Nonlinear Optimization: Characterization of Structural Stability

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Abstract. We study global stability properties for differentiable optimization problems of the type:

$$\mathcal{P}(f, H, G): \text{Min } f(x) \text{ on } M[H, G] = \{x \in \mathbb{R}^n \mid H(x) = 0, G(x) \geqq 0\}.$$

Two problems are called equivalent if each lower level set of one problem is mapped homeomorphically onto a corresponding lower level set of the other one. In case that $\mathcal{P}(\tilde{f}, \tilde{H}, \tilde{G})$ is equivalent with $\mathcal{P}(f, H, G)$ for all $(\tilde{f}, \tilde{H}, \tilde{G})$ in some neighbourhood of (f, H, G) we call $\mathcal{P}(f, H, G)$ structurally stable; the topology used takes derivatives up to order two into account. Under the assumption that $M[H, G]$ is compact we prove that structural stability of $\mathcal{P}(f, H, G)$ is equivalent with the validity of the following three conditions:

C.1. The Mangasarian–Fromovitz constraint qualification is satisfied at every point of $M[H, G]$.
C.2. Every Kuhn–Tucker point of $\mathcal{P}(f, H, G)$ is strongly stable in the sense of Kojima.
C.3. Different Kuhn–Tucker points have different (f-)values.

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1. Introduction, Main Result

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and $C^k(\mathbb{R}^n, \mathbb{R})$ the space of real valued, k-times continuously differentiable functions on $\mathbb{R}^n$. Moreover, we fix two finite index sets I, J, with $I = \{1, \ldots, m\}$, $J = \{1, \ldots, s\}$ and $m < n$. In the sequel, the functions $f, h_i, g_j, i \in I, j \in J$, belong to $C^2(\mathbb{R}^n, \mathbb{R})$, and H, resp. G, stands for $(h_1, \ldots, h_m)^T$, resp. $(g_1, \ldots, g_s)^T$.

The optimization problem under consideration will be of the following standard type:

$$\mathcal{P}(f, H, G): \text{Minimize } f \text{ on } M[H, G], \tag{1.1}$$

where the feasible set $M[H, G]$ is defined as

$$M[H, G] = \{x \in \mathbb{R}^n \mid h_i(x) = 0, g_j(x) \geqq 0, i \in I, j \in J\}. \tag{1.2}$$

The lower level set corresponding to the functional value t will be denoted as follows:

$$\mathcal{L}^t(f, H, G) := \{x \in M[H, G] \mid f(x) \leqq t\}. \tag{1.3}$$

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DEFINITION 1.1. The optimization problems $\mathscr{P}(f, H, G)$ and $\mathscr{P}(\tilde{f}, \tilde{H}, \tilde{G})$ are called *equivalent* if there exist continuous mappings $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $\psi : \mathbb{R} \to \mathbb{R}$ with the properties P1–P3:

P1. For every $t \in \mathbb{R}$ the mapping ϕ_t: $\mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism from $\mathbb{R}^n$ onto itself, where $\phi_t(x) := \phi(t, x)$.

P2. The mapping ψ is a homeomorphism from $\mathbb{R}$ onto itself and ψ is monotonically increasing.

P3. $\phi_t[\mathscr{L}^t(f, H, G)] = \mathscr{L}^{\psi(t)}(\tilde{f}, \tilde{H}, \tilde{G})$ for all $t \in \mathbb{R}$. □

In [1] it is pointed out that the above concept defines an equivalence relation on the set of optimization problems of type $\mathscr{P}(f, H, G)$. The above equivalence concept is very natural w.r.t. optimization since – *globally* – all descent flows in one problem are carried over into corresponding descent flows in the other one. The fact that in Definition 1.1 an one parameter family ϕ_t of homeomorphisms rather than one fixed homemorphism is chosen is due to the possible shift of stationary points from the boundary into the interior of the feasible set under small perturbations of the problem data (cf. also [1]).

DEFINITION 1.2. The optimization problem $\mathscr{P}(f, H, G)$ is called *structurally stable* if there exists a C_S^2-neighbourhood $\mathscr{O}$ of (f, H, G) with the property that $\mathscr{P}(f, H, G)$ and $\mathscr{P}(\tilde{f}, \tilde{H}, \tilde{G})$ are equivalent for all $(\tilde{f}, \tilde{H}, \tilde{G}) \in \mathscr{O}$. □

The C_S^2-topology above for the product $\Pi_{i=1}^r C^2(\mathbb{R}^n, \mathbb{R})$ will be the product-topology generated by the strong (or Whitney-)C^2-topology C_S^2 on each factor $C^2(\mathbb{R}^n, \mathbb{R})$ (cf. [2], [3]). A typical base-neighbourhood of $f \in C^2(\mathbb{R}^n, \mathbb{R})$ will be the set $f + \mathscr{W}_\varepsilon$, where $\mathscr{W}_\varepsilon$ is defined as follows with the aid of a continuous positive function $\varepsilon : \mathbb{R}^n \to \mathbb{R}$,

$$\mathscr{W}_\varepsilon = \Big\{ \phi \in C^2(\mathbb{R}^n, \mathbb{R}) \mid |\phi(x)| + \sum_i \left| \frac{\partial \phi}{\partial x_i}(x) \right| + \sum_{i,j} \left| \frac{\partial^2 \phi}{\partial x_i \, \partial x_j}(x) \right| < \varepsilon(x) \quad \text{for all } x \in \mathbb{R}^n \Big\} .$$

Since structural stability seems to be a very natural and basic concept, the next main theorem underlines the importance of the constraint qualification of Mangasarian and Fromovitz on the one hand, and the concept of strong stability of Kojima on the other hand.

MAIN THEOREM. *The optimization problem $\mathscr{P}(f, H, G)$ with compact feasible set $M[H, G]$ is structurally stable if and only if the following three conditions are satisfied*:

C1. *The Mangasarian–Fromovitz constraint qualification is satisfied at every point of $M[H, G]$.*

C2. *Every Kuhn–Tucker point of $\mathscr{P}(f, H, G)$ is strongly stable in the sense of Kojima.*

C3. *Different Kuhn–Tucker points have different (f-)values.* □

We proceed with a clarification of the notions used in the main theorem. Let Df (resp. D^2f) denote the row-vector of first partial derivatives (resp. the matrix of second partial derivatives).

Futhermore we put

$$J_0(x) = \{ j \in J \mid g_j(x) = 0 \} . \tag{1.4}$$

DEFINITION 1.3. The *Linear Independence Constraint Qualification* (shortly LICQ) is said to hold at $x \in M[H, G]$ if the vectors $Dh_i(x)$, $i \in I$, $Dg_j(x)$, $j \in J_0(x)$, are linearly independent. The *Mangasarian–Fromovitz Constraint Qualification* (shortly MFCQ) is said to hold at $x \in M[H, G]$ if the following two conditions are satisfied:

MF 1. The vectors $Dh_i(x)$, $i \in I$, are linearly independent.

MF 2. There exists a vector $\xi \in \mathbb{R}^n$ satisfying:

$$\left.\begin{array}{ll} Dh_i(x)\xi = 0\,, & i \in I \\ Dg_j(x)\xi > 0\,, & i \in J_0(x) \end{array}\right\} . \tag{1.5}$$

A vector ξ satisfying (1.5) will be called an *MF-vector.* □

DEFINITION 1.4. A point $\bar{x} \in M[H, G]$ is called a *Kuhn–Tucker point* if there exist real numbers λ_i, μ_j satisfying the following relations:

$$Df(\bar{x}) = \sum_{i \in I} \lambda_i Dh_i(\bar{x}) + \sum_{j \in J_0(\bar{x})} \mu_j Dg_j(\bar{x}) \tag{1.6a}$$

$$\mu_j \geqq 0\,, \quad j \in J_0(\bar{x})\,. \tag{1.6b}$$

The numbers λ_i, μ_j above are called *Lagrange multipliers.* □

The set of possible Lagrange multipliers at a Kuhn–Tucker point is compact (in fact a compact polyhedron) if and only if MFCQ is satisfied at $\bar{x}$ (cf. [4]). Of course, the latter set is a singleton if LICQ holds.

