Elasto-Plastic Deformations in Multibody Dynamics

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Abstract. The problem of formulating and numerically solving the equations of motion for a multibody system undergoing large motion and elasto-plastic deformations is considered here. Based on the principles of continuum mechanics and the finite element method, the equations of motion for a flexible body are derived. It is shown that the use of a lumped mass formulation and the description of the nodal accelerations relative to a nonmoving reference frame lead to a simple form of these equations. In order to reduce the number of coordinates that describe a deformable body, a Guyan condensation technique is used. The equations of motion of the complete multibody system are then formulated in terms of joint coordinates between the rigid bodies. The kinematic constraints that involve flexible bodies are introduced in the equations of motion through the use of Lagrange multipliers.

Key words: Multibody systems, finite elements, non-linear deformations, large rotations.

Notation

In this paper the following general rules will apply:
(a) Matrices and higher order tensors are in boldface upper-case characters.
(b) Column and algebraic vectors are in boldface lower-case characters.
(c) Scalars are in lightface characters.
(d) Summation convention is applied when tensors are written on component form.
(e) Left superscript denotes the configuration in which an event occurs.
(f) Left subscript denotes the configuration to which an event is refered to.

1. Introduction

Procedures based on multibody dynamics have shown to be powerful tools for the analysis of systems undergoing large displacements and rotations, and some level of structural deformations. Ride stability, maneuverability and crash simulation of vehicles are examples of such systems. Some of the objectives in the analysis of flexible multibody systems are evaluation of trajectories, impact forces, reaction forces, kinetic energy absorption, structural deformation, and injury and survivability indices for occupants. These objectives suggest that the methodology developed for the dynamic analysis of multibody systems must include some of the structural characteristics of the system components.

Hybrid models have been popular for representing the flexibility of multibody system components. This approach first introduced by Kamal [1, 2] and later explored by Ni and Song [3] considers a system as a collection of rigid masses and springs. The flexibility of the system is represented by nonlinear springs, which characteristics are obtained from experimental tests. Nikravesh and Chung [4, 5] proposed a 3-dimensional hybrid model where the relative motion between the rigid bodies is constrained by kinematic joints. This was the first model where a general multibody methodology and an elasto-plastic hinge model were used to represent the nonlinear structural characteristics of a system. As in the previous methodologies, this model can only be used for the simulation of cases where the plane or line of the deformation of a hinge coincides exactly with the plane (or line) of the deformation of the actual test specimen.

A more general formulation for the representation of the flexibility of the components of a multibody system is obtained through the use of the finite element method. In fact Thompson and Sung [6] stated that virtually all simulations of flexible multibody systems employ the finite element method. The problem of inertial coupling between the gross rigid body motion and the structural deformation of the system have been successfully addressed by Song and Haug [7], by Sunada and Dubowsky [8] and by Shabana [9]. However most of the developed methodologies can only deal with linear elastic behavior of the flexible components of a system.

With the use of finite element method, the number of degrees of freedom of the multibody system is greatly increased. Furthermore, the form of the equations of motion is such that none of the matrices involved in such equations is constant. These features add up to high computational costs. A popular approach for reducing the dimension and the size of the problem is the use of modal superposition technique [8–11]. However there are two important drawbacks in this technique. First, a high number of modes of vibration is required to describe the system if forces with high frequency components are applied to the flexible bodies, or if the spectrum of natural frequencies of the flexible bodies is compact. The second drawback is that the use of the modal superposition technique restricts the practical applications of such methodologies to cases where the components have an elastic behavior (generally linear).

Some attempts have been made to describe the nonlinear structural behavior in the flexible bodies of multibody systems. The problem of the coupling between the axial forces and transverse deformations has received considerable attention. Kane *et al.* [12] addressed this issue by developing a comprehensive model for a beam that accommodates for the effects of centrifugal stiffening and vibrations induced by gyroscopic forces. In a later paper, Ryan [13] compared this methodology with several methods using the modal superposition technique and concluded that due to the premature linearization of the equations of motion, the results provided by such methods may lead to very large errors.

