1992) proposed a heuristic for a two-part-type one-machine setup system, based on known results of a perfectly flexible system. Bai and Elhafsi (1993) studied the real-time scheduling of an unreliable one-machine two-part-type non-resumable setup system. They provided a continuous dynamic programming formulation of the problem which they discretized and solved numerically. Based on the numerical solution they provided two heuristics to solve the stochastic problem. Gallego (1989) studied the ELSP problem in the case of a machine subject to disruptions of small magnitude. He shows that the optimal policy selects the production lot sizes as a linear function of the current inventory levels.

In this paper, we study the production and setup control of a deterministic one-machine two-part- type system within a feedback control framework. Some of the results used in this paper are based on previous works by the authors (Bai and Elhafsi (1996), Elhafsi and Bai (1996a) and (1996b)), where a similar system was studied. The remainder of the paper is organized as follows: In Section 2, we present an optimal control formulation of the PSSP. In Section 3, we establish some preliminary results. In Section 4, we determine the optimal steady state production cycle of the system. The optimal transient behavior is presented in Section 5. In Section 6, we study the special case of zero setup times. We conclude our study with Section 7.

2. Problem Formulation

Consider a one-machine manufacturing system producing two distinct parts (or products) each has a constant demand rate $d_i (i = 1,2)$. When production is switched from Part Type j to Part Type i ($j \neq i$), a given constant setup time δ_i and setup cost $k_i (i = 1,2)$ are incurred. Our formulation follows the general framework introduced by Kimemia and Gershwin (1983), where the production flow is modeled as continuous rather than discrete. Let $x_i(t)$ be the production surplus of Part Type $i (i = 1,2)$ at time t; a positive value of $x_i(t)$ represents inventory while a negative value represents backlog. Let $u_i(t)$ be the controlled production rate of the machine producing Type i parts at time t. Let $\sigma(t) = (\sigma_1(t), \sigma_2(t), \sigma_{12}(t), \sigma_{21}(t))$ be the setup state vector of the machine at time t. Where, $\sigma_i(t), \sigma_{ij}(t) (j \neq i, i = 1,2, j = 1,2)$ are right continuous binary functions of t, such that $\sigma_i(t) = 1$ when the machine is ready to produce Type i parts and $\sigma_i(t) = 0$ otherwise; $\sigma_{ij}(t) = 1$ when the machine is undergoing a setup change from Part Type j to Part Type i and $\sigma_{ij}(t) = 0$ otherwise. Let $s(t)$ be a nonnegative right continuous function of t which takes on the value δ_i at the beginning of each setup change to Part Type i ($i = 1,2$) and decreases with time. $s(t)$ indicates whether a setup is completed or not. We assume that initially the machine is not set up for either part type.

2.1. System Dynamics and Constraints

The dynamics of the system can be described by:

$$\frac{dx_i(t)}{dt} = u_i(t) - d_i, \quad i = 1,2 \tag{1}$$

$$0 \le u_i(t) \le U_i\sigma_i(t), \quad i = 1,2 \tag{2}$$

where U_i is the maximum production rate of the machine when it is producing Type i parts.

The setup states of the machine obey the following set of constraints:

$$\sigma_1(t) + \sigma_2(t) + \sigma_{1,2}(t) + \sigma_{2,1}(t) = 1; \tag{3}$$

$$\text{if } \sigma_i(t^-) = 1 \text{and } \sigma_i(t) = 0, \text{ then } s(t) = \delta_j \text{ and } \sigma_{ij}(t) = 1; \tag{4}$$

$$\text{if } s(t^-) > 0 \text{ and } \sigma_{ij}(t^-) = 1, \text{ then } \dot{s}(t) = -1 \text{ and } \sigma_{ij}(t) = 1; \tag{5}$$

$$\begin{aligned} &\text{if } s(t^-) = 0 \text{ and } \sigma_{ij}(t^-) = 1, \text{ then } \sigma_{ij}(t) = 0 \\ &\quad \text{and } s(t) = 0 \text{ and } \sigma_j(t) = 1; \end{aligned} \tag{6}$$

for $i = 1, 2, j = 1, 2, i \neq j$. Where $\dot{s}(t)$ denotes the time derivative of $s(t)$.

2.2. Penalty Function

For mathematical convenience, we assume that setup costs are incurred at a constant rate $\chi_i = k_i/\delta_i$ (i=1,2) dollars per unit time, during a setup change. Hence, at the end of a setup change to Part Type i, we would have a total setup cost of k_i. The instantaneous cost which penalizes the production for being ahead of (i.e., $x_i > 0$) or being behind (i.e., $x_i < 0$) the demand is given by:

$$h(x) = \sum_{i=1}^{i=2}(c_i^+ x_i^+(t) + c_i^- x_i^-(t)),$$

where c_i^+ and c_i^- are the per unit instantaneous inventory holding and backlog costs respectively, and $x_i^+(t) = \max\{x_i(t), 0\}$ and $x_i^-(t) = \max\{-x_i(t), 0\}$. The total instantaneous cost is then given by:

$$\begin{aligned} g(x,\sigma) &= h(x) + \sum_{i=1, j\neq i}^{i=2} \chi_i \sigma_{ji}(t) \\ &= \sum_{i=1, j\neq i}^{i=2} (c_i^+ x_i^+(t) + c_i^- x_i^-(t) + (k_i/\delta_i)\sigma_{ji}(t)). \end{aligned} \tag{7}$$

2.3. State Variables and Control Variables

The state variable of the system is given by the vector $x(t) = (x_1(t), x_2(t))$. The variables $u(t) = (u_1(t), u_2(t))$ and $\sigma(t) = (\sigma_1(t), \sigma_2(t), \sigma_{12}(t), \sigma_{21}(t))$ are the control variables. We denote by (σ, u) the complete control vector.

2.4. Capacity Set

The capacity set represents the set of feasible production rates. When the setup state is $\sigma(t)$ at time t, it is given by:

$$\Omega(\sigma(t)) = \{u(t)|0 \leq u_i(t) \leq U_i\sigma_i(t), i = 1, 2\}.$$

2.5. Setup Constraints Set

The setup constraints set is the set of all possible setup vectors $\sigma(t) = (\sigma_1(t), \sigma_2(t), \sigma_{12}(t), \sigma_{21}(t))$ satisfying constraints (3)–(6). Let Φ denote this set.

2.6. Admissible Control Policies

We denote by $\Xi(\Phi, \Omega)$ the set of feasible controls, which depends on Φ and Ω. The set of admissible control policies, $\mathcal{A}$, is the set of all mappings $\mu, \mu : \mathcal{R}^2 \rightarrow \Xi(\Omega, \Phi)$ which satisfy $\mu(x) = (\sigma, u)$ and which are piecewise continuously differentiable. These admissible control policies are feedback controls that specify the control actions (setup and production rate of the machine) to be taken, given the state of the system.

2.7. Objective Function

The objective is to determine an optimal control policy $\mu^* \in \mathcal{A}$, corresponding to a setup control $\sigma^* = (\sigma_1^*, \sigma_2^*, \sigma_{12}^*, \sigma_{21}^*)$ and a production flow rate control $u^* = (u_1^*, u_2^*)$, that minimizes for each initial state $x(t)$ the following cost function:

$$J_\mu(x(t), t) = \int_t^{tf} g(x(s), \sigma(s))\, \mathrm{d}s \tag{8}$$

where the minimization is over all functions $\mu(x(\tau)) = (\sigma(\tau), u(\tau))$, such that $x(\tau)$, $\sigma(\tau)$ and $u(\tau)$ satisfy constraint (1) and $\sigma(\tau), u(\tau)) \in \Xi(\Phi, \Omega)$ for $t \leq \tau \leq t_f$, where t_f is assumed to be sufficiently large.

3. Preliminary

It can be shown (see Bai and Elhafsi, 1996) that the optimal solution to the above problem can be obtained in two parts by considering a transient period and a steady state period. The steady state corresponds to the case where the state of the system (inventory/backlog) has already reached a cyclic schedule, where the produced lots for each part type are of constant size over time. The transient period corresponds to the case where the state of the system still has not reached the cyclic schedule. In this section, we state and prove a theorem that will allow us to reduce the set of feasible production rates from an infinite set to a finite one with only three possible production rates.

Let t_s be the time instant the system reaches the steady state. The total cost can then be written as follows:

$$\begin{aligned} J_\mu(x(t),t) &= \int_t^{t_s} g(x(s),\sigma(s))\,\mathrm{d}s + \int_{t_s}^{t_f} g(x(s),\sigma(s))\,\mathrm{d}s \\ &= J_\mu^T(x(t),t) + (t_f - t_s)J_\mu^S(x(t_s),t_s). \end{aligned} \tag{9}$$

We refer to $J_\mu^T(x(t),t)$ as the transient cost,

$$J_\mu^T(xt),t) = \int_t^{t_f} g(x(s),\sigma(s))\,\mathrm{d}s \tag{10}$$

and $J_\mu^S(x(t_s),t_s)$ as the average steady state cost,

$$J_\mu^S(x(t_s),t_s) = (t_f - t_s)^{-1}\int_{t_s}^{t_f} g(x(s),\sigma(s))\,\mathrm{d}s \tag{11}$$

Throughout this paper, we assume that the following condition holds.

ASSUMPTION 1. *We assume that $(t_f - t)$, the planning horizon, is long enough so that the system reaches the steady state and stays there for a long period.*

The following Lemma is based on Assumption 1.

LEMMA 1. *Let t_s be the time the system reaches the steady state, and Let T be the length of the cyclic schedule. Then, we have:*

$$\lim_{(t_f - t_s)\to\infty} \frac{1}{t_f - t_s}\int_{t_s}^{t_f} g(x(s),\sigma(s))\,\mathrm{d}s = \frac{1}{T}\int_0^T g(x(s),\sigma(s))\,\mathrm{d}s$$

and by Assumption 1, we have $J_\mu^S(x(t_s),t_s) \cong \frac{1}{T}\int_0^T g(x(s),\sigma(s))\,\mathrm{d}s$.

The proof of Lemma 1 is straight forward and shall be omitted.

THEOREM 1. *The optimal production rate vector $u^*(s) = (u_1^*(s), u_2^*(s))$, $(t \leq s \leq t_f)$, belongs to the finite set of vectors $\Omega^* = \{(0,0), (U^*,0), (d_1,0), (0,U_2), (0,d_2)\}$. Where $u_i^*(s) = 0$ if the machine is not producing or undergoing a setup change to a Part Type; The machine produces at the demand rate ($u_i^*(s) = d_i$)) only when $x_i(s) = 0$ (i=1, 2).*

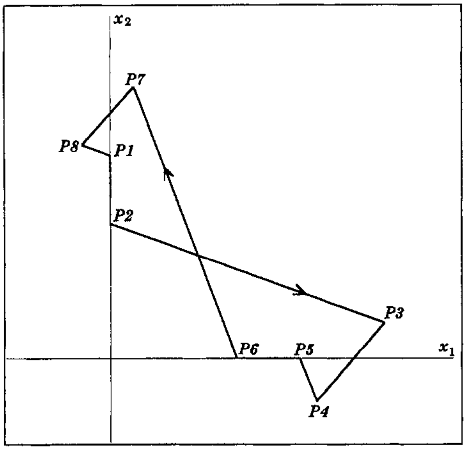

Figure 1. General structure of the cyclic schedule.

The proof of Theorem 1 is given in the appendix.

4. Optimal Steady State Solution

If the machine were perfectly flexible (i.e., with zero setup change times and costs), it would be optimal to produce both parts simultaneously at the demand rates (theoretically, it is optimal to infinitely switch production back and forth between the two products, since no costs or times are incurred when we switch). Thus, keeping the production surplus (inventory/backlog) at the zero level. In the case of significant setup times, it is not possible to produce both part types at the same time. Therefore, at the steady state the production surplus vector must follow a cyclic schedule that repeats itself over time.