For a given problem $\mathcal{P}(f, H, G)$ and a subset $\mathcal{U}$ of $\mathbb{R}^n$ we put:

$$\begin{aligned} &\mathrm{norm}[(f, H, G), \mathcal{U}] = \\ &= \sup_{x \in \mathcal{U}} \max_{\phi \in \{f, h_i, i \in I, g_j, j \in J\}} \left\{ |\phi(x)| + \sum_i \left| \frac{\partial \phi}{\partial x_i}(x) \right| + \sum_{i,j} \left| \frac{\partial^2 \phi}{\partial x_i \, \partial x_j}(x) \right| \right\} . \end{aligned} \tag{1.7}$$

For $x \in \mathbb{R}^n$ and $\rho > 0$ let $B(x, \rho) \subset \mathbb{R}^n$ denote the open Euclidean ball centered at x with radius ρ, and according to M. Kojima [5] we define:

DEFINITION 1.5. Let $x \in M[H, G]$ be a Kuhn–Tucker point for $\mathcal{P}(f, H, G)$. Then, x is called *strongly stable* if for some $\bar{\delta} > 0$ and each $\delta \in (0, \bar{\delta}]$ there exists an $\alpha > 0$ such that whenever $(\tilde{f}, \tilde{H}, \tilde{G})$ satisfies norm $[(f-\tilde{f}, H-\tilde{H}, G-\tilde{G}), B(x, \delta)] \leqq \alpha$, the ball $B(x, \delta)$ contains a Kuhn–Tucker point for $\mathcal{P}(\tilde{f}, \tilde{H}, \tilde{G})$ which is unique in $B(x, \bar{\delta})$. □

In this paper we shall actually work with another formulation of strong stability which is equivalent to the above one under the assumption that MFCQ is valid (cf. [5]). For this we need some more definitions.

Let $\bar{x} \in M[H, G]$ be a point at which (1.6a) is fulfilled. With respect to a set of Lagrange multipliers λ_i, $i \in I$, μ_j, $j \in J_0(\bar{x})$, satisfying the relation (1.6a) we introduce the Lagrange function $L_{[\lambda,\mu]}$:

$$L_{[\lambda,\mu]}(x) = f(x) - \sum_{i \in I} \lambda_i h_i(x) - \sum_{j \in J_0(\bar{x})} \mu_j g_j(x) . \tag{1.8}$$

Let $\Lambda(\bar{x})$ denote the polyhedron formed by the set of vectors $(\lambda, \mu) := (\ldots, \lambda_i, \ldots, \mu_j, \ldots)_{i \in I, j \in J_0(\bar{x})}$ which satisfy both (1.6a) and (1.6b); here, some fixed ordering of the components of (λ, μ) is assumed. So we have

$$\Lambda(\bar{x}) = \{(\lambda, \mu) \in \mathbb{R}^{|I|+|J_0(\bar{x})|} \mid (\lambda, \mu) \text{ satisfies (1.6a) and (1.6b)}\} . \tag{1.9}$$

Finally, for $x \in \mathbb{R}^n$ and $\tilde{J} \subset J$ we put:

$$W(x, \tilde{J}) = \{\xi \in \mathbb{R}^n \mid Dh_i(x)\xi = 0, i \in I, Dg_j(x)\xi = 0, j \in \tilde{J}\} . \tag{1.10}$$

LEMMA 1.1. (M. Kojima, [5]). *Let $\bar{x} \in M[H, G]$ be a Kuhn–Tucker point for $\mathcal{P}(f, H, G)$.*

(i) *If LICQ is satisfied at $\bar{x}$, then $\bar{x}$ is strongly stable if and only if the matrix $D^2 L_{[\lambda,\mu]}(\bar{x})$ has nonvanishing determinants with a common sign on the subspaces $W(\bar{x}, \tilde{J})$, for all $\tilde{J}$ with $J_+(\bar{x}) \subset \tilde{J} \subset J_0(\bar{x})$, where*

$$J_+(\bar{x}) = \{j \in J_0(\bar{x}) \mid \mu_j > 0\} . \tag{1.11}$$

(ii) *Let MFCQ be satisfied at $\bar{x}$, but LICQ not. Then, $\bar{x}$ is strongly stable if and only if for every $(\lambda, \mu) \in \Lambda(\bar{x})$ the matrix $D^2 L_{[\lambda,\mu]}(\bar{x})$ is positive definite on the subspace $W(\bar{x}, J_+(\bar{x}))$, with $J_+(\bar{x})$ as in* (1.11). □

We call a Kuhn–Tucker point $\bar{x}$ *nondegenerate*, if LICQ holds at $\bar{x}$ together with $J_+(\bar{x}) = J_0(\bar{x})$ and with nonsingularity of $D^2 L_{[\lambda,\mu]}(\bar{x})$ on $W(\bar{x}, J_0(\bar{x}))$.

The necessity part of the Main Theorem has been proved in [1] with special perturbation techniques and tools from algebraic topology. So, in this paper we turn to the sufficiency part. Its proof is divided into three parts which correspond to the next three sections. This main body of the proof is contained in Section 4.

2. Reduction to the Case $I = \emptyset$ and ψ = Identity

In this section we first reduce the sufficiency part of the proof of the Main Theorem to the case with inequality constraints only. Let the optimization problem $\mathscr{P}(f, H, G)$, with compact feasible set $M[H, G]$, be given and suppose that the conditions C1–C3 as formulated in the Main Theorem, hold. From the definition of strong stability together with our constraint qualification MFCQ it follows that strongly stable stationary points are isolated and altogether describe a closed set. Hence, on the compact set $M[H, G]$ there is a finite number of them, say $\bar{x}_1, \ldots, \bar{x}_l$.

Starting with a sufficiently small C_S^2-neighbourhood $\mathscr{O}$ of (f, H, G) we proceed as follows. For each $(\tilde{f}, \tilde{H}, \tilde{G}) \in \mathscr{O}$ we perform a coordinate transformation $\phi_{(\tilde{f},\tilde{H},\tilde{G})}$ in x-space of compact support, shortly $\tilde{\phi}$, such that $\tilde{\phi}$ is of class C^1 and, moreover, $\tilde{\phi}$ is of class C^2 in a neighbourhood of the points $\bar{x}_1, \ldots, \bar{x}_l$; the coordinate transformation $\tilde{\phi}$ maps the set $M[\tilde{H}]$ in a neighbourhood of $M[\tilde{H}, \tilde{G}]$ onto the corresponding set of the unperturbed problem, where $M[\tilde{H}] = \{x \in \mathbb{R}^n \mid \tilde{H}(x) = 0\}$.

Moreover, the coordinate transformations $\tilde{\phi}$ are performed in an uniform way for all $(\tilde{f}, \tilde{H}, \tilde{G}) \in \mathscr{O}$; hence, their construction depends, more or less, only on the neighbourhood $\mathscr{O}$.

Note that a (local) C^2-coordinate transformation in x-space does not affect the characterization of strong stability of stationary points in the sense of Lemma 1.1. Having established the above coordinate transformation $\tilde{\phi}$ we may assume that $\tilde{H} = H$, and further coordinate transformations can be performed such that they leave the set $M[H]$ in a neighbourhood $M[H, G]$ invariant; in particular, certain flows can be taken to be tangential to the zero set of H, which then have a parallel extension to a neighbourhood of the latter set. Therefore, after the construction of the transformations $\tilde{\phi}$, we may assume that there are no equality constraints present, i.e. $I = \emptyset$.

The local C^2-coordinate transformation. Choose $\bar{x} \in \{\bar{x}_1, \ldots, \bar{x}_l\}$, and $(\tilde{f}, \tilde{H}, \tilde{G}) \in \mathscr{O}$. From the condition MF1 in Definition 1.3 it follows that the zero-sets $M[H]$ and $M[\tilde{H}]$ are C^2-manifolds in some neighbourhood U of $\bar{x}$. In fact, put $y = \zeta(x)$, where $y_i = h_i(x)$, $i = 1, \ldots, m$, $y_j = \xi_j^T(x - \bar{x})$, $j = m + 1, \ldots, n$, and where ξ_j, $j = m + 1, \ldots, n$, form a basis for the orthogonal complement of the set $\{D^T h_i(\bar{x}), i = 1, \ldots, m\}$. Then, ζ is a local C^2-coordinate transformation around $\bar{x}$, mapping the zero set $M[H]$ to the set $\{y \mid y_i = 0, i = 1, \ldots, m\}$. The set $M[\tilde{H}]$ in these new coordinates takes the form $\{y \mid y_i = \eta_i(y_{m+1}, \ldots, y_n), i = 1, \ldots, m\}$, where η_i are C^2-functions. Consider the shift-mapping $\gamma : y_i \mapsto y_i - \eta_i \cdot \beta_i$, $i = 1, \ldots, m$, $y_j \mapsto y_j$, $j = m + 1, \ldots, n$, where each $\beta_i = \beta_i(y_1, \ldots, y_n)$ is of class C^2 having compact support, being equal to one in a neighbourhood of the origin, and only depending on the above C_S^2-neighbourhood $\mathscr{O}$ of (f, H, G). This shift-mapping γ induces a C^2-coordinate transformation in the original x-space thereby *locally* mapping the set $M[\tilde{H}]$ onto $M[H]$.