All the methodologies that are referred to in this paper have two points in common. First they only deal with linear elastic material laws. Second they confine the analysis to problems of small strain. Several advances in the finite element method have provided tools for the analysis of nonlinear structural systems [14–17]. Bathe *et al.* [17–19] developed a general formulation for finite elements which accommodates for large displacements, large strains and material nonlinearities. Such a methodology contains most of the ingredients necessary for kinematic description of flexible bodies. They discussed the advantages and drawbacks of using a total Lagrangian formulation or an updated Lagrangian formulation for formulating this type of problem. They concluded that though both formulations theoretically lead to the same result, there are computational advantages to use the updated Lagrangian formulation when the material law is elasto-plastic.

This paper presents an updated Lagrangian formulation of the equations of motion for a flexible body undergoing large displacements, rotations and deformations. The deformation of the flexible body is always related to a corotated configuration, to which a body fixed referential is attached. Special attention is paid to the evaluation of elasto-plastic constitutive equations to describe the material behavior. In order to simplify the form of the equations of motion, absolute nodal accelerations are used instead of accelerations relative to the body fixed coordinate system. Furthermore, a lumped mass formulation is used to calculate the mass matrix of a flexible body.

The reduction of the number of degrees of freedom necessary to the description of the flexible body deformation is achieved through the use of a Guyan condensation technique. Finally the formulation of the equations of motion of the flexible multibody system in terms of a set of joint coordinates is discussed with special emphasis on the treatment of the nodal coordinates of the flexible bodies.

2. Kinematics of a Flexible Body

Let XYZ denote a fixed cartesian coordinate reference frame. In order to define the position and the orientation of a flexible body with respect to XYZ let a new coordinate system $\xi\eta\zeta$ be defined. This new coordinate system is rigidly attached to a point of the flexible body as depicted in Figure 1.

The motion of the flexible body involves a continuous change of its shape and generally large displacements and rotations. Such complex motion is described by Figure 2. The initial configuration of the flexible body is denoted as configuration *0*. The configuration $t + \Delta t$ denotes the current equilibrium configuration, which is generally unknown. The last known equilibrium configuration is here denoted as t and referred in the text as the updated configuration.

The principle of the virtual displacements can be used to express the equilibrium of the flexible body in the current configuration. Such principle is written as

$$\int_{{}^{t+\Delta t}V} (\delta\, {}_{t+\Delta t}\mathbf{e})^T\, {}^{t+\Delta t}\tau\, {}^{t+\Delta t}\mathrm{d}v = {}^{t+\Delta t}\mathbf{R}\,, \tag{1}$$

where the external virtual work ${}^{t+\Delta t}\mathbf{R}$ is given by:

$$\begin{aligned} {}^{t+\Delta t}\mathbf{R} = & \int_{{}^{t+\Delta t}V} {}^{t+\Delta t}\rho(\delta\mathbf{h})^T\, {}^{t+\Delta t}\ddot{\mathbf{h}}\, {}^{t+\Delta t}\mathrm{d}v + \int_{{}^{t+\Delta t}V} {}^{t+\Delta t}\rho(\delta\mathbf{h})^T\, {}^{t+\Delta t}_{t+\Delta t}\mathbf{f}_b\, {}^{t+\Delta t}\mathrm{d}v \\ & + \int_{{}^{t+\Delta t}A} (\delta\mathbf{h})^T\, {}^{t+\Delta t}_{t+\Delta t}\mathbf{f}_s\, {}^{t+\Delta t}\mathrm{d}a\,. \end{aligned} \tag{2}$$