Based on Theorem 1, it is not difficult to see that a feasible cyclic schedule has the general structure shown in Figure 1, where x_i represents the inventory/backlog axis of Part Type *i* (*i*=1,2). Now, assume that we start the cycle at the point *P1*, we then progress toward the point *P2* by producing Part Type 1 at the demand rate along segment [*P1*,*P2*], where x_2 decreases until we reach point *P2*. At this point, we increase the production rate to the maximum and continue producing Part Type 1. x_1 increases while x_2 decreases until we reach Point *P3*, where we switch production to Part Type 2. During the setup of the machine for Part Type 2, both inventory levels decrease. Once the machine is ready to produce Part Type 2 (Point *P4*), the production begins with the maximum allowable rate in order to eliminate backlog as soon as possible (since after the set up, we end up with a backlog for Part Type 2). Once the backlog of Part Type 2 is completely eliminated, we decrease the production rate to the demand rate so that the system moves along [*P5*,*P6*]. When we reach *P6*, the production rate of Part Type 2 is increased to the maximum and a certain inventory is built to hedge against future shortages brought about by setups and production of Part Type 1. At the point *P7*, we set up the machine for Part Type 1. At the point *P8*, we produce Part Type 1 at maximum production rate to eliminate backlog as soon as possible until we reach point *P1*, where we start the cycle all over again.

The steady state solution is completely characterized when the optimal location and shape (in x-space) of the cyclic schedule is known. Using Lemma 1, it can be seen that determining the cyclic schedule is equivalent to solving a special two-product Economic Lot Scheduling Problem (ELSP). Graphically, the optimization problem can be seen as one of locating and determining the shape of the cyclic schedule in the x-space so as to minimize the average setup, inventory holding and backlog costs per unit time. In Elhafsi and Bai (1996a) the authors studied a special version of the ELSP where they considered controllable production rates at the beginning as well as within a production run. They determined the optimal solution for the two-product case (steady state case in this paper) analytically. The optimal solution consists of the times the machine spends producing at the demand rate and at the maximum rate as well as the maximum inventory and backlog levels. In this section, we determine the optimal coordinates of the points *P1*, *P2*, *P3*, *P4*, *P5*, *P6*, *P7* and *P8* in x-space. We also show that depending on the parameters, the cyclic schedule will have four different shapes. But before we present the solution, we introduce the following notation:

For product i (i=1, 2)

t_i	time spent producing at maximum rate within a cyclic schedule
τ_i	time spent producing at the demand rate within a cyclic schedule
S_i	maximum inventory level
s_i	maximum backlog level
γ_i	$=c_i^+ c_i^- / (c_i^+ + c_i^-)$ cost factor
ρ_i	$=d_i/U_i$ utilization factor of the machine by product i
A_i	$=\gamma_i/2d_i(1-\rho_i)$
T	$=\sum_{i=1}^{i=2}(t_i+\tau_i+\delta_i)$ length of the cyclic schedule
δ	$=\delta_1+\delta_2$ total setup time during T
K	$=k_1+k_2$ total setup cost during T
ρ	$=\rho_1+\rho_2$ total utilization factor of the machine
α_i	$=(1-\rho_i)/(1-\rho)$

It can be shown (see Elhafsi and Bai, 1996a) that S_1, S_2, τ_1 and τ_2 are the solution of the following optimization problem:

$$\text{Minimize} \quad F(S,\tau)$$
$$= \sum_{i=1}^{i=2}\left(k_i + \frac{1}{2d_i(1-\rho_i)}(c_i^+ S_i^2 + c_i^-(S_i - Q_i)^2)\right) / \left(\sum_{i=1}^{i=2}\frac{Q_i}{d_i} - \delta\right)$$

$$\text{Subject to:} \quad Q_i = q_i\left(1 + \sum_{j=1}^{j=2}(1-\rho_j)\tau_j/\delta - (1-\rho)\tau_i/\delta\right), \quad i = 1,2$$

$$S_i \geq 0, \quad \tau_i \geq 0, \quad Q_i \geq 0, \quad i = 1,2$$

where $Q_i = S_i - s_i$ and $q_i = d_i\delta(1-\rho_i)/(1-\rho)$ for $i = 1,2$.

The optimal solution to the above optimization problem is given as follows:

$$S_i = Q_i c_i^- /(c_i^+ + c_i^-), i = 1,2; \tag{12a}$$

$$s_i = -Q_i c_i^+ /(c_i^+ + c_i^-), i = 1,2. \tag{12b}$$

Substituting S_i in $F(S, \tau)$ gives

$$F(\tau) = (K + A_1Q_1^2 + A_2Q_2^2)/(Q_1/d_1 + Q_2/d_2 - \delta). \tag{13}$$

The quantities Q_1, Q_2, τ_1, and τ_2 are calculated as follows:

$$Q_1^u = q_1d_2\gamma_2(1 + \sqrt{1 + 2K(1-\rho)(\alpha_1/\gamma_1d_1 + \alpha_2/\gamma_2d_2)/\delta^2})/ (\alpha_2d_1\gamma_1 + \alpha_1d_2\gamma_2); \tag{14a}$$

$$Q_2^u = q_2d_1\gamma_1(1 + \sqrt{1 + 2K(1-\rho)(\alpha_1/\gamma_1d_1 + \alpha_2/\gamma_2d_2)/\delta^2})/ (\alpha_2d_1\gamma_1 + \alpha_1d_2\gamma_2); \quad (14a)$$

$$\tau_1^u = Q_2^u/d_2 - Q_1^u\rho_1/(1-\rho_1)d_1 - \delta; \tag{15a}$$

$$\tau_2^u = Q_1^u/d_1 - Q_2^u\rho_2/(1-\rho_2)d_2 - \delta. \tag{15b}$$

IF $\tau_1^u \geq 0$ *and* $\tau_2^u \geq 0$ *THEN*

$\tau_1^* = \tau_1^u, \tau_2^* = \tau_2^u, Q_1^* = Q_1^u$ and $Q_2^* = Q_2^u$;

STOP
ELSE

$$Q_1^u = (1-\rho_1)d_1\sqrt{(K + A_2(\delta d_2)^2)/(d_1^2(1-\rho_1)^2A_1 + d_2^2\rho_1^2A_2)}; \tag{16a}$$

$$Q_2^u = d_2\delta + \rho_1d_2\sqrt{(K + A_2(\delta d_2)^2)/(d_1^2(1-\rho_1)^2A_1 + d_2^2\rho_1^2A_2)}; \tag{16b}$$

and compute τ_1^u and τ_2^u, using (15a) and (15b).
IF $\tau_2^u > 0$ *THEN*
Calculate $C2 = F(\tau_1^u, \tau_2^u)$ using (13).
ELSE
$C2 = \infty$.
END

Let

$$Q_1^u = d_1\delta + \rho_2 d_1\sqrt{(K + A_1(\delta d_1)^2)/(d_2^2(1-\rho_2)^2 A_2 + d_1^2\rho_2^2 A_1)}, \tag{17a}$$

$$Q_2^u = (1-\rho_2)d_2\sqrt{(K + A_1(\delta d_1)^2)/(d_2^2(1-\rho_2)^2 A_2 + d_1^2\rho_2^2 A_1)}, \tag{17b}$$

and compute τ_1^u and τ_2^u, using (15a) and (15b).
IF $\tau_1^u > 0$ *THEN*
Calculate $C1 = F(\tau_1^u, \tau_2^u)$ using (13).
ELSE
$C1 = \infty$.
END
Let $Q_1^u = q_1$ and $Q_2^u = q_2 \mapsto \tau_1^u = 0$ and $\tau_2^u = 0$;
Calculate $C0 = F(\tau_1^u, \tau_2^u)$ using (13).
$((\tau_1^*, \tau_2^*); (Q_1^*, Q_2^*)) = \text{argmin}_{(\tau_1^u, \tau_2^u),(Q_1^u, Q_2^u)}\{C0, C1, C2\}$
ENDIF

Now, based on Figure 1, PJ (J=1,2,..., 8) are given as follows:

$$P1 = \begin{pmatrix} 0 \\ S_2 - \delta_1 d_2 + s_1\rho_1 d_2/d_1(1-\rho_1) \end{pmatrix},$$
$$P2 = \begin{pmatrix} 0 \\ S_2 - \delta_1 d_2 + s_1 d_2\rho_1/d_1(1-\rho_1) - \tau_1 d_2 \end{pmatrix}, \tag{17a}$$

$$P3 = \begin{pmatrix} S_1 \\ s_2 + \delta_2 d_2 \end{pmatrix}, \quad P4 = \begin{pmatrix} S_1 - \delta_2 d_1 \\ s_2 \end{pmatrix},$$
$$P5 = \begin{pmatrix} S_1 - \delta_2 d_1 + s_2 d_1\rho_2/d_2(1-\rho_2) \\ 0 \end{pmatrix}, \tag{17b}$$

$$P6 = \begin{pmatrix} S_1 - \delta_2 d_1 + s_2 d_1\rho_2/d_2(1-\rho_2) - \tau_2 d_1 \\ 0 \end{pmatrix},$$
$$P7 = \begin{pmatrix} s_1 + \delta_1 d_1 \\ S_2 \end{pmatrix}, \quad P8 = \begin{pmatrix} s_1 \\ S_2 - \delta_1 d_2 \end{pmatrix}. \tag{17c}$$

Here, s_i, S_i, Q_i and $\tau_i (i = 1, 2)$ are the optimal values calculated above.

The four possible cases of the optimal cyclic schedule are shown in Figures 2, 3, 4 and 5. The following theorem gives a condition for which the cyclic schedule of Figure 2 is optimal in the case of zero setup costs. This result is useful for the remainder of the paper.

THEOREM 2. *Without loss of generality, let Part 2 be the part such that* $\gamma_2 d_2 \geq \gamma_1 d_1$. *If the setup costs* k_1 *and* k_2 *are zero, then* $\tau_1^* = 0$ *and* $\tau_2^* = 0$ *if and only*

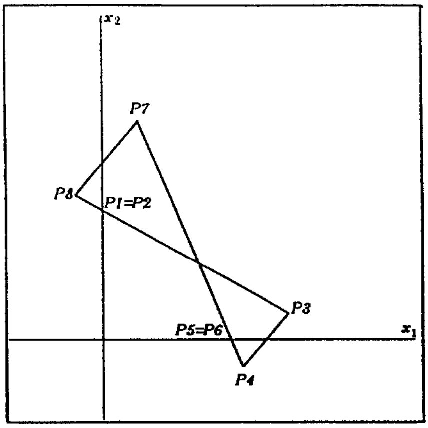

Figure 2. Case where $\tau_1 = 0$ and $\tau_2 = 0$.

if $\rho + \rho_1 - 1 \geq 0$. *If* $\rho + \rho_1 - 1 < 0$, *then* $\tau_1^* = 0$ *and* $\tau_2^* = 0$ *if and only if* $\gamma_2 d_2/\gamma_1 d_1 \leq (1 - \rho_1)/(1 - \rho - \rho_1)$ *(see appendix for proof).*

The following corollary follows immediately.

COROLLARY 1. *Assuming that Part 2 is the part such that* $\gamma_2 d_2 \geq \gamma_1 d_1$. *If the setup costs* k_1 *and* k_2 *are nonzero, then* $\tau_1^* = 0$ *and* $\tau_2^* = 0$ *if and only if*

$$0 \leq K \leq (T_0^2/2) min\{(1-\rho_2)(\gamma_2 d_2 - \gamma_1 d_1) + \rho\gamma_1 d_1; (\rho+\rho_1-1)\gamma_2 d_2 + (1-\rho_1)\gamma_1 d_1\}.$$

The following theorem shows that, when the setup costs are zero for both parts, at most one part type will be produced at the demand rate, during the cyclic schedule.