The global C^1-coordinate transformation. Now we finish the construction of the desired coordinate transformation $\phi_{(\tilde{f},\tilde{H},\tilde{G})}$. From the preceding part we may assume that the mapping $\tilde{H}$ vanishes on the set $M[H]$ in some neighbourhood of the points $\bar{x}_1, \ldots, \bar{x}_l$. Moreover, we may assume that $M[\tilde{H}, \tilde{G}]$ is contained in a neighbourhood $\mathcal{U}$ of $M[H, G]$, where $\mathcal{U}$ only depends on the chosen C_S^2-neighbourhood $\mathcal{O}$ of (f, H, G). By means of the flow of a suitable C^1-vectorfield we transform (in a neighbourhood of $M[H, G]$) the zero set $M[\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_m]$ to the set $M[h_1, \tilde{h}_2, \ldots, \tilde{h}_m]$. Then, similarly, $M[h_1, \tilde{h}_2, \tilde{h}_3, \ldots, \tilde{h}_m]$ is transformed into the set $M[h_1, h_2, \tilde{h}_3, \ldots, \tilde{h}_m]$, and so, after m steps we have transformed $M[\tilde{H}]$ into $M[H]$ (around $M[H, G]$). In particular, points at which $\tilde{H}(x) = H(x)$ remain fixed during the latter transformation; hence, the essence of the former local C^2-transformation is not disturbed. We only need to indicate the first step, i.e. the transformation of $M[\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_m]$ into the set $M[h_1, \tilde{h}_2, \ldots, \tilde{h}_m]$ within the neighbourhood $\mathcal{U}$ of $M[H, G]$. To this aim we consider the homotopy $\hat{H}(x, v) = vh_1(x) + (1 - v)\tilde{h}_1(x)$. The above neighbourhood $\mathcal{U}$ is assumed to be small enough in order that the following constructions can be made:

The vectors $D_x\hat{H}(x, v)$, $Dh_i(x)$, $i = 2, \ldots, m$, are linearly independent on $\mathcal{U} \times (-\varepsilon, 1 + \varepsilon)$, some $\varepsilon > 0$. For $x \in M[h_2, \ldots, h_m] \cap \mathcal{U}$, let $\pi(x)$ denote the orthogonal projection (matrix) of $\mathbb{R}^n$ onto the tangent space $\{\xi \in \mathbb{R}^n \mid Dh_i(x)\xi = 0,\ i = 2, \ldots, m\}$ corresponding to the manifold $M[h_2, \ldots, h_m]$. Next, consider the vectorfield $\mathcal{F}$ on $\mathcal{U} \times (-\varepsilon, 1 + \varepsilon)$:

$$\mathcal{F}(x, v) = \left(-D_v\hat{H} \cdot \frac{\pi \cdot D_x^T\hat{H}}{D_x\hat{H} \cdot \pi \cdot D_x^T\hat{H}}, 1\right). \tag{2.1}$$

Note that the term $D_x\hat{H} \cdot \pi \cdot D_x^T\hat{H}$ is positive on $\mathcal{U} \times (-\varepsilon, 1 + \varepsilon)$. From (2.1) we see that $D\hat{H} \cdot \mathcal{F} \equiv 0$ on $\mathcal{U}$, hence $\hat{H}$ remains constant on the trajectories of $\mathcal{F}$. So, integrating $\mathcal{F}$ in time one, the zero set of $\hat{H}(\cdot, 0)$ $(= \tilde{h}_1)$ is mapped to the zero set of $\hat{H}(\cdot, 1)$ $(= h_1)$. Moreover, the x-component of the flow of $\mathcal{F}$, starting at $M[\tilde{h}_2, \ldots, \tilde{h}_m]$, remains on the latter set. So, in particular, the set $M[\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_m]$ is mapped (around $M[H, G]$) in integration time one to the set $M[h_1, \tilde{h}_2, \ldots, \tilde{h}_m]$. (Technically, the vector field $\mathcal{F}$ should be cut off to zero outside a suitable neighbourhood of $M[H, G] \times [0, 1]$.)

Now we explain the reduction to the case $\psi =$ identity. The only task of the mapping ψ is to map the values of the stationary points of the unperturbed problem to the corresponding values of the perturbed problem. From strong stability it follows that the stationary points, and hence also their corresponding objective function values, depend continuously on C_S^2-perturbations of the problem data.

From condition C3 we see that the stationary values $f(\bar{x}_1), \ldots, f(\bar{x}_l)$ are pairwise different. So, we only have to perform a shift on $\mathbb{R}$, taking the values $f(\bar{x}_1), \ldots, f(\bar{x}_l)$ to neighbouring values $\gamma_1, \ldots, \gamma_l$. Such a shift can be performed by means of integrating (in time one) the vectorfield

$$\xi_\gamma(x) = \sum_{i=1}^{l} (f(\bar{x}_i) - \gamma_i) \cdot \eta_i(x), \tag{2.2}$$

where $\eta_i(x)$ is smooth, nonnegative, identically equal to one in a (γ-independent) neighbourhood of $f(\bar{x}_i)$ and identically equal to zero outside a slightly larger neighbourhood of $f(\bar{x}_i)$, $i = 1, \ldots, l$.

So, from now on we may assume that the critical values of $\mathcal{P}(f, H, G)$ and the perturbed problem $\mathcal{P}(\tilde{f}, \tilde{H}, \tilde{G})$ coincide for $(\tilde{f}, \tilde{H}, \tilde{G})$ in some C_S^2-neighbourhood $\mathcal{O}$ of (f, H, G), and, in addition, that there are no equality constraints present, i.e. $I = \emptyset$.

3. Global Construction, Outside the Stationary Points

In virtue of Section 2 we may omit the equality constraints. Put

$$M[G, F, t] = \{x \in \mathbb{R}^n \mid g_j(x) \geqq 0,\ j \in J,\ t - f(x) \geqq 0\}\ . \tag{3.1}$$

Obviously, we have $M[G, F, t] = \mathcal{L}^t(f, G)$ (cf. (1.3), omitting H). For fixed t, the set $M[G, F, t]$ is a usual constraint set with one special inequality constraint, namely $t - f(x) \geqq 0$. We say that MFCQ is fulfilled at $x \in M[G, F, t]$ if MFCQ is fulfilled with respect to the (active) inequality constraints g_j, $j \in J$, and the additional inequality constraint $t - f(\cdot)$. In the latter case, an MF-vector is defined analogously according to (1.5). The following easy lemma will be crucial in this section, where $M[G]$ denotes the set $M[H, G]$ with H omitted.

LEMMA 3.1. *Let $\bar{x} \in M[G]$ be given, and suppose that MFCQ is fulfilled. Then, $\bar{x}$ is a Kuhn–Tucker point for $\mathcal{P}(f, G)$ if and only if MFCQ is violated for $\bar{x}$ viewed at as an element of $M[G, F, f(\bar{x})]$.* □

Let $\bar{x}_1, \ldots, \bar{x}_l$ again denote the stationary points of the unperturbed problem $\mathcal{P}(f, G)$, equality constraints being omitted. We make the following assumption.