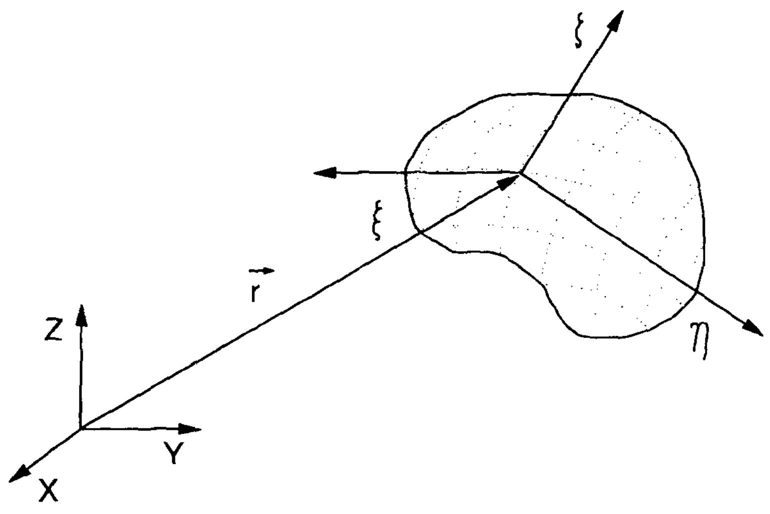

Fig. 1. Inertial reference frame XYZ and body fixed coordinate system $\xi\eta\zeta$.

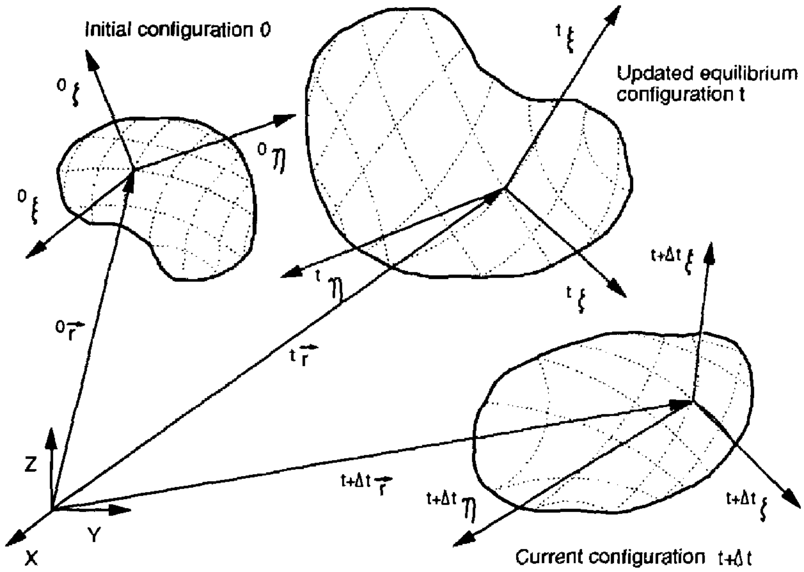

Fig. 2. Gross motion of a flexible body.

In equations (1) and (2) $\delta\mathbf{h}$ is a virtual variation in the current displacements, i.e., $\delta\mathbf{h} = \delta^{t+\Delta t}\mathbf{h}$, while $\delta_{t+\Delta t}\mathbf{e}$ is the correspondent variation in the vector of infinitesimal strains. The vector ${}^{t+\Delta t}\tau$ denotes the Cauchy stresses which are measured in the current configuration. ${}^{t+\Delta t}V$, ${}^{t+\Delta t}A$ and ${}^{t+\Delta t}\rho$ are respectively the volume, area and mass density of the flexible body in the equilibrium configuration $t + \Delta t$. The vectors of body forces and surface forces are respectively denoted by ${}^{t+\Delta t}_{t+\Delta t}\mathbf{f}_b$ and ${}^{t+\Delta t}_{t+\Delta t}\mathbf{f}_s$.

Equation (1) cannot be solved in the form presented if it is referred to the current configuration which is unknown. However, if the quantities involved in the virtual displacement equation are referred to a previously known configuration, then a solution for such equation may be obtained. In practice the choice for such reference configuration is taken between initial configuration and the previously known equilibrium configuration. The option for the former leads to a total Lagrangian formulation while the latter leads to an updated Lagrangian formulation. In this paper an updated Lagrangian formulation is adopted.