THEOREM 3. *If the setup costs* k_1 *and* k_2 *are zero, then at least one of the* τ_i*'s will be equal to zero (see appendix for proof).*

In the next section, we derive the optimal solution for the transient case. We show how the optimal solution of the case shown in Figure 2 is adapted to the other three cases shown in Figures 3, 4 and 5.

5. Optimal Transient Solution

As indicated by Theorem 2, for a heavily loaded system (i.e., ρ is close to one), the condition $\rho + \rho_1 - 1 \geq 0$ will hold true. Hence, τ_1^* and τ_2^* will be equal to zero. In this case, the cyclic schedule is as shown in Figure 2. In order to apply the results to the other cases, we need to extract a schedule from the cyclic schedule of Figure 5 (which represents the general case). The extracted schedule has exactly the solution structure shown in Figure 2. Figure 6 shows how to extract this schedule. Now, let

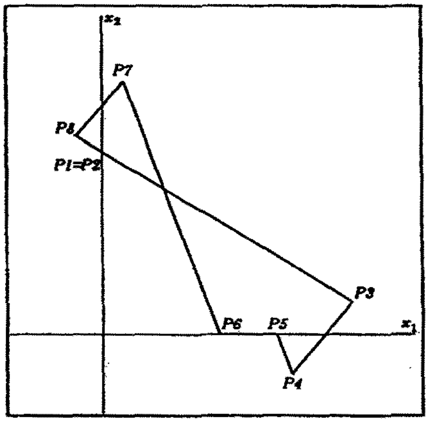

Figure 3. Case where $\tau_1 = 0$ and $\tau_2 > 0$.

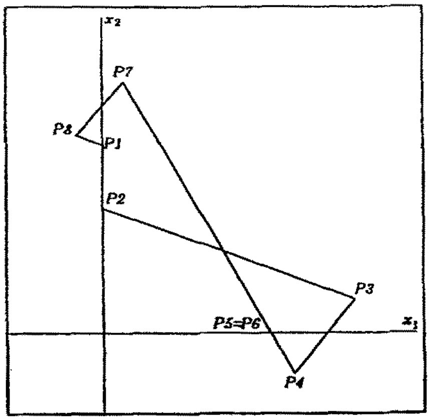

Figure 4. Case where $\tau_1 = 0$ and $\tau_2 = 0$.

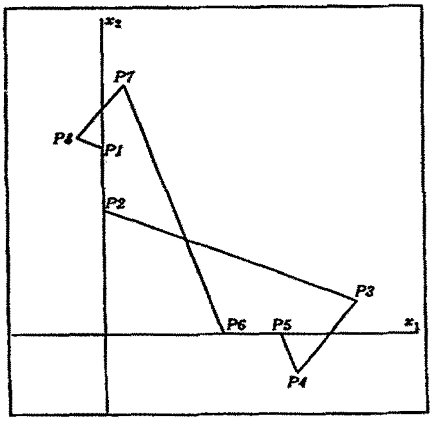

Figure 5. Case where $\tau_1 = 0$ and $\tau_2 = 0$.

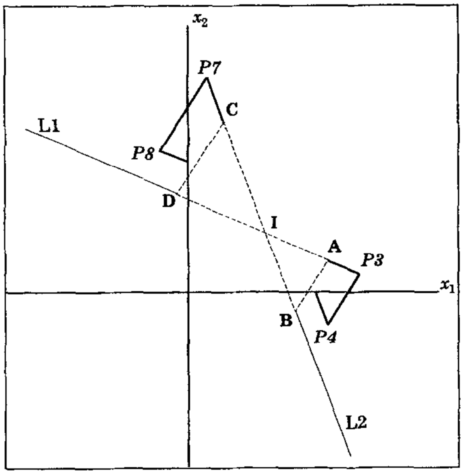

Figure 6. Extraction of the extracted schedule.

us determine the location of the extracted schedule. The latter is characterized by the coordinates of the points A, B, C and D in x-space (see Figure 6).

Let $A = \binom{A_1}{A_2}, B = \binom{B_1}{B_2}, C = \binom{C_1}{C_2}$, and $D = \binom{D_1}{D_2}$.
Denote by $L1$, the line containing Points D and $P3$ and by $L2$, the line containing Points B and $P7$ (see Figure 6). Then, $L1$ and $L2$ are given as follows:

$$LineL1 : d_2\rho_1(x_1 - P_{31}) + d_1(1 - \rho_1)(x_2 - P_{32}) = 0; \tag{18}$$

$$LineL2 : d_2(1 - \rho_2)(x_1 - P_{71}) + d_1\rho_2(x_2 - P_{72}) = 0. \tag{19}$$

Notice that $B = A - \binom{\delta_2 d_1}{\delta_2 d_2}$ and $D = C - \binom{\delta_1 d_1}{\delta_1 d_2}$ and that $A \in L1, B \in L2, C \in L2$, and $D \in L1$.

Hence, the coordinates of Point A and Point C can be determined by solving the following system of linear equations, respectively:

$$\begin{cases} d_2\rho_1(A_1 - P_{31}) + d_1(1 - \rho_1)(A_2 - P_{32}) = 0, \\ d_2(1 - \rho_2)(A_1 - \delta_2 d_1 - P_{71}) + d_1\rho_2(A_2 - \delta_2 d_2 - P_{72}) = 0; \end{cases}$$

$$\begin{cases} d_2\rho_1(C_1 - \delta_1 d_1 - P_{31}) + d_1(1 - \rho_1)(C_2 - \delta_1 d_2 - P_{32}) = 0, \\ d_2(1 - \rho_2)(C_1 - P_{71}) + d_1\rho_2(C_2 - P_{72}) = 0. \end{cases}$$

The solution of the above two systems is given by:

$$A = \begin{pmatrix} S_1 + (d_1/d_2)\rho_2\alpha_1 Q_2 - \alpha_1(1 - \rho_2)(Q_1 - \delta d_1) \\ s_2 + \delta_2 d_2 + (d_2/d_1)\rho_1\alpha_2(Q_1 - \delta d_1) - Q_2\rho_1\rho_2/(1 - \rho) \end{pmatrix}; \tag{20}$$

$$B = \begin{pmatrix} S_1 - \delta_2 d_1 + (d_1/d_2)\rho_2\alpha_1 Q_2 - \alpha_1(1 - \rho_2)(Q_1 - \delta d_1) \\ s_2 + (d_2/d_1)\rho_1\alpha_2(Q_1 - \delta d_1) - Q_2\rho_1\rho_2/(1 - \rho) \end{pmatrix}; \tag{21}$$

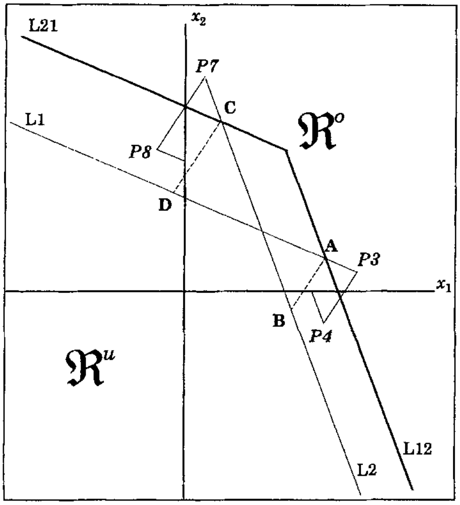

Figure 7. Regions $\mathcal{R}^u$ and $\mathcal{R}^o$.

$$C = \begin{pmatrix} s_1 + \delta_1 d_1 + (d_1/d_2)\rho_2\alpha_1(Q_2 - \delta d_2) - Q_1\rho_1\rho_2/(1-\rho) \\ S_2 + (d_2/d_1)\rho_1\alpha_2 Q_1 - \alpha_2(1-\rho_1)(Q_2 - \delta d_2) \end{pmatrix}; \qquad (22)$$

$$D = \begin{pmatrix} s_1 + (d_1/d_2)\rho_2\alpha_1(Q_2 - \delta d_2) - Q_1\rho_1\rho_2/(1-\rho) \\ S_2 - \delta_1 d_2 + (d_2/d_1)\rho_1\alpha_2 Q_1 - \alpha_2(1-\rho_1)(Q_2 - \delta d_2) \end{pmatrix}. \qquad (23)$$

Notice that, if there is no production at the demand rate for both parts (i.e., $\tau_1 = 0$ and $\tau_2 = 0$), then $Q_1 = q_1$ and $Q_2 = q_2$. In this case, we recover the coordinates of the cyclic schedule studied in Bai and Elhafsi (1996) and Elhafsi and Bai (1996b) which are given below:

$$A = \begin{pmatrix} S_1 \\ s_2 + \delta_2 d_2 \end{pmatrix}, \; B = \begin{pmatrix} S_1 - \delta_2 d_1 \\ s_2 \end{pmatrix}, \; C = \begin{pmatrix} s_1 + \delta_1 d_1 \\ S_2 \end{pmatrix}$$
$$\text{and } D = \begin{pmatrix} s_1 \\ S_2 - \delta_1 d_2 \end{pmatrix}.$$

Now, to obtain the optimal solution of the transient case, the x-space is divided into two mutually exclusive major regions ($\mathcal{R}^u$ and $\mathcal{R}^o$ in Figure 7) which are defined as follows:

$$\mathcal{R}^u = \{(x_1, x_2) | d_2(1-\rho_2)(x_1 - A_1) + d_1\rho_2(x_2 - A_2) < 0; d_2\rho_1(x_1 - C_1) + d_1(1-\rho_1)(x_2 - C_2) < 0\};$$

$$\mathcal{R}^o = \{(x_1, x_2) | d_2(1-\rho_2)(x_1 - A_1) + d_1\rho_2(x_2 - A_2) \geq 0; d_2\rho_1(x_1 - C_1) + d_1(1-\rho_1)(x_2 - C_2) \geq 0\}.$$

In Figure 7, Lines $L12$ and $L21$ are defined as follows:

$$LineL12 : d_2(1-\rho_2)(x_1-A_1)+d_1\rho_2(x_2-A_2)=0; \quad (24)$$

$$LineL21 : d_2\rho_1(x_1-C_1)+d_1(1-\rho_1)(x_2-C_2)=0. \quad (25)$$

We carry out the transient analysis in two parts. In the first part, we determine the optimal transient solution for all initial surplus levels in region $\mathcal{R}^o$. In the second part, the optimal transient solution for initial surplus levels in Region $\mathcal{R}^u$ is derived. Without loss of generality, we index the parts such that Part Type 1 is the part type with the larger setup time and setup cost (i.e., $\delta_1 \geq \delta_2$ and $k_1 \geq k_2$).