ASSUMPTION A. *There exists an $\varepsilon > 0$, such that the closed balls $B(\bar{x}_i, \varepsilon)$, $i = 1, \ldots, l$, are pairwise disjoint and such that the following holds:*

for all t the sets $M[G, F, t]$ and $M[\tilde{G}, \tilde{F}, t]$ coincide on $\overline{B(\bar{x}_i, \varepsilon)} \backslash B(\bar{x}_i, \frac{1}{2}\varepsilon)$, $i = 1, \ldots, l$. □

Put $B = \bigcup_{i=1}^{l} \overline{B(\bar{x}_i, \frac{1}{2}\varepsilon)}$. At each point $\hat{x} \in \partial M[G, F, t] \backslash B$ we choose an MF-vector $\xi_{\hat{x}}$ (which then is an MF-vector in a neighbourhood of $\hat{x}$ with respect to both $\partial M[\tilde{G}, \tilde{F}, t]$ and $\partial M[G, F, t]$). By means of a smooth partition of unity we obtain a smooth vectorfield $\xi(x)$ having the property that $\xi(x)$ is an MF-vector for all $x \in \partial M[G, F, t] \backslash B$ resp. $x \in \partial M[\tilde{G}, \tilde{F}, t] \backslash B$. Next, consider the normalized vectorfield $\eta(x) = \xi(x) / \| \xi(x) \|$ and cut it smoothy off to zero outside a neighbourhood of $M[G] \backslash B$. Now, for each fixed t we rescale η per integral curve and obtain a Lipschitz continuous vectorfield such that in time-one integration the set $\partial M[G, F, t] \backslash B$ is mapped to $\partial M[\tilde{G}, \tilde{F}, t] \backslash B$ (see also [6]). This gives a Lipschitzian homeomorphism ϕ_t sending $M[G, F, t] \backslash B$ to $M[\tilde{G}, \tilde{F}, t] \backslash B$. It is not difficult to see that $\phi_t(x)$ is *jointly* Lipschitzian in both t and x.

Note that $\phi_t(x) = x$ on the set $\bigcup_{i=1}^{l} \overline{B(\bar{x}_i, \varepsilon)} \backslash B(\bar{x}_i, \frac{1}{2}\varepsilon)$.

4. The Local Construction

The following construction, in which we omit equality constraints again, consists of four parts. In each of the first three parts we describe appropriate mappings in x-space, due to specific *positions* (boundary, resp. interior) and *characteristics* of our Kuhn–Tucker points before and after a sufficiently small C_S^2-perturbation $(f, G) \to (\tilde{f}, \tilde{G})$. Let us call these points $(\bar{x}_1=)\bar{x}_1^u, \ldots, (\bar{x}_l=)\bar{x}_l^u$, and $\bar{x}_1^d, \ldots, \bar{x}_l^d$, resp. As our work will be local, we may restrict to one single undisturbed stationary point $\bar{x} = \bar{x}^u$ and one single (corresponding) disturbed stationary point $\bar{x}^d$.

Recall that the properties of MFCQ and strong stability are fulfilled at $\bar{x}^u$ and at $\bar{x}^d$. In order to express the mentioned characteristics in a brief manner we recall Kojima's *stationary index* at $\bar{x}^u$, denoted by s.index $(\bar{x}^u, (f, G))$ (cf. [5]). This is the number of negative eigenvalues of $D^2L_{[\mu]}(\bar{x}^u) \mid W(\bar{x}^u, J_+(\bar{x}^u))$ provided LICQ holds at $\bar{x}^u$, and s.index $(\bar{x}^u, (f, G))$ to be 0 otherwise. It is known that s.index$(\bar{x}^d, (\tilde{f}, \tilde{G}))$ = s.index$(\bar{x}^u, (f, G))$ (cf. [5]).

In the fourth part we fit our local constructions into the global construction established in Section 3.

One has to focus the exposition upon the dynamical aspect of homeomorphical steering our lower level sets. So, we keep the description of the statical aspect (before and after perturbation) short. However, we explain several main features of the above steering with a certain representative example, and indicate a typical way of locally reducing higher dimensional situations to lower dimensional ones.

Case 1 (*local construction*). We assume: both Kuhn–Tucker points are lying in the interior; i.e., $\bar{x}^u \in M[G] \backslash \partial M[G]$ and $\bar{x}^d \in M[\tilde{G}] \backslash \partial M[\tilde{G}]$. Lemma 1.1 tells us that the two stationary points are nondegenerate.

We may assume $\bar{x}^d$ to be inside of a ball $B(\bar{x}^u, \delta)$ $(\delta > 0)$, and $B(\bar{x}^u, 3\delta)$ to be disjoint from both boundaries $\partial M[G]$, $\partial M[\tilde{G}]$, and from the 3δ-balls around the other undisturbed Kuhn–Tucker points. As our *first step* we perform a small C^∞-shift in $B(\bar{x}^u, \delta)$ taking $\bar{x}^u$ to $\bar{x}^d$. So, we may assume $\bar{x}^u = \bar{x}^d = 0$, and $\bar{t} = 0$. In the *second step* with the aid of a (locally) linear transformation we may identify $D^2f(0)$ with $A := D^2\tilde{f}(0)$. Here we remark that the latter transformation depends continuously on the data.

The *third step* is based on the following linear homotopies:

$$\left.\begin{aligned} F_q(x, v) &= \tfrac{1}{2} v x^T A x + (1 - v) f(x) \\ \tilde{F}_q(x, v) &:= \tfrac{1}{2} v x^T A x + (1 - v) \tilde{f}(x) \end{aligned}\right\} ((x, v) \in \mathbb{R}^n \times \mathbb{R}) .$$

By means of F_q and $\tilde{F}_q$ we define, with some $\tau > 0$ and $\delta_0 \in (0, \delta)$, two vectorfields on $B(0, 2\delta_0) \times (-\tau, 1 + \tau)$:

$$F_{F_q}(x, v) := \left\{ \begin{array}{ll} \left(-D_v F_q(x\, v) \dfrac{D_x^T F_q(x, v)}{\|D_x^T F_q(x, v)\|^2}, 1 \right), & \text{if } x \neq 0 \\ (0, 1), & \text{if } x = 0 \end{array} \right\}, \tag{4.1}$$

and $F_{\tilde{F}_q}(x, v)$ in the same way replacing F_q by $\tilde{F}_q$. These vectorfields are of class C^1 and we glue them in $B(0, 2\delta_0)\backslash\overline{B(0, \delta_0)}$ with the constant vectorfield $(0, 1) \in \mathbb{R}^n \times \mathbb{R}$. With an integration in time one and reducing to $\mathbb{R}^n$ again, we arrive at two C^1-diffeomorphisms, θ_{F_q} and $\theta_{\tilde{F}_q}$, taking for an appropriate $\delta_0' \in (0, \delta_0)$ the level sets of $F_q(\cdot, 0)$ $(=f)$ and $\tilde{F}_q(\cdot, 0)$ $(=\tilde{f})$ in $B(0, \delta_0')$ onto the corresponding level sets of $F_q(\cdot, 1)$: $x \mapsto \frac{1}{2}x^TAx$ in $\theta_{F_q}(B(0, \delta_0'))$ and $\theta_{\tilde{F}_q}(B(0, \delta_0'))$, resp. For more details see [7]. Now we conclude

$$(\theta_{\tilde{F}_q}^{-1} \circ \theta_{F_q})(x) \begin{cases} = x\,, & \text{for } x \in \mathbb{R}^n\backslash B(0, 2\delta_0) \\ \in M[\tilde{G}]\backslash\partial M[\tilde{G}]\,, & \text{otherwise} \end{cases}.$$

Let us set $\theta := \theta_{\tilde{F}_q}^{-1} \circ \theta_{F_q}$, $\mathcal{U} := \theta^{-1}(B(0, \delta_0')) \cap B(0, \delta_0')$, and $\mathcal{V} := \theta(\mathcal{U})$. Then the C^1-transformation θ maps the t-level of f in $\mathcal{U}$ onto the t-level of $\tilde{f}$ in $\mathcal{V}$, simultaneously with respect to the parameter t. Moreover, θ is the identity outside $B(0, 2\delta_0)$. The latter transformation again depends continuously on the data. Now we may assume that the functions f and $\tilde{f}$ coincide in $\mathcal{V} \subset B(0, \delta_0)$.

In the *fourth step*, given at the end of Section 4, the construction ouside $\mathcal{V}$ will be completed. There, the appearing functions are only assumed to be of class C^1.

Case 2 (*local construction*). Now we turn to the case that $\bar{x}^u$ lies on the boundary.

Case 2(*a*). We assume

$$\bar{x}^u \in \partial M[G]\,, \tag{4.2}$$

$$\text{s.index}(\bar{x}^u, (f, G)) \geqq 1\,. \tag{4.3}$$

We note that in this case our stationary points $\bar{x}^u$ and $\bar{x}^d$ must be saddle points and that LICQ holds. This Case 2(a) contains two subcases, defined by

Subcase 2(a)1: $\bar{x}^d \in M[\tilde{G}]\backslash\partial M[\tilde{G}]$,

Subcase 2(a)2: $\bar{x}^d \in \partial M[\tilde{G}]$.