The Cauchy stresses, that appear in equation (1), are always referred to the configuration in which they occur and therefore they cannot be referred to any previous equilibrium configurations. This implies that another stress measure must be used if equation (1) is to be referred to the updated equilibrium configuration. The 2nd Piola–Kirchhoff stress is the appropriate stress measure that must be adopted for this formulation. The strain measure must be energy conjugate to the stress measure. In this sense the Green–Lagrange strain must be used instead of the infinitesimal strain. Equation (1) becomes

$$\int_{{}^{t+\Delta t}V} (\delta\ _{t+\Delta t}\mathbf{e})^T\ {}^{t+\Delta t}\tau\ {}^{t+\Delta t}\mathrm{d}v = \int_{{}^{t}V} (\delta\ {}^{t+\Delta t}_{t}\varepsilon)^T\ {}^{t+\Delta t}_{t+\Delta t}\mathbf{S}\ {}^{t}\mathrm{d}v\ , \tag{3}$$

where ${}^{t+\Delta t}_{t}\mathbf{S}$ is the 2nd Piola–Kirchhoff stress tensor corresponding to the current configuration but measured in the updated configuration. The components of such stress tensor are

$$ {}^{t+\Delta t}_{\;\;t}S_{ij} = \frac{{}^{t}\rho}{{}^{t+\Delta t}\rho} \frac{\partial\, {}^{t}h_i}{\partial\, {}^{t+\Delta t}h_s} \, {}^{t+\Delta t}\tau_{sr} \, \frac{\partial\, {}^{t}h_j}{\partial\, {}^{t+\Delta t}h_r} \, . \tag{4} $$

The variation of the Green–Lagrange strain tensor is denoted by $\delta\, {}^{t+\Delta t}_{\;\;t}\varepsilon$. The components of this tensor are written as:

$$ {}^{t+\Delta t}_{\;\;t}\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial\, {}^{t+\Delta t}u_i}{\partial\, {}^{t}h_j} + \frac{\partial\, {}^{t+\Delta t}u_j}{\partial\, {}^{t}h_i} + \frac{\partial\, {}^{t+\Delta t}u_k}{\partial\, {}^{t}h_j} \frac{\partial\, {}^{t+\Delta t}u_k}{\partial\, {}^{t}h_i} \right), \tag{5} $$

where the vector of displacements ${}^{t+\Delta t}\mathbf{u}$ is calculated as:

$$ {}^{t+\Delta t}\mathbf{u} = {}^{t+\Delta t}\mathbf{h} - {}^{t}\mathbf{h} \, . \tag{6} $$

It has been shown by several researchers [20–22] that referring equation (3) to a corotational coordinate system presents great advantages over the traditional updated Lagrangian formulation. If finite elements with rotational degrees of freedom are used to discretize the body, the use of such system is particularly important. A natural corotational frame is the $\xi\eta\zeta$ body fixed coordinate system. Furthermore, let a ghost configuration of the flexible body be associated with such coordinate system. For the purpose of this alternative updated Lagrangian formulation, let such ghost configuration be equal to the updated configuration, but translated and rotated like a rigid body as depicted in Figure 3. The updated ghost configuration is referred to as t'.

Nygard and Bergan [22] show that the Green–Lagrange strain tensor components ${}^{t+\Delta t}_{\;\;t}\varepsilon_{ij}$ measured in the current configuration and referred to the updated configuration are equal to the components of the same strain tensor referred to the corotated ghost configuration and expressed in the body fixed coordinate system, i.e.,

$$ {}^{t+\Delta t}_{\;\;t}\varepsilon_{ij} = {}^{t+\Delta t}_{\;\;t'}\varepsilon'_{ij} \, . $$