6. Optimal Transient Solution in Region $\mathcal{R}^o$

To determine the optimal solution for initial surplus levels in Region $\mathcal{R}^o$, we apply the same technique used in Bai and Elhafsi (1996). First, we partition Region $\mathcal{R}^o$ into three mutually exclusive regions, G, G1 and G2. Then, we further partition Region G into eight mutually exclusive regions G11, G12, G21, G22, H11, H12, H21, and H22. Figures 8 and 9 show the partition of Region $\mathcal{R}^o$ and G respectively. These regions are defined as follows:

$$G11=\{(x_1,x_2)|x_1-\delta_1 d_1 \geq 0; -d_2(x_1-P_{71})+d_1(x_2-P_{72})\geq 0\};$$

$$\begin{aligned} G12 &= \{(x_1,x_2)|d_2(x_1-P_{71})-d_1(x_2-P_{72})>0; -d_2(x_1-E_1) \\ &\quad +d_1(x_2-E_2)\geq 0; d_2(1-\rho_2)(x_1-A_1)+d_1\rho_2(x_2-A_2)\geq 0\}; \end{aligned}$$

$$\begin{aligned} G21 &= \{(x_1,x_2)|-d_2(x_1-P_{31})+d_1(x_2-P_{32})\geq 0; d_2(x_1-E_1) \\ &\quad -d_1(x_2-E_2)>0; d_2\rho_1(x_1-C_1)+d_1(1-\rho_1)(x_2-C_2)\geq 0\}; \end{aligned}$$

$$G22=\{(x_1,x_2)|x_2-\delta_2 d_2\geq 0; d_2(x_1-P_{31})-d_1(x_2-P_{32})>0\};$$

$$\begin{aligned} H11 &= \{(x_1,x_2)|x_1-\delta_1 d_1\geq 0; d_2(x_1-P_{71})-d_1(x_2-P_{72}>0; \\ &\quad -d_2(x_1-g_{11})+d_1(x_2-g_{12})\geq 0; -d_2(1-\rho_2)(x_1-A_1) \\ &\quad -d_1\rho_2(x_2-A_2)\geq 0\}; \end{aligned}$$

$$\begin{aligned} H12 &= \{(x_1,x_2)|d_2\rho_1(x_1-C_1)-d_1(1-\rho_1)(x_2-C_2)>0; \\ &\quad -d_2(x_1-g_{11})+d_1(x_2-g_{12})\geq 0; -d_2(1-\rho_2)(x_1-A_1) \\ &\quad -d_1\rho_2(x_2-A_2)\geq 0\}; \end{aligned}$$

$$\begin{aligned} H21 &= \{(x_1,x_2)|d_2(x_1-I_1)-d_1(x_2-I_2)>0; d_2(1-\rho_2)(x_1-A_1) \\ &\quad -d_1\rho_2(x_2-A_2)\geq 0; \\ &\quad -d_2\rho_1(x_1-C_1)-d_1(1-\rho_1)(x_2-C_2)>0; \\ &\quad -d_2(x_1-g_{21})+d_1(X_2-g_{22})\geq 0\}; \end{aligned}$$

$$
\begin{aligned}
H22 = \{(x_1, x_2) | x_2 - \delta_2 d_2 \geq 0; -d_2(x_1 - P_{31}) - d_1(x_2 - P_{32}) > 0; \\
-d_2\rho_1(x_1 - C_1) - d_1(1-\rho_1)(x_2 - C_2) > 0; -d_2(x_1 - g_{21}) \\
+d_1(x_2 - g_{22}) > 0\};
\end{aligned}
$$

$$G1 = \{(x_1, x_2) | -x_1 + \delta_1 d_1 > 0; d_2\rho_1(x_1 - P_{71}) + d_1(1-\rho_1)(x_2 - P_{72}) \geq 0\};$$

$$G2 = \{(x_1, x_2) | -x_2 + \delta_2 d_2 > 0; d_2(1-\rho_2)(x_1 - P_{31}) + d_1\rho_2(x_2 - P_{32} \geq 0\};$$

where $g_1 = (g_{11}, g_{12})^T$ is the point in x-space given by:

$$
\begin{aligned}
g_{11} &= \delta_1 d_1; \\
g_{12} &= C_2 + (C_1 - \delta_1 d_1) d_2 \rho_1 / d_1(1-\rho_1).
\end{aligned}
$$

$g_2 = (g_{21}, g_{22})^T$ is the point in x-space given by:

$$
\begin{aligned}
g_{21} &= A_1 + (A_2 - \delta_2 d_2) d_1 \rho_2 / d_2(1-\rho_2); \\
g_{22} &= \delta_2 d_2.
\end{aligned}
$$

and $E = (E_1, E_2)^T$ is the point in x-space given by:

$$E_1 = (-\theta_2 + \sqrt{\theta_2^2 + \theta_1\theta_3})/\theta_1 \text{ and } E_2 = \xi_1(-\theta_2 + \sqrt{\theta_2^2 + \theta_1\theta_3})/\theta_1 + \xi_2.$$

where,

$$
\begin{aligned}
\theta_1 &= (c_1^+/d_1)(\xi_1^2 - \xi_3^2) + (c_2^+/d_2)(1 - \xi_5^2); \\
\theta_2 &= (c_1^+/d_1)(\xi_1\xi_2 - \xi_3\xi_4) - (c_2^+/d_2)\xi_5\xi_6; \\
\theta_3 &= 2(k_1 - k_2) + (c_1^+/d_1)(\xi_4^2 - \xi_2^2) + (c_2^+/d_2)\xi_6^2; \\
\xi_1 &= -(d_1/d_2)(1-\rho_1)/\rho_1; \\
\xi_2 &= A_1 + A_2(d_1/d_2)(1-\rho_1)/\rho_1; \\
\xi_3 &= -(d_1/d_2)(\rho_2/\rho_1); \\
\xi_4 &= \rho_2 A_1 + (\rho_2/\rho_1)(1-\rho_1)A_2 + (1-\rho_2)C_1 + \rho_2(d_1/d_2)C_2; \\
\xi_5 &= (1-\rho_2)/\rho_1; \\
\xi_6 &= \rho_2 C_2 - A_2(1-\rho_2)(1-\rho_1)/\rho_1 + (d_2/d_1)(1-\rho_2)(C_1 - A_1).
\end{aligned}
$$

REMARK. Line LE is the boundary in x-space separating Region G12 and G21. Since the direction of Line LE is known (given by the vector $(-d_1, -d_2)$ in x-space), it is sufficient to find the coordinates of Point E in x-space to completely define Boundary LE. To determine Point E, we note that for initial surplus levels belonging to G12∩G21 (i.e., on Line LE), the cost of reaching the cyclic schedule at Point E with a setup change to Part Type 2 and the cost of reaching the cyclic schedule at Point F with a setup change to Part Type 1 must be equal. Mathematically, this can be written as follows:

$$k_1 + J = k_2 + J + (c_1^+/d_1)(E_1^2 - F_1^2) + (c_2^+/d_2)(E_2^2 - F_2^2).$$

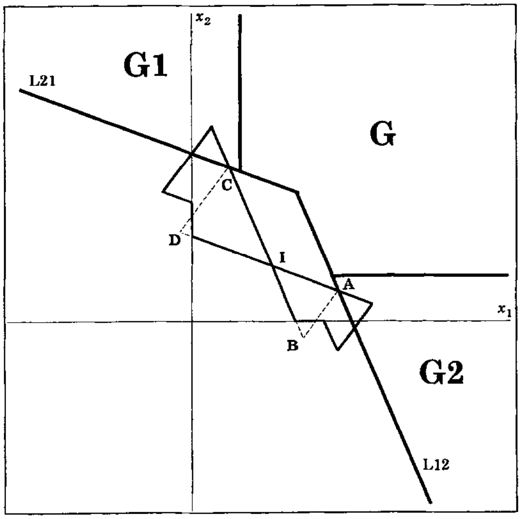

Figure 8. Partition of Region $\mathcal{R}^o$.

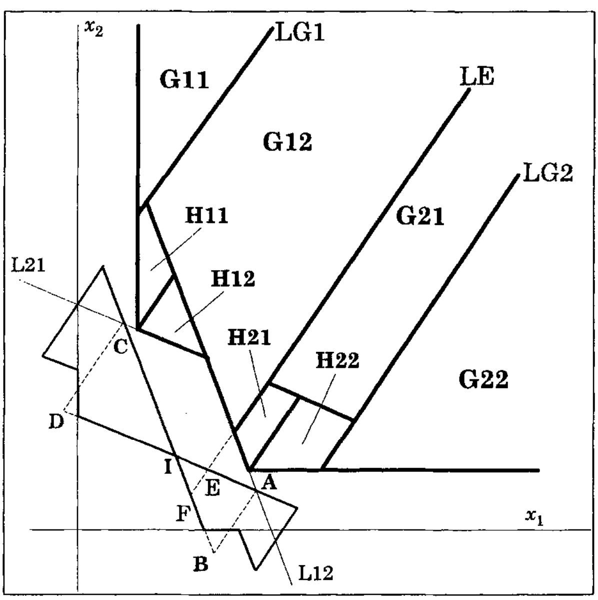

Figure 9. Partition of Region G.

Here, J represents the cost of reaching Point E with a setup change to either part type. Now, noticing that $E \in L1$, $F \in L2$, $E \in LE$ and $F \in LE$, we express F_1, F_2, and E_1 as a function of E_2 and then substitute in the above cost equality. The result is a quadratic in E_2, E_1 is then obtained by substituting E_2 by its expression from the quadratic solution as shown above.

In the case where $E_1 > A_1$, Region G21 becomes empty (i.e., G21=$\emptyset$). This case corresponds to a very high setup cost of Part Type 1 compared to that of Part Type 2. Therefore it is more economical to setup the machine to Part Type 2 before the cyclic schedule is reached.

In the case of equal setup costs for both part types, Point *E* coincides with Point *I*. In this case, we obtain the exact partition obtained in Bai and Elhafsi (1996). The optimal control for initial surplus levels in Region $\mathcal{R}^o$ is given in the appendix. In the next sub-section, we provide the optimal transient solution in Region $\mathcal{R}^u$.

6.1. Optimal Transient Solution in Region $\mathcal{R}^u$

The optimal solution for initial surplus levels in Region $\mathcal{R}^u$ is not obvious and is more involved mathematically. The following results were established in Bai and Elhafsi (1996).

FACT 1. For initial surplus levels in Region $\mathcal{R}^u$, the optimal way to progress toward the cyclic schedule is by producing at maximum machine speed whenever it is possible. That is, $u^* = (U_1, 0)$ or $(0, U_2)$ if the machine is producing and $u^* = (0, 0)$ if the machine is undergoing a setup change. Therefore, given the current setup state, we know the direction of the surplus trajectory in $\mathcal{R}^u$.

DEFINITION 1. We say that a trajectory is following Direction D_i, if it moves parallel to Line L*i* in the direction of increasing x_i. That is, the machine is producing Part Type $i (i = 1, 2)$ at maximum machine speed. If the machine is undergoing a setup change, then the trajectory follows Direction D_0, where both surplus levels deplete (Direction $(-d_1, -d_2)$).

REMARK. Since for surplus levels in Region $\mathcal{R}^u$, the machine is either producing at its maximum rate or being set up, the trajectories will move either along Direction D_1, along Direction D_2 or along Direction D_0.

DEFINITION 2. We call a $D_i - n - step$ trajectory $(i = 1, 2; n > 1)$, a trajectory that performs alternately m_i setup change-production runs of Part Type *i* and m_j setup change-production runs of Part Type *j* (*j*≠*i*); with the initial setup change to Part Type *i* and the last segment touching the cyclic schedule at point *B* or *D*. If *n* is even then $m_i = m_j = n/2$. If *n* is odd then $m_i = (n + 1)/2$ and $m_j = (n - 1)/2$. A $D_1 - 3 - step$ trajectory is shown in Figure 10.

FACT 2. To reach the cyclic schedule in finite time, starting with initial surplus levels in Region $\mathcal{R}^u$, the trajectory leading to the cyclic schedule must touch the boundary L*ij* (*i*, *j* =1,2, and *i*≠*j*) of Region $\mathcal{R}^o$, just before switching to Part Type *j* and reaching the cyclic schedule at one of the points *B* or *D*. That is, the last

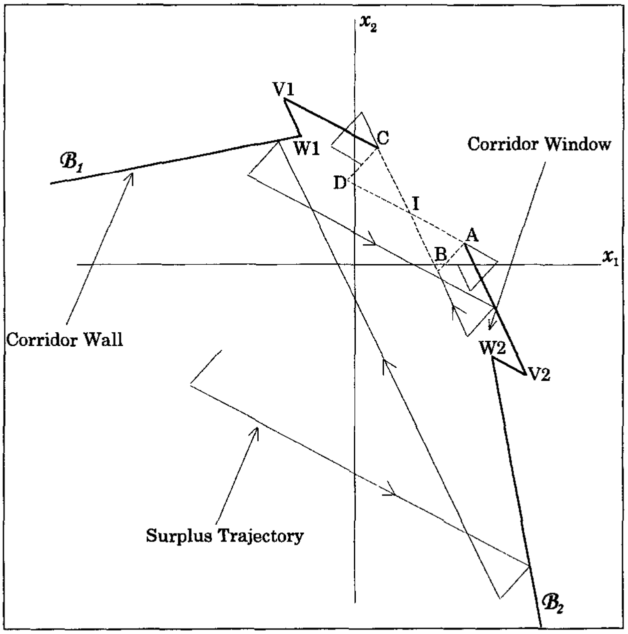

Figure 10. Illustration of the setup switching policy.

setup change before reaching the cyclic schedule is initiated on the boundary L12 or L21.