As f and $\tilde{f}$ are C^1-near, in *Subcase* 2(*a*)1 we have $D^Tf(\bar{x}^u) = D^T\tilde{f}(\bar{x}^d) = 0$. Without restrictions we assume $\bar{x}^d$ to be in the interior of $M[G]$ (shifting otherwise). The main local *strategy* can be described in the following way. We transform our unperturbed situation into an interior situation; i.e., keeping the inequality constraints fixed we replace $\bar{x}^u$ by a (fictive) implanted inner saddle point $\hat{x}^u$ near $\hat{x}^d := \bar{x}^d$. But as the situations around $\hat{x}^u$ and $\hat{x}^d$ are similar now, we are back in Case 1. Note, that the result of such an *implantation* can be realized by means of a small C^∞-shift, corresponding with a small perturbation of f. Moreover, taking arguments as in Case 1 into account, we may assume that f and $\tilde{f}$ around $\hat{x}^u$ and $\hat{x}^d$, resp., are *quadratic forms*, and at last even sums of squares (Morse-type normal forms).

In *Subcase* 2(*a*)2, however, we perform *two implantations*, the first for the unperturbed problem: $\bar{x}^u \to \hat{x}^u$, and the second for the perturbed one: $\bar{x}^d \to \hat{x}^d$.

But now it might happen that one of the Lagrange multipliers does not vanish. So, a nesting of a nucleus (germ) of implantation structure might become more complicated than in Subcase 2(a)1. Nevertheless we essentially reduced our task to Case 1 again.

Now, let us be concerned with some operational *details*. Let $J_0(\bar{x}^u)$ be the set $\{1, \ldots, p\}$ $(p \leqq s)$. Choose vectors $\xi_{p+1}, \ldots, \xi_n \in \mathbb{R}^n$ such that $\{D^T g_j(\bar{x}^u)$ $(j \in J_0(\bar{x}^u))$, $\xi_{p+1}, \ldots, \xi_n\}$ is a basis for $\mathbb{R}^n$. By means of the C^2-transformation ζ_G, defined by

$$y_1 = g_1(x), \ldots, y_p = g_p(x), y_{p+1} = \xi_{p+1}^T(x - \bar{x}^u), \ldots, y_n = \xi_n^T(x - \bar{x}^u)$$

one locally linearizes $M[G]$ around $\bar{x}^u$, to $\mathbb{H}^p \times \mathbb{R}^{n-p}$ around 0. In Subcase 2(a)2 we work with a special local linearization $\zeta_{\tilde{G}}$ for the perturbed problem, namely with $J_0(\bar{x}^d) = \{1, \ldots, q\}$ $(q \leqq p)$ (as one many assume) near $\bar{x}^d$ (and $\bar{x}^u$) by

$$\left.\begin{aligned} y_1 = \tilde{g}_1(x), \ldots, y_q = \tilde{g}_q(x), y_{q+1} = \tilde{g}_{q+1}(x), \ldots, y_p = \tilde{g}_p(x)\,, \\ y_{p+1} = \xi_{p+1}^T(x - \bar{x}^d), \ldots, y_n = \xi_n^T(x - \bar{x}^d) \end{aligned}\right\}.$$

We note, that around $\bar{x}^d$ the transformed set $M[\tilde{G}]$ becomes a small relative neighbourhood in $\mathbb{H}^q \times \mathbb{R}^{n-q}$, around the point $\zeta_{\tilde{G}}(\bar{x}^d)$ (perhaps being 0, too) and that ζ_G and $\zeta_{\tilde{G}}$ are C^2-near.

See Figure 4.1 for the most relevant pictures in dimension three with vanishing gradient $D^T f(\bar{x}^u)$ $(\{f = t\} := \{x \in \mathbb{R}^n \mid f(x) = t\})$.

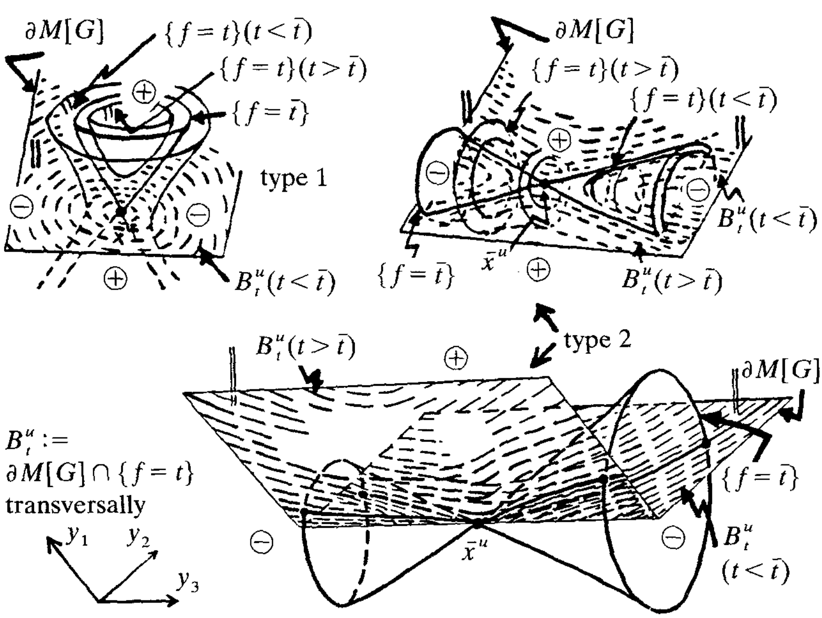

Fig. 4.1. Strongly stable saddle points (Case 2(a)) with $D^T f(\bar{x}^u) = 0$ $(n = 3)$: in a pointed neighbourhood of the stationary boundary point $\bar{x}^u$ the level sets of f meet $\partial M[G]$ (locally linearized) without any tangential effect (transversality). Different types naturally arise by particular growth behaviours.

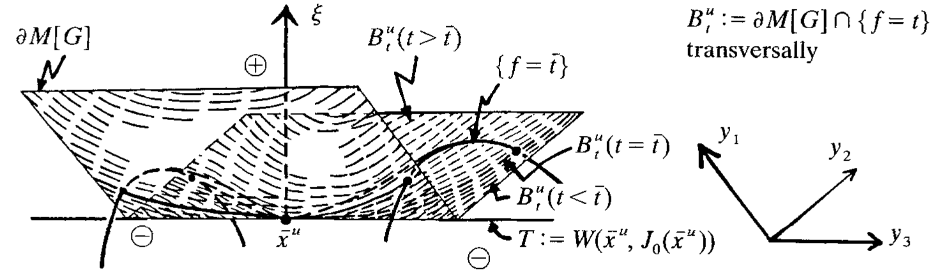

Fig. 4.2. A strongly stable saddle point (Subcase 2(a)2) with $D^Tf(\bar{x}^u) \neq 0$ $(n = 3)$: in a pointed neighbourhood of the stationary boundary point $\bar{x}^u$ the level sets of f meet $\partial M[G]$ (locally linearized) without any tangential effect (transversality). Example (cf. type 2 in Figure 4.1): $|J_0(\bar{x}^u)| = 2$, T denoting the tangent space of $\partial M[G]$ at $\bar{x}^u$.

As some significant realization of Subcase 2(a)2 we look at Figure 4.2.

Moreover, from the (feasible) level structure near a saddle point $\bar{x}$ with $D^Tf(\bar{x}) \neq 0$ we can always, in a *fictive* sense, extrapolate to a strongly stable saddle point $\hat{z}$ *beyond* $\bar{x}$ with vanishing gradient of a *quadratic* objective function at $\hat{z}$, such that $\bar{x}$ occurs as a saddle point induced by $\hat{z}$. We refer to Figure 4.2 merely to illustrate for the unperturbed problem *what* should be the *result* of implantation. One knows the boundary $\partial M[G]$ to be an $(n-1)$-dimensional Lipschitzian manifold (cf. [8]) with MF-vector ξ, say $(1, 1, 0)^T$, in Figure 4.2. So, we can always interpret Figure 4.2 in higher dimensions. We assume from now on that we are in the linearized situation. With $\delta > 0$ sufficiently small we choose

$$\hat{z} := -\delta(\underbrace{1, \ldots, 1}_{p}, \underbrace{0, \ldots, 0}_{n-p})^T .$$

Note that

$$z \mapsto (\underbrace{0, \ldots, 0}_{p}, \underbrace{z_{p+1}, \ldots, z_n}_{n-p})^T$$

is a projection to the tangent space T, and the quadratic function

$$z \mapsto \sum_{j=1}^{p} (z_j + \delta)^2 + \frac{1}{2}(0, \ldots, 0, z_{p+1}, \ldots, z_n)C(0, \ldots, 0, z_{p+1}, \ldots, z_n)^T \tag{4.4}$$

is a fictive second order "approximation" for f (more precisely for $f \circ \zeta_G^{-1}$) around $\hat{z}$; here $C = (D\zeta_G^{-1}(0))^T D^2 L_{[\mu]}(\bar{x}) D\zeta_G^{-1}(0)$. Now, taking account of geometry and a linear transformation (cf. [7]), there is no loss of second order information if we replace (4.4) by (4.5):

$$z \mapsto \sum_{j=1}^{p} (z_j + \delta)^2 + \sum_{k=p+1}^{n} (\pm z_k^2) . \tag{4.5}$$

Here the numbers of signs ⊖ is just s.index $(\bar{x}^u, (f, G))$, hence $\geqq 1$. Then we switch that arbitrarily small neighbourhood of $\hat{z}$ into the interior; i.e., by translation we arrive at

$$\hat{x}^u := \delta(\underbrace{1,\ldots,1}_{p},\underbrace{0,\ldots,0}_{n-p})^T$$

which is the critical point of the quadratic form

$$\hat{f}^u : x \mapsto \sum_{j=1}^{p} (x_j - \delta)^2 + \sum_{k=p+1}^{n} (\pm x_k^2)\,, \tag{4.6}$$

locally around $\bar{x}^u$. Outside of that neighbourhood we can *t-wise* "nest" $\hat{f}^u$ into the local level structure of f. In a *coupled* way (shift of $\hat{f}^u$) we also achieve a fictive saddle point $\hat{x}^d$ and a nested function $\hat{f}^d$ for our disturbed problem.

So, we know what we have to implant.

Our above illustration refers to a small neighbourhood of $\hat{x}^u$ and to an arbitrarily small neighbourhood of $\hat{x}^u$ (and $\hat{x}^d$) therein. The way of nesting of $\hat{f}^u$ and $\hat{f}^d$ around $\bar{x}^u$ and $\bar{x}^d$, resp., is nothing else than a (fictive) shift of a stationary point into the interior and it will be handled as such in the subsequent example. The MF-vector ξ gives a *vertical* flow structure for both the unperturbed and the perturbed problem defined by the *constant* vector field ξ.

Now, let us turn to the *technical aspect* of the two implantations; we may choose the unperturbed situation (f, G). For our *example* in dimension $n = 3$ we restrict on an implantation within the case $D^T f(\bar{x}^u) = 0$. Moreover, $\hat{x}^u$ may be interpreted as being shifted into the interior after some perturbation of f and it is placed on the trajectory of the constant vector field ξ which runs through $\bar{x}^u$. We assume as above that the new function $\hat{f}^u$ near $\hat{x}^u$ has the normal form $\Sigma_{j=1}^n \pm x_j^2$ (up to translation in x-space). As there is still a piecewise linearity of $\partial M[G]$ in Figure 4.1, *type* 2, and this creased Lipschitzian manifold can be made almost flat (by a linear transformation), we actually proceed for it with construction-ideas similar to the ones for *type* 1. So for our example we prefer $\bar{x}^u$ to be of the (with respect to ξ) *radial* type 1 instead of the more conical type 2. Then, however, $|J_0(\bar{x}^u)| = 1$ necessarily holds, and we can give an impression how our later dimensional reduction might also work. Indeed, a careful reflection justifies the following assumption.

ASSUMPTION. *There is a family of hypersurfaces, say even hyperplanes, parametrized by the rotation-angle $\alpha \in [0, \pi)$, which meet in the axis through the origin generated by ξ. Moreover they transversally intersect in some small feasible neighbourhood of $\bar{x}^u$ all the t-levels of f (for $t \neq \bar{t}$ near $\bar{t}$) and the critical $\bar{t}$-level: namely, in two C^2-components (one component) for $t < \bar{t}$ ($t > \bar{t}$, resp.) and, for $t = \bar{t}$, in two C^2-*manifolds *meeting in $\bar{x}^u$.* □

Thus, our radial surface structure is compatible with the vertical flow-box. Now, we may *locally reduce* our 3-dimensional situation to a family of 2-dimensional situations (cf. Figure 4.3).

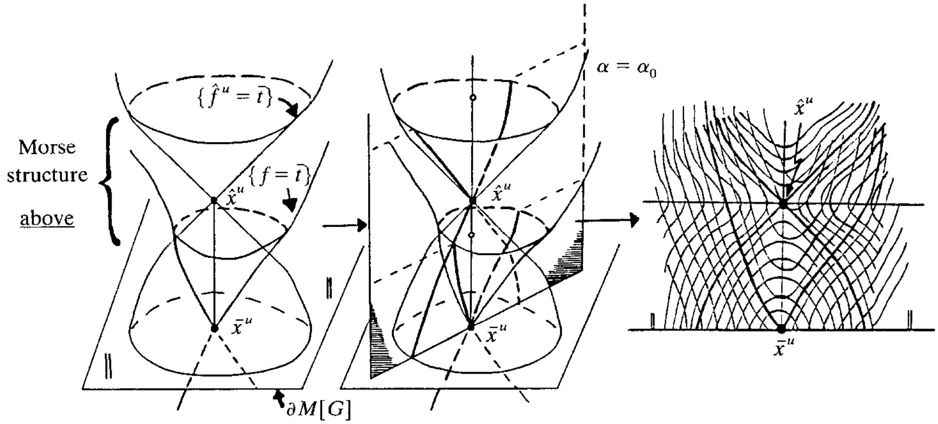

Fig. 4.3. Reduction before perturbation ($\alpha_0 \in [0, \pi)$): from the 3-dimensional situation of our example we turn to the codim.1 situation given by a specific vertical hyperplane, which conserves the sums of squares (Morse) structure around the fictive stationary point $\hat{x}^u$.

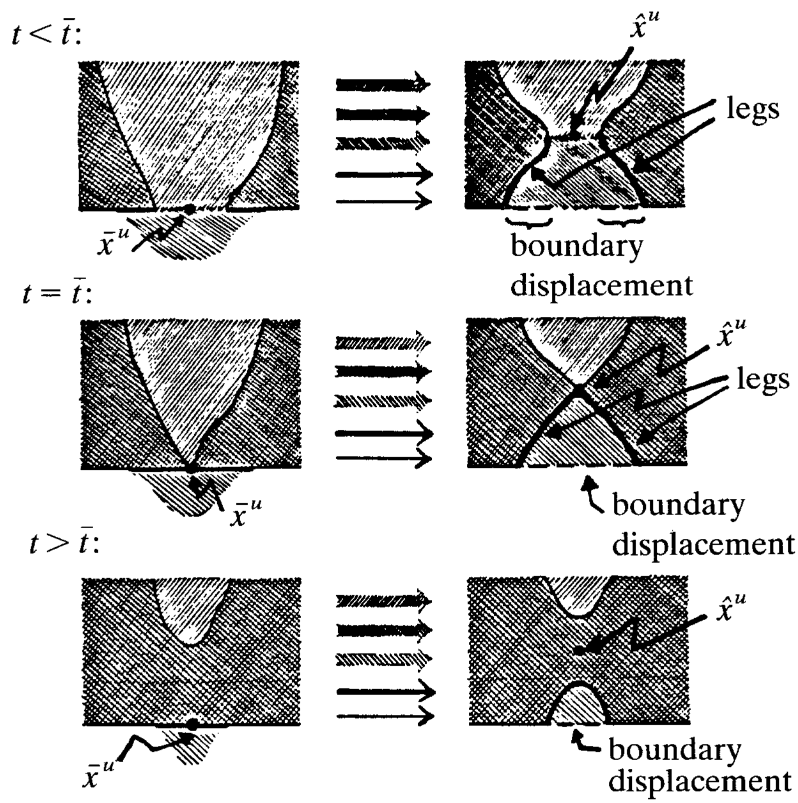

Fig. 4.4. Mapping of area and boundaries in a reduced situation, before perturbation ($\alpha_0 \in [0, \pi)$): we continue Figure 4.3 for our example. Boundary displacement: The plotted mappings imply that parts of $\partial M[G]$ wander inwards to pieces of the fictive level set. For $t \leq \bar{t}$ there pieces are composed by "legs".