In Elhafsi and Bai (1996b), it has been established (for the case of Figure 2) that the optimal switching policy is a special corridor policy with two windows. In the case where production at the demand rates is possible in the cyclic schedule, this result is still valid and in fact is optimal when applied to the extracted schedule characterized by its location in x-space given by A, B, C, and D (equations (20)-(23)). Figure 10 illustrates this corridor policy which is completely characterized by its corridor walls defined by boundaries $\mathcal{B}1$ and $\mathcal{B}2$ and its corridor windows defined by the pairs of points ($W1,V1$) and ($W2,V2$). Using the technique developed in Elhafsi and Bai (1996b), we can determine the corridor boundaries and windows as follows:

Equations of Boundaries $\mathcal{B}1$ and $\mathcal{B}2$:

$$\mathcal{B}1 : M_1x_1 + M_2x_2 + M_3 = 0; \tag{26}$$

$$M_1 = c_1^-/(1-\rho_1)-\rho_1\rho_2c_1^+/(1-\rho_1)(1-\rho)-(d_1/d_2)(\rho_1/\rho_2)\alpha_2c_2^-; \tag{27a}$$

$$\begin{aligned} M_2 = & -(d_1/d_2)\rho_2c_1^+/(1-\rho) \\ & -c_2^+/\rho_2 - \alpha_1(1-\rho_2)c_2^-/\rho_2; \end{aligned} \tag{27b}$$

$$M_3 = -(M_1C_1 + M_2C_2). \tag{27c}$$

and

$$\mathcal{B}2 : N_1x_1 + N_2x_2 + N_3 = 0; \tag{28}$$

$$N_1 = -(d_2/d_1)\rho_1 c_2^+/(1-\rho) - c_1^+/\rho_1 - \alpha_1(1-\rho_2)c_1^-/\rho_1; \tag{29a}$$

$$N_2 = c_2^-/(1-\rho_2) - \rho_1\rho_2 c_2^+/(1-\rho_2)(1-\rho) - (d_2/d_1)(\rho_2/\rho_1)\alpha_1 c_1^-; \tag{29b}$$

$$N_3 = -(N_1A_1 + N_2A_2). \tag{29c}$$

Corridor windows:
Let $W1 = (w_{11}, w_{12})$, $V1 = (v_{11}, v_{12})$, $W2 = (w_{21}, w_{22})$, and $V2 = (v_{21}, v_{22})$ in x-space. $W1$ satisfies the following system:

$$\begin{cases} a_1 w_{11}^2 + a_2 w_{11} + a_3 = 0; \\ w_{12} = -(M_1/M_2)w_{11} - (M_3/M_2); \end{cases} \tag{30}$$

$$\begin{cases} v_{11} = w_{11} + \rho_1\rho_2(w_{11} - C_1)/(1-\rho) + (d_1/d_2)\rho_2(1-\rho_1) \\ \quad (w_{12} - C_2)/(1-\rho); \\ v_{12} = C_2 + \rho_1\rho_2(C_2 - w_{12})/(1-\rho) + (d_2/d_1)\rho_1(1-\rho_2) \\ \quad (C_1 - w_{11})/(1-\rho); \end{cases} \tag{31}$$

where,

$$a_1 = \frac{(\gamma_{11}^2 - 1)c_1^- - \eta_{11}^2 c_1^+}{2d_1(1-\rho_1)} + \frac{(\gamma_{12}^2 - (M_1/M_2)^2)c_2^+ - \eta_{12}^2 c_2^-}{2d_2(1-\rho_2)} \tag{32a}$$

$$a_2 = \frac{(\gamma_{11}(\mu_{11} - \delta_1 d_1) + \delta_1 d_1)c_1^- - \eta_{11}v_{11}c_1^+}{d_1(1-\rho_1)} + \frac{(\gamma_{12}\mu_{12} - M_1M_3/M_2^2)c_2^+ - \eta_{12}(v_{12} - \delta_2 d_2)c_2^-}{d_2(1-\rho_2)}; \tag{32b}$$

$$a_3 = \frac{\mu_{11}(\mu_{11} - 2\delta_1 d_1)c_1^- - (v_{11}^2 - I_1^2)c_1^+}{2d_1(1-\rho_1)} + \frac{(\mu_{12}^2 - (M_3/M_2)^2 - I_2^2)c_2^+ - (v_{12} - \delta_2 d_2)^2 c_2^-}{2d_2(1-\rho_2)} - k_2. \tag{32c}$$

$$\eta_{11} = \rho_2(d_1/d_2)(M_1/M_2)\alpha_1 - \rho_1/(1-\rho); \tag{33a}$$

$$\eta_{12} = \alpha_2\{(d_2/d_1)\rho_1 - (M_1/M_2)(1-\rho_1)\}; \tag{33b}$$

$$v_{11} = \alpha_1(1-\rho_2)A_1 + \alpha_1\rho_2(d_1/d_2)(A_2 + (M_3/M_2)) + \delta_1 d_1 \rho_2/(1-\rho); \quad (34a)$$

$$v_{12} = -\rho_1\alpha_2(d_2/d_1)A_1 - \rho_1\rho_2 A_2/(1-\rho) - (1-\rho_2)\alpha_1(M_3/M_2) - \alpha_2(d_2/d_1)\delta_1 d_1; \quad (34b)$$

$$\gamma_{11} = \alpha_{11}\{(1-\rho_2) - \rho_2(d_1/d_2)(M_1/M_2)\}; \quad (35a)$$

$$\gamma_{12} = \rho_1\rho_2(M_1/M_2)/(1-\rho) - \rho_1\alpha_2(d_2/d_1); \quad (35b)$$

$$\mu_{11} = -\rho_1\rho_2 C_1/(1-\rho) - \rho_2\alpha_1(d_1/d_2)(C_2 + (M_3/M_2)); \quad (36a)$$

$$\mu_{12} = \rho_1\alpha_2(d_2/d_1)C_1 + (1-\rho_1)a_2C_2 + \rho_1\rho_2(M_3/M_2)/(1-\rho); \quad (36b)$$

and $W2$ satisfies the following system:

$$\begin{cases} b_1 w_{22}^2 + b_2 w_{22} + b_3 = 0; \\ w_{21} = -(N_2/N_1)w_{22} - (N_3/N_1); \end{cases} \quad (37)$$

$$\begin{cases} v_{21} = A_1 + \rho_1\rho_2(A_1 - w_{21})/(1-\rho) + (d_1/d_2)\rho_2(1-\rho_1) \\ \quad (A_2 - w_{22})/(1-\rho); \\ v_{22} = w_{22} + \rho_1\rho_2(w_{22} - A_2)/(1-\rho) + (d_2/d_1)\rho_1(1-\rho_2) \\ \quad (w_{21} - A_1)/(1-\rho); \end{cases} \quad (38)$$

where,

$$b_1 = \frac{(\gamma_{21}^2 - (N_2/N_1)^2)c_1^+ - \eta_{21}^2 c_1^-}{2d_1(1-\rho_1)} + \frac{(\gamma_{22}^2 - 1)c_2^- - \eta_{22}^2 c_2^+}{2d_2(1-\rho_2)}; \quad (39a)$$

$$b_2 = \frac{(\gamma_{21}\mu_{21} - N_2N_3/N_1^2)c_1^+ - \eta_{21}(v_{21} - \delta_1 d_1)c_1^-}{d_1(1-\rho_1)} + \frac{(\gamma_{22}(\mu_{22} - \delta_2 d_2) + \delta_2 d_2)c_2^- - \eta_{22}v_{22}c_2^+}{d_2(1-\rho_2)}; \quad (39b)$$

$$b_3 = \frac{(\mu_{21}^2 - (N_3/N_1)^2 - I_1^2)c_1^+ - (v_{21} - \delta_1 d_1)^2 c_1^-}{2d_1(1-\rho_1)} + \frac{\mu_{22}(\mu_{22} - 2\delta_2 d_2)c_2^- - (v_{22}^2 - I_2^2)c_2^+}{2d_2(1-\rho_2)} - k_1; \quad (39c)$$

$$\eta_{21} = \alpha_1\{(d_1/d_2)\rho_2 - (N_2/N_1)(1-\rho_2)\}; \quad (40a)$$

$$\eta_{22} = \rho_1\alpha_2(d_2/d_1)(N_2/N_1) - \rho_1\rho_2/(1-\rho); \quad (40b)$$

$$v_{21} = -\rho_1\rho_2 C_1/(1-\rho) - \rho_2\alpha_1(d_1/d_2)C_2 - (1-\rho_2)\alpha_1(N_3/N_1) -\alpha_1(d_1/d_2)\delta_2 d_2; \quad (41a)$$

$$v_{22} = \rho_1\alpha_2(d_2/d_1)(C_1 + (N_3/N_1)) + (1-\rho_1)\alpha_2 C_2 +\delta_2 d_2\rho_1/(1-\rho); \quad (41b)$$

$$\gamma_{21} = \rho_1\rho_2(N_2/N_1)/(1-\rho) - \rho_2\alpha_1(d_1/d_2); \quad (42a)$$

$$\gamma_{22} = \alpha_2\{(1-\rho_1) - \rho_1(d_2/d_1)(N_2/N_1)\}; \quad (42b)$$

$$\mu_{21} = (1-\rho_2)\alpha_1 A_1 + \rho_2\alpha_1(d_1/d_2)A_2 + \rho_1\rho_2(N_3/N_1)/(1-\rho); \quad (43a)$$

$$\mu_{22} = -\rho_1\alpha_2(d_2/d_1)(A_1 + (N_3/N_1)) - \rho_1\rho_2 A_2/(1-\rho). \quad (43b)$$

Notice that in (30), w_{11} must be a real root of the quadratic such that $w_{11} \leq C_1$. Similarly, w_{22} in (37) must be a real root such that $w_{22} \leq A_2$. If either quadratic in (30) or (37) has no real root, then the corridor window does not exist in this case meaning that it is always optimal to reach the extracted schedule from the other side (i.e., the side with the corridor window).

Now based on Facts 1 and 2, the optimal trajectory emanating from a point in Region $\mathcal{R}^u$ and leading to the extracted schedule in finite time can be obtained as follows: Given an initial surplus point in Region $\mathcal{R}^u$, we choose the first setup and calculate the cost of the trajectory leading to the extracted schedule with two setup changes only. At this point, we have a $D_i - 2 - step$ trajectory, where *i* is the initial setup for Part Type *i*. The next step is to try to lower the cost of the current trajectory by introducing a setup change to the other part type so that the extracted schedule is reached at the opposite side. If the cost can be reduced, then the obtained new trajectory is a $D_i - 3 - step$ trajectory. We keep trying to reduce the cost of the current trajectory by introducing, each time, a setup change before the extracted schedule is reached until we cannot lower the cost anymore. At this point we have an optimal $D_i - n - step$ trajectory emanating in Region $\mathcal{R}^u$ and reaching the extracted schedule in finite time. In the same manner, we obtain the optimal $D_j - n - step$ ($j \neq i$) trajectory starting with a setup to Part type *j* first. The optimal trajectory would be the one with the lowest cost. The numerical solution of various examples suggests that the above procedure be further simplified based on the following Conjecture.

CONJECTURE. The initial setup of the optimal trajectory is given by $i^* = \text{argmin}_{i=1,2}\{C_i^2(x)\}$, where $C_i^2(x)$ is the cost of the $D_i - 2 - step$ trajectory starting with a setup change to Part Type *i* and reaching the extracted schedule with two setup changes only.

Based on the above Conjecture, The optimal trajectory emanating from $X=(a,b)$ in Region $\mathcal{R}^u$ is obtained using the following procedure.

PROCEDURE

STEP 0: $X_1(0) := a, X_2(0) := b; k := 1;$
IF $C_1^2(X(0)) \geq C_2^2(X(0))$ THEN
$X_1(k) := X_1(0) - \delta_2 d_1;$
$X_2(k) := X_2(0) - \delta_2 d_2;$
GOTO STEP 1;
ELSE
$X_1(k) := X_1(0) - \delta_1 d_1;$
$X_2(k) := X_2(0) - \delta_1 d_2;$
ENDIF;

STEP 1: $k := k + 1;$
$X_1(k) := (d_1\rho_2(M_3 + X_2(k-1)M_2) + d_2(1-\rho_2)X_1(k-1)M_2)/$
$(d_2(1-\rho_2)M_2 - d_1\rho_2 M_1)$
$X_2(k) := -(M_1/M_2)X_1(k) - (M_3/M_2);$
IF $X_1(k) \geq w_{11}$ THEN
$X_1(k) := \gamma_{11}X_1(k-1) + \mu_{11};$
$X_2(k) := \gamma_{12}X_2(k-1) + \mu_{12};$
$k := k + 1;$
$X_1(k) := X_1(k-1) - \delta_1 d_1;$
$X_2(k) := X_2(k-1) - \delta_1 d_2;$
$k := k + 1;$
$X(k) := I;$
GOTO STEP 3;
ELSE
GOTO STEP 2;
ENDIF;

STEP 2: $k := k + 1;$
$X_2(k) := (d_2\rho_1(N_3 + X_1(k-1)N_1) + d_1(1-\rho_1)X_1(k-1)N_1)/$
$(d_1(1-\rho_1)N_1 - d_2\rho_1 N_2)$
$X_1(k) := -(N_2/N_1)X_2(k) - (N_3/N_1);$
IF $X_2(k) \geq w_{22}$ THEN
$X_1(k) := \gamma_{21}X_1(k-1) + \mu_{21};$
$X_2(k) := \gamma_{22}X_2(k-1) + \mu_{22};$
$k := k + 1;$
$X_1(k) := X_1(k-1) - \delta_2 d_1;$
$X_2(k) := X_2(k-1) - \delta_2 d_2;$
$k := k + 1;$
$X(k) := I;$
GOTO STEP 3;
ELSE

GOTO STEP 1;
ENDIF;

STEP 3: OUTPUT: OPTIMAL TRAJECTORY=
$\{X(0), X(1), \cdots, X(k-1), X(k)\}$;

Compared to the case of zero setup costs, the only change is in the expressions of a_3 and b_3 given above by (32c) and (39c) respectively, where $-k_2$ was added to a_3 and $-k_1$ was added to b_3 originally obtained in Elhafsi and Bai (1996b). This result is intuitive since in the case of non zero setup costs, we will have wider corridor windows, which means that if the setup costs are high then it might not be optimal to add another setup change and therefore reach the extracted schedule producing the same part type that the machine is set up for and therefore avoiding a rather expensive setup change. Also, note that the setup costs do not appear explicitly in the equations of the boundaries $\mathcal{B}1$ and $\mathcal{B}2$, but the latter depend on the setup costs implicitly through the coordinates of the points A and C of the extracted schedule.

6.2. A Numerical Example

Consider the following system:
Part Type 1:
$c_1^+ = \$2$/unit/day, $c_1^- = \$10$/unit/day, $U_1 = 8$/day, $d_1 = 2.5$/day, $\delta_1 = 1$ day, and $k_1 = \$100$;
Part Type 2:
$c_2^+ = \$2$/unit/day, $c_2^- = \$10$/unit/day, $U_2 = 10$/day, $d_2 = 3$/day, $\delta_2 = 1$ day, and $k_2 = \$120$.

Figure 11 shows the optimal location and shape of the cyclic schedule as well as the optimal trajectory emanating at the point (–20,–20) (i.e., we start 20 units short of both part types) in Region $\mathcal{R}^u$. Also, Figure 11 shows the boundaries $\mathcal{B}1$ and $\mathcal{B}2$ as well as the two pairs ($W1$,$V1$) and ($W2$,$V2$) defining the two corridor windows. The optimal production and setup planning for this system can be described as follows (after rounding to integers): Set up the machine and produce Part Type 2 at the rate of 10 units/day until its surplus level reaches 14 units (i.e., when the surplus trajectory hits Boundary $\mathcal{B}1$); set up the machine and produce Part Type 1 at the rate of 8 units/day until its surplus level reaches 10 units (i.e., when the surplus trajectory hits Boundary $\mathcal{B}2$); set up the machine and produce Part Type 2 at the rate of 10 units/day until its surplus level reaches 15 units (i.e., when the surplus trajectory enters Window ($W1$,$V1$)); set up the machine and produce Part Type 1 at the rate of 8 units/day until its surplus level reaches 16 units (which corresponds to Point $P3$ on the cyclic schedule); switch to the control actions of the cyclic schedule given as follows: Set up the machine and produce Part Type 2 at the rate of 10 units/day until its surplus level reaches 0; continue producing Part Type 2 at the demand rate of 3 units/day until the surplus level of Part Type 1

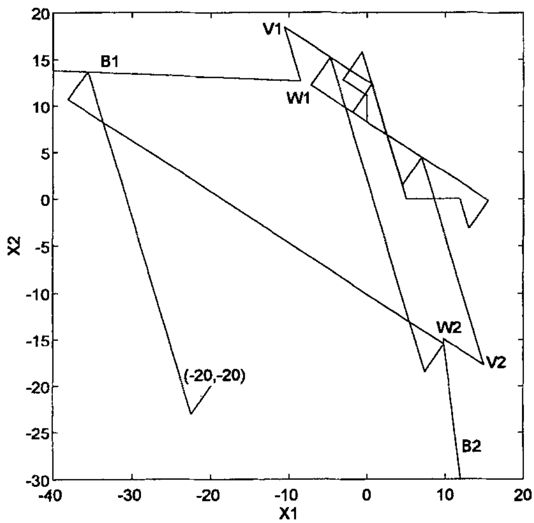

Figure 11. Numerical illustration.

drops to 5 units; at this moment, increase the production rate of Part Type 2 to 10 units/day until its surplus level reaches 16 units; set up the machine and produce Part Type 1 at the rate of 8 units/day until its surplus level reaches 0; continue producing Part Type 1 at the demand rate of 2.5 units/day until the surplus level of Part Type 2 drops to 8 units; at this moment, increase the production rate of Part Type 1 to 8 units/day until its surplus level reaches 16 units, which is the point we started the cyclic schedule at.

7. The Case of Zero Setup Times and Nonzero Setup Costs

In this section, we consider the case where setup times are very small or are order of magnitude smaller than the other parameters of the system (especially setup costs). This kind of situation may arise when the setup changes can be accomplished relatively quickly, but the equipment, the material (cleaning products for instance) or the labor used for setup changes are very expensive. As in the general case, we divide the analysis into two parts: steady state and transient cases.

7.1. Steady State Optimal Solution

The steady state solution for this case can be obtained using the same procedure used for the case where setup times are nonzero. The following theorem shows that in the case of zero setup times, the cyclic schedule must have at least one segment with demand production rate (i.e., it is not possible to have the case of Figure 2).

THEOREM 4. *If the setup time is zero for both part types, then there is at least one segment within the cyclic schedule corresponding to a production at the demand rate. That is, if $\delta_1 = \delta_2 = 0$, then it is not possible to have $\tau_1^* = \tau_2^* = 0$.*

Proof. Assuming that $\tau_1^* = 0$ and $\tau_2^* = 0$, and letting $\delta \to 0$ in the objective function introduced in the proof of Theorem 2, gives:

$$\begin{aligned} F(0,0) &= \lim_{\delta \to 0} \frac{(K + H_1 T_0^2/2 + H_2 T_0^2/2)}{T_0} = \frac{K}{0} \\ &= \infty \text{ (since } T_0 = \delta/(1-\rho) \text{ and } K > 0). \end{aligned}$$

Hence, this solution cannot be optimal. ■

In the case of zero setup times, the general structure of the cyclic schedule is as shown in Figure 12. The coordinates of the cyclic schedule in this case are given as follows:

$$P1 = \begin{pmatrix} 0 \\ S_2 + s_1(d_2/d_1)\rho_1/(1-\rho_1) \end{pmatrix},$$

$$P2 = \begin{pmatrix} 0 \\ S_2 + s_1(d_2/d_1)\rho_1/(1-\rho_1) - \tau_1 d_2 \end{pmatrix}, P3 = P4 = \begin{pmatrix} S_1 \\ s_2 \end{pmatrix},$$

$$P5 = \begin{pmatrix} S_1 + s_2(d_1/d_2)\rho_2/(1-\rho_2) \\ 0 \end{pmatrix},$$

$$P6 = \begin{pmatrix} S_1 + s_2(d_1/d_2)\rho_2/(1-\rho_2) - \tau_2 d_1 \\ 0 \end{pmatrix}, P7 = P8 = \begin{pmatrix} s_1 \\ S_2 \end{pmatrix}.$$

Where s_i, S_i, Q_i and τ_i (i=1,2) are the optimal values calculated using the same procedure as in the case of non zero setup times.

The extracted schedule in this case collapses to a single point in the x-space (Point *I* in Figure 12). To see this, let $\delta_1 = \delta_2 = 0$ in the coordinates of the extracted schedule calculated for the nonzero setup times case (Equations (20)-(23)). This gives:

$$A = B = \begin{pmatrix} S_1 + (d_1/d_2)\rho_2\alpha_1 Q_2 - (1-\rho_2)\alpha_1 Q_1 \\ s_2 + (d_2/d_1)\rho_1\alpha_2 Q_1 - \rho_1\rho_2 Q_2/(1-\rho) \end{pmatrix};$$

$$C = D = \begin{pmatrix} s_1 + (d_1/d_2)\rho_2\alpha_1 Q_2 - \rho_1\rho_2 Q_1/(1-\rho) \\ S_2 + (d_2/d_1)\rho_1\alpha_2 Q_1 - (1-\rho_1)\alpha_2 Q_2 \end{pmatrix}.$$

But, $s_i = S_i - Q_i$. Substituting in the coordinates of *A* and *B*, gives the Points *C* and *D*. Therefore,

$$A = B = C = D = I = \begin{pmatrix} S_1 + (d_1/d_2)\rho_2\alpha_1 Q_2 - (1-\rho_2)\alpha_1 Q_1 \\ s_2 + (d_2/d_1)\rho_1\alpha_2 Q_1 - \rho_1\rho_2 Q_2/(1-\rho) \end{pmatrix}.$$

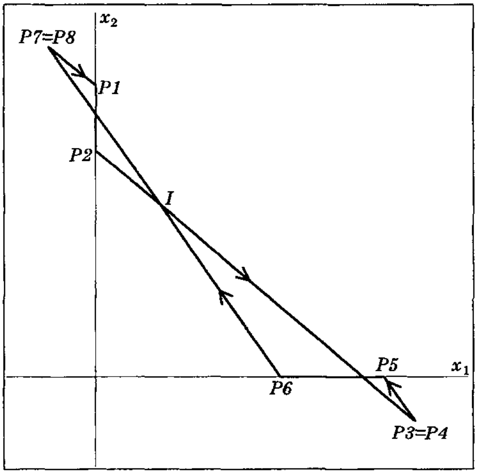

Figure 12. General structure of the cyclic schedule in the case of no setup times.