So, our full-dimensional control will just integrate a family of well-coupled controls with dimension decreased by 1. Each of the latter steerings has to perform the mappings depicted in Figure 4.4. In this figure three levels are distinguished. In every case the hatched areas have to be mapped onto the corresponding hatched areas.

Although we focus on the hyperplane section corresponding to an arbitrary fixed $\alpha_0 \in [0, \pi)$, the whole local construction will be done *simultaneously* w.r.t. α.

In our example two difficulties arise. On the one hand it is possible that for some t near $\bar{t}$ the transversal intersection $\hat{B}_t^u$ between the t-level of $\hat{f}^u$ and the plane through $\hat{x}^u$, locally being parallel $\partial M[G]$, does *not* precisely lie *above* the corresponding transversal intersection B_t^u of the t-level of f and $\partial M[G]$ (geometrically *below*). On the other hand, at some $x \in B_t^u$ $(t < \bar{t})$ a trajectory of our flow-box might be tangent to the t-level of f (the analogous difficulty will arise *above*). However, for each $t < \bar{t}$ near $\bar{t}$ we dynamically perform a horizontal C^2-shift (*positioning*) from B_t^u to the projection of the set $\hat{B}_t^u$ (locally onto $\partial M[G]$) and, moreover, for each $t \leqq \bar{t}$ we dynamically realize a (continuous) sharpening of the t-level of f and its intersection with the boundary. So we received points of intersection, which describe kinks and full-dimensionally form our so-called *fundamental domain* (for t). We also need such fundamental domains *above*, for the Morse structure. As it is made clear in Figure 4.4 an *incomparability* between the level structure for our unperturbed and our fictive inner problem appears. Therefore, we make with respect to each $t \leqq \bar{t}$, a *raising* of $\partial M[G]$ in our flow-box. In this way we "sucked" area from outside of $M[G]$. For $t > \bar{t}$ the raising from $\bar{t}$ is continuously (in t) reduced to zero.

Now, having Figure 4.4 in mind we know how to proceed in the new situation from Figure 4.5. For each t near $\bar{t}$ this picture exhibits pairs of comparable

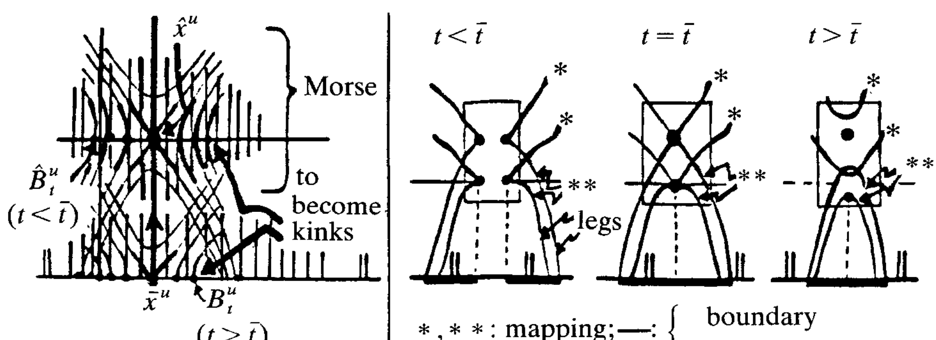

Fig. 4.5. A radial section ($\alpha_0 \in [0, \pi)$) and its significant profiles: for our example we transfer the mapping (of area and boundaries) from Figure 4.4 into the situations where some positioning and sharpening and (in the original unperturbed situation) a raising of $\partial M[G]$ are made. Now, especially: $* \longrightarrow *$, $** \longrightarrow **$.

elements, one due to the unperturbed problem and the other due to the fictive problem. Thus, we established situations being similar with the manifold's comparability which underlies the MF-technique from Section 3. Indeed, in a small neighbourhood of a pair of corresponding elements from the two fundamental domains (for $t = \bar{t}$, especially, of both saddle points) we have in hand transversality of our vertical flow with respect to both the "manifold of start" and the "manifold of termination", and finally arrive at our desired steering there, namely by a C^2-flow followed by a continuous (horizontal) tapered deviation-straightening.

Figure 4.6 gives an insight, how this quite local steering completes to the intended local transformation (implantation).

Let us leave our example with the remark that for examples of *type* 2 we would better do the positioning *above*, namely in certain projective sense from $\hat{B}_t^u$ to B_t^u. So, our fictive problems are quite appropriate (as the right links between the perturbed and the unperturbed problem), because for them we have "place to work". We emphasize this advantage especially for the case of a nonsmooth boundary $\partial M[G]$, where our local construction actually works, too. Moreover, every saddle point $\bar{x}^u$ with $D^T f(\bar{x}^u) \neq 0$ allows an almost similar exposition because of similar local geometry relative to type 1 or type 2.

Now, in order to indicate briefly in Case 2(a) how for an implantation with respect to any $n > 3$ the *local reduction* to a lower dimensional case happens, we work again with (f, G). We use an approach which systematically exploits radiality (and transversality). For simplicity we assume $|J_0(\bar{x}^u)| = 1$ and that our function f restricted to $\partial M[G]$ possesses a normal form around $\bar{x}^u$. In Subcase 2(a)2 this germ in $\mathbb{R}^{n-1}$ reveals the same stationary index as the extrapolated, then switched and implanted, saddle point $\bar{x}^u$ of $\hat{f}^u$ in $\mathbb{R}^n$. In the boundary $\partial M[G]$ we cross one ball of rotation (in the normal form) by means of an intersection with a transversal hyperplane (cf. Figure 4.7); i.e., we cross out one $\oplus$ or one $\ominus$ from

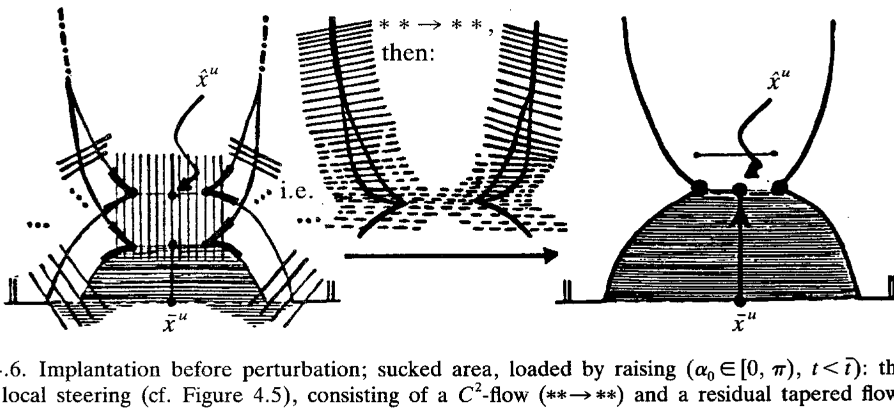

Fig. 4.6. Implantation before perturbation; sucked area, loaded by raising ($\alpha_0 \in [0, \pi)$, $t < \bar{t}$): the quite local steering (cf. Figure 4.5), consisting of a C^2-flow ($** \rightarrow **$) and a residual tapered flow, evolves into a local transformation. (Cf. Figure 4.4; we refer to the situations after some positioning, sharpening and a raising.)