7.2. Transient Optimal Solution

As for the case of nonzero setup times, we divide the x-space into two mutually exclusive major regions. Region $\mathcal{R}^u$ and Region $\mathcal{R}^o$ defined as follows:

$$\begin{aligned}
\mathcal{R}^u &= \{(x_1, x_2) | d_2(1-\rho_2)(x_1 - I_1) + d_1\rho_2(x_2 - I_2) < 0; d_2\rho_1(x_1 - I_1) \\
&\quad + d_1(1-\rho_1)(x_2 - I_2) < 0\}; \\
\mathcal{R}^o &= \{(x_1, x_2) | d_2(1-\rho_2)(x_1 - I_1) + d_1\rho_2(x_2 - I_2) \geq 0; d_2\rho_1(x_1 - I_1) \\
&\quad + d_1(1-\rho_1)(x_2 - I_2) \geq 0\}.
\end{aligned}$$

This partition of the x-space is shown in Figure 13. In this case, Lines L12 and L21 coincide with Lines L2 and L1 respectively. Lines L1 and L2 are defined as follows:

$$\begin{aligned}
\text{Line L1}: &\quad d_2\rho_1(x_1 - I_1) + d_1(1-\rho_1)(x_2 - I_2) = 0; \\
\text{Line L2}: &\quad d_2(1-\rho_2)(x_1 - I_1) + d_1\rho_2(x_2 - I_2) = 0.
\end{aligned}$$

The optimal transient solution for initial surplus levels in Region $\mathcal{R}^o$ can be obtained by inspection in a similar way as in the case of nonzero setup times. The optimal switching policy in Region $\mathcal{R}^u$ for the case of nonzero setup times is a special corridor policy characterized by the boundaries $\mathcal{B}1$ and $\mathcal{B}2$ of the corridor wall, and the two windows of the corridor defined by the points $W1$ and $W2$. In the case of zero setup times, to obtain the optimal policy, all we need to do is to let $\delta_1 = 0$ and $\delta_2 = 0$ in the equations of Boundaries $\mathcal{B}1$ and $\mathcal{B}2$ as well as in the coordinates of Points $W1$, $V1$, $W2$ and $V2$.

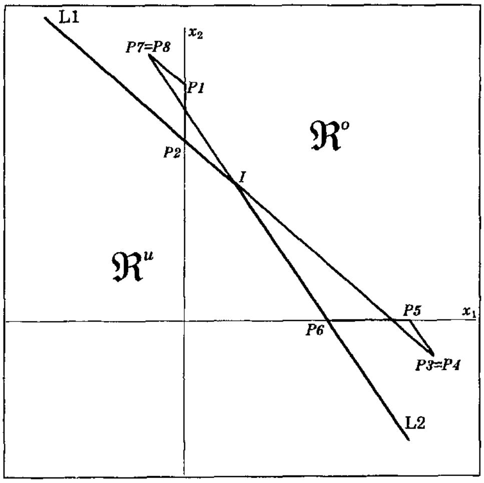

Figure 13. Partition of the x-space in the case of zero setup times.

8. Conclusion

In this paper, we studied a two-product manufacturing system incurring setup times as well as setup costs when production is switched from one product to the other. The production rates are controllable. We provided a feedback control formulation of the problem. We distinguished between two different periods: a steady state period (for which the problem reduces to the two-product Economic Lot Scheduling Problem), and a transient period. The steady state optimal solution consists of a cyclic schedule, where the two products are produced alternatively. We showed that in the steady state, within the production run of a product, it might be optimal to reduce the production rate from the maximum to the demand rate so as to keep its inventory at the zero level for a certain amount of time, and therefore, delay as much as possible the setup cost of the next setup change. This is particularly true in the case of high setup costs, since we do not want to switch setups frequently. The optimal transient solution consists of a trajectory in x-space leading to the cyclic schedule in finite time and with minimum cost. The former is obtained by partitioning the x-space into two mutually exclusive major regions. For initial surplus levels (inventory/backlog) in one of the regions, the optimal solution is obtained by inspection. For initial surplus levels in the other region a procedure is provided to determine the optimal state-space trajectory. Also, we studied the case of zero setup times and nonzero setup costs and showed that at least one product should be produced at the demand rate for a certain time within the cyclic schedule in this case.

Appendix

A. Proof of Theorem 1

First, we prove the theorem for the transient case. Then, we show that a similar argument can be used for the steady state case. The proof is based on the Hamilton–Jacobi–Bellman (HJB) equation. Throughout the proof, we assume that the optimal cost functional is differentiable in x and t. In fact, the optimal state trajectory is continuous piecewise linear. Hence, the optimal cost will not depend explicitly on t and will be the sum of quadratics in x (since the cost rate is linear in x) and therefore differentiable in x.

$$\text{Let } J^*_{T_f}(x,t) = \min_{\sigma, u \epsilon \Xi(\phi,\Omega)} \int_t^{T_f} g(x(s),\sigma(s))\, \mathrm{d}s, \text{ where } T_f \text{ is finite.}$$

Transient case: In this case, we have $T_f = t_s$ in the expression of $J^*_{T_f}$ to obtain the optimal transient cost component. The HJB equation (see Gershwin (1993) for a formal derivation) is given by:

$$-\frac{\partial J^*_{T_f}(x,t)}{\partial t} = \min_{\sigma, u \epsilon \Xi(\phi,\Omega)} \left\{ g(x,\sigma) + \frac{\partial J^*_{T_f}(x,t)}{\partial x_1}(u_1 - d_1) + \frac{\partial J^*_{T_f}(x,t)}{\partial x_2}(u_2 - d_2) \right\}$$

It is clear that, when the machine is undergoing a setup change to a Part Type, there is no decision to make and (u_1^*, u_2^*) is forced to be equal to (0,0). Now, assume that we know the optimal setup state of the machine. Let $\sigma = (1,0,0,0)$ be this setup state. That is, the machine can produce Part Type 1. In this case, the HJB equation can be rewritten as follows:

$$-\frac{\partial J^*_{T_f}(x,t)}{\partial t} = \min_{u \epsilon \Omega(1,0)} \left\{ g(x,\sigma) + \frac{\partial J^*_{T_f}(x,t)}{\partial x_1}(u_1 - d_1) + \frac{\partial J^*_{T_f}(x,t)}{\partial x_2}(u_2 - d_2) \right\}$$

Now, notice that at each time instant t, if we knew $J^*_{T_f}(x,t)$, we would solve a linear programming problem for which u_1 and u_2 are the decision variables, $\partial J^*_{T_f}/\partial x_1$ and $\partial J^*_{T_f}/\partial x_2$ are the cost coefficients and $\Omega(1,0)$ is the constraints set, $\Omega(1,0) = \{(u_1,u_2) | 0 \le u_1 \le U_1, u_2 = 0\}$, which is bounded and convex. We know that the solution of the above linear programming problem is always at a vertex of the constraint set $\Omega(1,0)$. That is, (u_1^*, u_2^*) is either equal to (0,0) (if $\partial J^*_{T_f}/\partial x_1 > 0$) or equal to $(U_1, 0)$ (if $\partial J^*_{T_f}/\partial x_1 < 0$). Furthermore, the solution is unique if the cost coefficient $\partial J^*_{T_f}/\partial x_1$ is nonzero. In the case $\partial J^*_{T_f}/\partial x_1 = 0$, the solution is not unique anymore since any solution (u_1^*, u_2^*) will not affect the objective function of the linear programming problem at time instant t. However, to keep the cost coefficient $\partial J^*_{T_f}/\partial x_1$ equal to zero at time instant $t + \delta t$, we should produce Part Type 1 at the demand rate d_1 so as to minimize the rate of increase of the cost function $J^*_{T_f}$. In this case (u_1^*, u_2^*) is equal to $(d_1, 0)$. A similar argument is used when the optimal setup state is $\sigma = (0, 1, 0, 0)$.

Steady state case: In this case, let $t_s = t$ and $T_f = t_f$ in the expression of $J^*_{T_f}$ above. Let $J^* = \min_\mu J^S_\mu(x,t) = \min_{\sigma,u\epsilon\Xi(\phi,\Omega)}(t_f - t_s)^{-1}\int_{t_s}^{t_f} g(x(s), \sigma(s))\, ds$. Here, we have an average cost formulation. The HJB equation (see Kushner and Dupuis (1992) for a formal derivation) in this case is given by:

$$J^* = \min_{\sigma,u\epsilon\Xi(\phi,\Omega)} \left\{ g(x,\sigma) + \frac{\partial V(x,t)}{\partial x_1}(U_1 - d_1) + \frac{\partial V(x,t)}{\partial x_2}(u_2 - d_2) \right\};$$

where

$$V(x,t) = \lim_{(t_f - t)\to\infty} J^*_{T_f} - (t_f - t)J^*.$$

As in the previous case, for each time instant t, if we knew $V(x,t)$, we would solve a linear programming problem which decision variables are the production rates u_1 and u_2 and which cost coefficients are $\partial V/\partial x_1$ and $\partial V/\partial x_2$ respectively. Hence, using a similar argument as for the transient case and using Lemma 1, the result follows immediately.

B. Proof of Theorem 2

The proof is based on the Karush–Kuhn–Tucker (KKT) optimality conditions. It is not difficult to show that the optimal τ_1 and τ_2 can be obtained by solving the following nonlinear optimization problem:

$$\text{minimize} \quad F(\tau_1, \tau_2) = \frac{(K + H_1(T_0 + (\alpha_1 - 1)\tau_1 + \alpha_2\tau_2)^2/2 + H_2(T_0 + \alpha_1\tau_1 + (\alpha_2 - 1)\tau_2)^2/2)}{T_0 + \alpha_1\tau_1 + \alpha_2\tau_2}$$

$$\text{subject to}: \quad \tau_i \geq 0 \quad i = 1, 2;$$

where,

$$\begin{aligned} H_i &= \gamma_i d_i(1 - \rho_i) \quad i = 1, 2; \\ \alpha_i &= (1 - \rho_i)/(1 - \rho) \quad i = 1, 2; \\ T_0 &= \delta/(1 - \rho). \end{aligned}$$

Also, it can be shown (see Elhafsi and Bai, 1996a) that $F(\tau_1, \tau_2)$ is strictly convex in τ_1 and τ_2. Since the constraint domain is convex, it follows that the KKT optimality conditions are necessary and sufficient (which establishes the if and only if part of the theorem).

Let $g_i(\tau_1, \tau_2) = -\tau_i (i = 1, 2)$. Then, the KKT optimality conditions are given as follows:

$$\nabla F(\tau_1^*, \tau_2^*) + \mu_1 \nabla g_1(\tau_1^*, \tau_2^*) + \mu_2 \nabla g_2(\tau_1^*, \tau_2^*) = 0$$

$$\mu_1 g_1(\tau_1, \tau_2) = 0$$
$$\mu_2 g_2(\tau_1, \tau_2) = 0$$
$$\mu_i \geq 0 \text{ for } i = 1, 2$$

where, τ_i^* is the optimal value of $\tau_i (i = 1, 2)$. $\nabla F(\tau_1, \tau_2)$, $\nabla g_1(\tau_1, \tau_2)$ and $\nabla g_2(\tau_1, \tau_2)$ are the gradients of $F(\tau_1, \tau_2)$, $g_1(\tau_1, \tau_2)$, and $g_2(\tau_1, \tau_2)$ respectively.

Now, letting $\tau_i^* = 0 (i = 1, 2)$, calculating the gradients and substituting in the KKT conditions, we get:

$$(\alpha_1/2 - 1)H_1 + \alpha_1 H_2/2 - \alpha_1 K/T_0^2 - \mu_1 = 0;$$
$$\alpha_2 H_1/2 + (\alpha_2/2 - 1)H_2 - \alpha_2 K/T_0^2 - \mu_2 = 0.$$

For the KKT conditions to hold true, we must have $\mu_i \geq 0 (i = 1, 2)$. Hence, we must have

$$\alpha_1 K/T_0^2 \leq (\alpha_1/2 - 1)H_1 + \alpha_1 H_2/2;$$
$$\alpha_2 K/T_0^2 \leq \alpha_2 H_1/2 + (\alpha_2/2 - 1)H_2.$$

Substituting α_1, α_2, H_1 and H_2 by their expressions gives:

$$K \leq T_0^2((1 - \rho_2)(\gamma_2 d_2 - \gamma_1 d_1) + \rho\gamma_1 d_1)/2;$$
$$K \leq T_0^2((\rho + \rho_1 - 1)\gamma_2 d_2 + (1 - \rho_1)\gamma_1 d_1)/2.$$

Now letting K=0, we get the following conditions:

$$(1 - \rho_2)(\gamma_2 d_2 - \gamma_1 d_1) + \rho\gamma_1 d_1 \geq 0; \quad (a)$$
$$(\rho + \rho_1 - 1)\gamma_2 d_2 + (1 - \rho_1)\gamma_1 d_1 \geq 0. \quad (b)$$

Notice that condition (a) is always satisfied since $\gamma_2 d_2 \geq \gamma_1 d_1$. For condition (b), if $\rho + \rho_1 - 1 \geq 0$, then it is always satisfied. If $\rho + \rho_1 - 1 < 0$, then it is satisfied only if $\gamma_2 d_2/\gamma_1 d_1 \leq (1 - \rho_1)/(1 - \rho - \rho_1)$. Which completes the proof.

C. Proof of Theorem 3

We prove the theorem by contradiction. Assume that $\tau_1^* > 0$ and $\tau_2^* > 0$, then Q_1^* and Q_2^* are given by (14a) and (14b), where K=0. In this case, we have:

$$Q_1^* = 2q_1 d_2\gamma_2/(\alpha_2 d_1\gamma_1 + \alpha_1 d_2\gamma_2), \text{ and } Q_2^* = 2q_2 d_1\gamma_1/(\alpha_2 d_1\gamma_1 + \alpha_1 d_2\gamma_2).$$

Now, using (15a) and (15b), we get:

$$\begin{aligned}\tau_1^* &= 2q_2\gamma_1 d_1/d_2(\alpha_2\gamma_1 d_1 + \alpha_1\gamma_2 d_2) - 2q_1\gamma_2 d_2\rho_1/d_1(1 - \rho_1) \\ &\quad (\alpha_2\gamma_1 d_1 + \alpha_1\gamma_2 d_2) - \delta > 0; \\ \text{and } \tau_2^* &= 2q_1\gamma_2 d_2/d_1(\alpha_2\gamma_1 d_1 + \alpha_1\gamma_2 d_2) - 2q_2\gamma_1 d_1\rho_2/d_2(1 - \rho_2) \\ &\quad (\alpha_2\gamma_1 d_1 + \alpha_1\gamma_2 d_2) - \delta > 0.\end{aligned}$$

substituting q_1 and q_2 by their expressions, simplifying and rearranging terms gives:

$$(1-\rho_2)\gamma_1 d_1 - (1+\rho_1)\gamma_2 d_2 > 0 \text{ and } (1-\rho_1)\gamma_2 d_2 - (1+\rho_2)\gamma_1 d_1 > 0.$$
$$\Leftrightarrow \quad \gamma_2 d_2/\gamma_1 d_1 < (1-\rho_2)/(1+\rho_1) < 1 \text{ and } \gamma_2 d_2/\gamma_1 d_1 > (1+\rho_2)/(1-\rho_1) > 1$$

Which cannot be. Hence, in the case of no setup costs, τ_1^* and τ_2^* cannot be nonzero simultaneously.

D. Optimal Control for Initial Surplus Levels in Region $\mathcal{R}^o$

To summarize the control actions in Region $\mathcal{R}^o$, let $x = (a, b)$ be the vector of initial surplus levels in Region $\mathcal{R}^o$.

- If $x \in G1$:
  1: Setup the machine for Part Type 1;
  2: After the setup change, produce Part Type 1 at the rate U_1;
  3: When the surplus level of Part Type 1 becomes 0, change the production rate to d_1;
  4: When the surplus level of Part Type 2 becomes $x_2 = A_2 + A_1 d_2 \rho_1 / d_1(1-\rho_1)$, switch to the control actions of the cyclic schedule.

- If $x \in G11 \cup H11$:
  1: Do not produce either part type;
  2: When the surplus level of Part Type 1 becomes $\delta_1 d_1$, start a setup change for Part Type 1;
  3: After the setup change, produce Part Type 1 at the demand rate d_1;
  4: When the surplus level of Part Type 2 becomes $x_2 = A_2 + A_1 d_2 \rho_1 / d_1(1-\rho_1)$, switch to the control actions of the cyclic schedule.

- If $x \in G12 \cup H21$:
  1: Do not produce either part type;
  2: When the surplus level of Part Type 2 reaches level l_2, immediately start a setup change for Part Type 2. l_2 is given by
  $l_2 = (1-\rho_2)(b - (d_2/d_1)a) + (d_2/d_1)(1-\rho_2)A_1 + \rho_2 A_2$;
  3: at the end of the setup change, switch to the cyclic schedule control actions.

- If $x \in G21 \cup H12$:
  1: Do not produce either part type;
  2: When the surplus level of Part Type 1 reaches the level l_1, immediately start a setup change for Part Type 1. l_1 is given by

$l_1 = \rho_1(b - (d_2/d_1)a) + (d_2/d_1)\rho_1 C_1 + (1-\rho_1)C_2$;
3: at the end of the setup change, switch to the cyclic schedule control actions.

- If $x \in G22 \cup H22$:
  1: Do not produce either part type;
  2: When the surplus level of Part Type 2 becomes $\delta_2 d_2$, start a setup change for Part Type 2;
  3: After the setup change, produce Part Type 2 at the demand rate d_2;
  4: When the surplus level of Part Type 1 becomes $x_1 = C_1 + C_2 d_1 \rho_2 / d_2(1-\rho_2)$, switch to the control actions of the cyclic schedule.

- If $x \in G2$:
  1: Setup the machine for Part Type 2;
  2: After the setup change, produce Part Type 2 at the rate U_2;
  3: When the surplus level of Part Type 2 becomes 0, change the production rate to d_2;
  4: When the surplus level of Part Type 1 becomes $x_1 = C_1 + C_2 d_1 \rho_2 / d_2(1-\rho_2)$, switch to the control actions of the cyclic schedule.

References

Bai, S. X. and Elhafsi, M. (1993), "Real-Time Scheduling of a Manufacturing System with Non-Resumable Setup Changes," Research Report 93–28. Department of Industrial and Systems Engineering. University of Florida.

Bai, S. X. and Elhafsi, M. (1996), "Transient and Steady State Analysis of a Manufacturing System with Setup Changes," *Journal of Global Optimization* **8**, 349–378.

Buzacott, J. A. and Ozkarahan, I. A. (1983), "One- and Two-Stage Scheduling of Two Products with Distributed Inserted Idle Time: The Benefits of a Controllable Production Rate," *Naval Research Logistics Quarterly* **30**, 675–696.

Caramanis, M., Sharifnia, A., Hu, J. and Gershwin, S. B. (1991), "Development of a Science Base for Planning and Scheduling Manufacturing Systems," *Proceedings of the 1991 NSF Design and Manufacturing Systems Conference*, The University of Texas at Austin, January 9–11, 1991.

Carreno, J. J. (1990), "Economic Lot Scheduling for Multiple Products on Parallel Identical Processors," *Management Science* **36**, 348–358.

Connolly, S. (1992), "A Real-Time Policy for Performing Setup Changes in a Manufacturing System," Master's Thesis, MIT Operations Research Center, May, 1992; MIT Laboratory for Manufacturing and Productivity. Report LMP-92–0005.

Dobson, G. (1987), "The Economic Lot Scheduling Problem: Achieving Feasibility Using Time-Varying Lot Sizes," *Operations Research* **35**, 764–771.

Elhafsi, M. and Bai, S. X. (1996a), "The Common Cycle Economic Lot Scheduling Problem with Backorders: Benefits of Controllable Production Rates." To appear in *Journal of Global Optimization.*

Elhafsi, M. and Bai, S. X. (1996b), "Optimal Production and Setup Control of a Dynamic Two-Product Manufacturing System: Analytical Solution." To appear in *Journal of Computers and Mathematical Modelling.*

Elmaghraby, S. E. (1978), "The Economic Lot Scheduling Problem (ELSP): Review and Extensions," *Management Science* **24**, 587–598.

Gallego, G. (1989), "Real-Time Scheduling of Several Products on a Single Facility With Setups and Backorders," Ph.D. Thesis, School of OR&IE, Cornell University, Ithaca, NY.

Gershwin, S. B. (1993), *Manufacturing Systems Engineering*. Prentice Hall, Inc.

Goyal, S. K. (1984), "Determination of Economic Production Quantities for a Two-Product Single Machine System," *International Journal of Production Research* **22**, 121–126.

Hu, J. and Caramanis, M. (1992), "Near Optimal Set-up Scheduling for Flexible Manufacturing Systems," *Proceedings of the Third Renselaer Polytechnic Institute International Conference on Computer Integrated Manufacturing*, Troy, New York, May 20–22, 1992.

Hu, J. and Caramanis, M. (1995), "Dynamic Set-up Scheduling of Flexible Manufacturing Systems: Design and Stability of Near Optimal General Round Robin Policies," To appear in *Discrete Event Systems, IMA Volumes in Mathematics and its Applications Series*, 1995.

Jones, P. and Inman, R. (1989), "When Is the Economic Lot Scheduling Problem Easy?," *IIE Transactions* **21**, 11–20.

Kimemia, J. and Gershwin, S. B. (1983), "An Algorithm for the Computer Control of a Flexible Manufacturing System," *IIE Transactions* **15**(4), 353–362.

Kumar, P. R., and Seidman, T. I. (1990), "Dynamic Instabilities and Stabilization Methods in Distributed Real-Time Scheduling of Manufacturing Systems," *IEEE Transactions on Automatic Control* **35**, 289–338.

Kushner, H. J. and Dupuis, P. G. (1992), *Numerical Methods for Stochastic Control Problems in Continuous Time*. Springer-Verlag.

Moon, I., Gallego, G., and Simchi-Levi, D. (1991), "Controllable Production Rates in a Family Context," *International Journal of Production Research* **29**, 2459–2470.

Olsder, G. J. and Suri, R. (1980), "The Optimal Control of Part Routing in a Manufacturing System with Failure Prone Machines," *Proceedings of the 19th IEEE Conference on Decision and Control*.

Perkins, J. and Kumar, P. R. (1989), "Stable, Distributed, Real-Time Scheduling of Flexible Manufacturing/Assembly/Disassembly Systems," *IEEE Transactions on Automatic Control* **34**, 139–148.

Roundy, R. (1985), "98%-Effective Integer-Ratio Lot Sizing for One-Warehouse Multi-Retailer Systems," *Management Science* **31**, 1416–1430.

Sharifnia, A., Caramanis, M., and Gershwin, S. B. (1991), "Dynamic Setup Scheduling and Flow Control in Manufacturing Systems," *Discrete Event Dynamic Systems: Theory and Applications*, **1**, 149–175.

Silver, E. (1990), "Deliberately Slowing Down Output in a Family Production Context," *International Journal of Production Research* **28**, 17–27.

Srivatsan, N., and Gershwin, S. B. (1990), "Selection of Setup Times in a Hierarchically Controlled Manufacturing System," *Proceedings of the 29th IEEE Conference on Decision and Control*, Hawaii, December 1990.