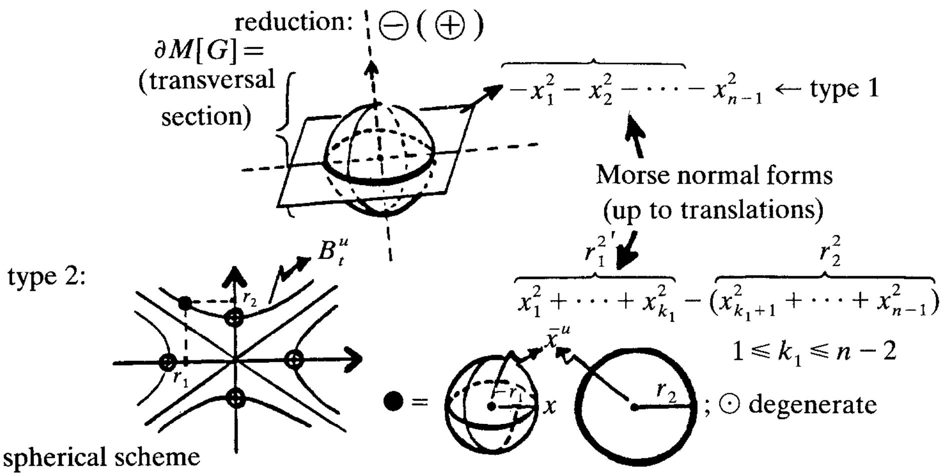

Fig. 4.7. Crossing (before lifting) in the boundary, before perturbation (Case 2(a)): Example: $|J_0(\bar{x}^u)| = 1$. We "cross" out one $\oplus$ or one $\ominus$ be means of a suitable relative hyperplane in $\partial M[G]$ (which then has to be lifted in vertical direction). Degeneracy: a sphere becomes a single point.

the normal form at each of its levels t near $\bar{t}$. This can suitably proceed, step by step, until we arrive at $n - 1 = 2$ and together with a further $\oplus$ at the characteristic $\{\oplus, \ominus\}$-distribution, at least of our fictive structure, reduced to dimension $n = 3$. In fact, this crossing has to be lifted from the boundary into the (full) space, let us say in direction ξ.

Hence, one is back in a well understood dimension where one now *simultaneously* works with respect to a family of unperturbed and fictive problems parametrized by $(\alpha_1, \ldots, \alpha_{n-3}) \in [0, \pi)^{n-3}$. So we finished our reduction in a t-independent way.

Now, we proceed with the third part of our proof. We are in *Case* $2(b)$ with the assumption:

$$\bar{x}^u \in \partial M[G], \tag{4.7}$$

$$\text{s.index}(\bar{x}^u, (f, G)) = 0. \tag{4.8}$$

In particular, $\bar{x}^u$ must be a local minimum. Whereas the main lines of construction presented for saddle points in Case 2(a) remain the same, in detail we often must argue in a more technical way because LICQ is not always guaranteed at $\bar{x}^u$ (perhaps loss of piecewise smoothness of $\partial M[G]$).

Again for each $n \in \mathbb{N}$, $n \geqq 2$, we would start with one or two implantations, one for the unperturbed and maybe another in the perturbed problem, reducing to $n = 2$. Then we essentially proceed as in Case 1. Now, *Subcase* $2(b)1$ and *Subcase* $2(b)2$ naturally arise as subcases as in Case 2(a). Indeed, for implantation we *always* nest ball-structures of (lower) levels. Hence, our elaborated steering is finally based on a *spherical argumentation*; for instance, the appropriate dimensional reduction sophistically happens sphere −, i.e. t-wise. A stationary point $\bar{x}$

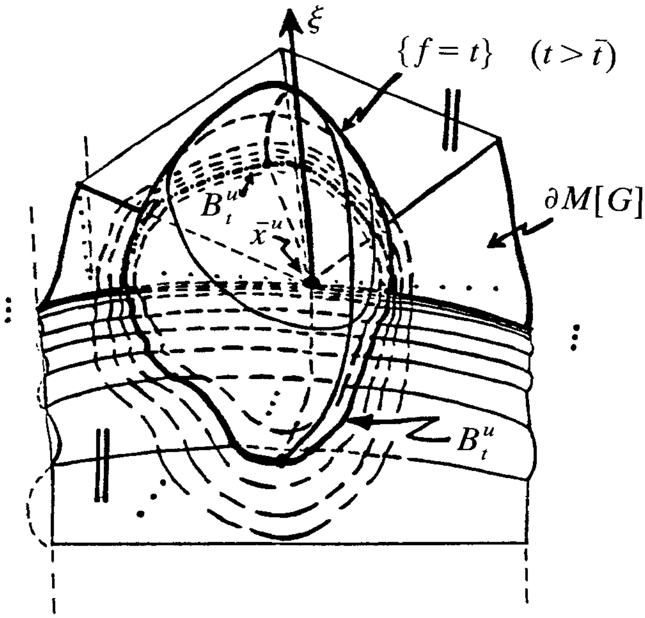

Fig. 4.8. A strongly stable point in Case 2(b) with $D^T f(\bar{x}^u) = 0$ ($n = 3$): in a pointed neighbourhood of the stationary boundary point $\bar{x}^u$ the level sets of f and $\partial M[G]$ meet without any tangential effect (transversality; $B_t^u = \{f = t\} \cap \partial M[G]$). Example (in bird's-eye view): variously structured and highly nonsmooth boundary.

with (4.7), (4.8), and $D^T f(\bar{x}) = 0$, is called of *type* 0. This type is totally *radial* (like type 1; cf. Figure 4.8). Let us again choose $\bar{x}^u$ instead of $\bar{x}^d$.

Now, we work out an adequate perturbation technique (geometrically *below*) for the MF-vector induced flow-box. In fact, we only need to sharpen *above* (i.e. for our fictive structure) and we can avoid the positioning of "kinks". These kinks at our Lipschitzian fundamental domains, however, geometrically point away from $\bar{x}^u$ and $\hat{x}^u$, resp. But, while for saddle points we mapped $\bar{x}^u \mapsto \hat{x}^u$ with respect to all $t \leqq \bar{t}$, now $\bar{x}^u \mapsto \hat{x}^u$ is assured only for $t = \bar{t}$.

Since from a radial point of view we have a dynamical smoothing technique in hand, we may state that Figure 4.9 essentially reflects our radial steering.

At this moment of entrance into the *last part* of the proof we can state as a résumé that we have finished our local constructions around all our Kuhn–Tucker points $\bar{x}_i$ $(i = 1, \ldots, l)$. Moreover in Case 2(a), (b) we could guarantee that outside of some very small common neighbourhood of $\bar{x}_i$ and $\bar{x}_i^d$, say $B(\bar{x}_i, \rho)$ $(\rho > 0)$, in our unperturbed and in our perturbed situation the fictive new level structure coincides with the one of f and $\tilde{f}$, resp. Namely, we worked in a more set theoretical than (due to (f, G)) functional way of transformations, similar to the MF-technique from Section 3 (details omitted). Having always turned to Case 1, we locally proceed with such a t-wise dynamical treatment of (lower) level sets outside of an arbitrary small (already treated) neighbourhood of $\hat{x}_i^d$. So, with a

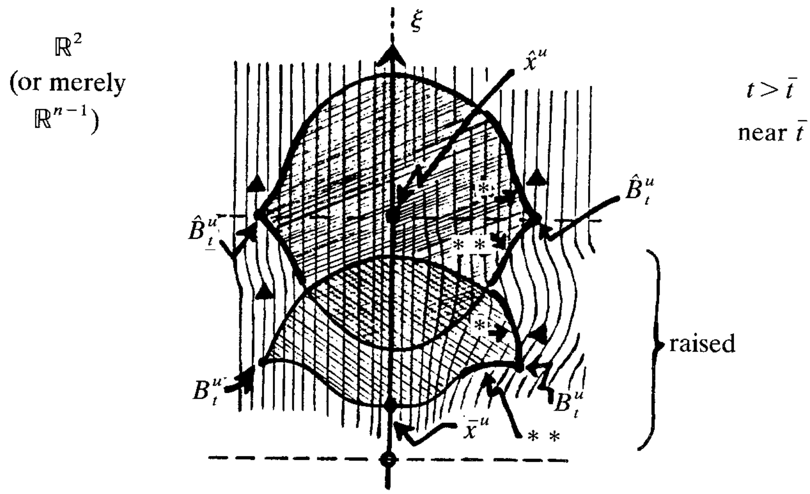

Fig. 4.9. A radial section and its significant profile (type 0): The lower hatched area has to be transformed onto the upper one. Moreover: $* \rightarrow *$, $** \rightarrow **$. (Cf. Figure 4.3–4.6).

shrinking of ρ, if necessary, in the outer subset $\overline{B(\bar{x}_i, \varepsilon)} \backslash B(\bar{x}_i, \frac{1}{2}\varepsilon)$ of a closed little ball around $\bar{x}_i$ ($\varepsilon > 0$) – *in particular* – we can make $M[G, F, t]$ and $M[\tilde{G}, \tilde{F}, t]$ for all t to coincide, actually by MF-technique being ready to evolve into the complement of $\overline{B(\bar{x}_i, \varepsilon)}$. But this just constitutes our *Assumption A* from Section 3.

Now, the local constructions are fitted in the global constructions, and we reached our parametrized pair of transformations (ϕ_t, ψ) due to a sufficiently slight perturbation of our optimization problem. □

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