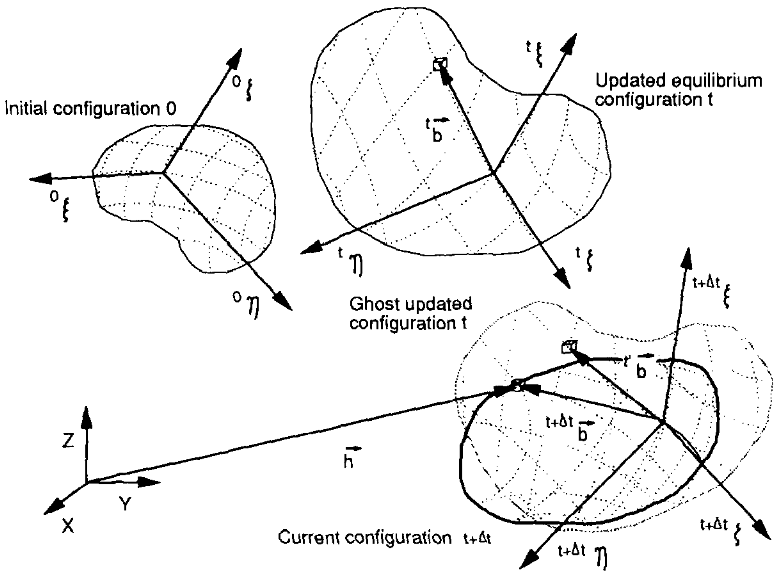

Fig. 3. Representation of the ghost configuration associated with the body fixed coordinate system.

In such case the strain tensor components ${}^{t+\Delta t}_{t'}\varepsilon'_{ij}$ are written as

$$ {}^{t+\Delta t}_{t'}\varepsilon'_{ij} = \frac{1}{2}\left(\frac{\partial^{t+\Delta t}u'_i}{\partial^{t'}b'_j} + \frac{\partial^{t+\Delta t}u'_j}{\partial^{t'}b'_i} + \frac{\partial^{t+\Delta t}u'_k}{\partial^{t'}b'_j}\,\frac{\partial^{t+\Delta t}u'_k}{\partial^{t'}b'_i}\right), \tag{7} $$

where the displacement vector ${}^{t+\Delta t}\mathbf{u}'$ is now defined as

$$ {}^{t+\Delta t}\mathbf{u}' = {}^{t+\Delta t}\mathbf{b}' - {}^{t'}\mathbf{b}' . \tag{8} $$

A similar relation holds for the components of the 2nd Piola–Kirchhoff stress tensor. The components of such tensor referred to the corotated ghost configuration and expressed in the body fixed coordinate frame are written as

$$ {}^{t+\Delta t}_{t'}S'_{ij} = \frac{{}^{t'}\rho}{{}^{t+\Delta t}\rho}\,\frac{\partial^{t'}b'_i}{\partial^{t+\Delta t}b'_s}\,{}^{t+\Delta t}\tau'_{sr}\,\frac{\partial^{t'}b'_j}{\partial^{t+\Delta t}b'_r} . \tag{9} $$

Equation (1) is finally written as:

$$ \int_{{}^{t+\Delta t}V} (\delta\,{}_{t+\Delta t}\mathbf{e})^T\,{}^{t+\Delta t}\tau\,{}^{t+\Delta t}dv = \int_{{}^{t'}V} (\delta\,{}^{t+\Delta t}_{t'}\varepsilon')^T\,{}^{t+\Delta t}_{t'}\mathbf{S}'\,{}^{t'}dv . \tag{10} $$

At this point let the stresses ${}^{t+\Delta t}_{t'}\mathbf{S}'$ be decomposed into

$$ {}^{t+\Delta t}_{t'}\mathbf{S}' = {}^{t'}\tau' + {}_{t'}\mathbf{S}' , \tag{11} $$

where ${}^{t'}\tau'$ is the Cauchy stress tensor of configuration t' in body fixed coordinates, and ${}_{t'}\mathbf{S}'$ denotes the increment of the 2nd Piola–Kirchhoff stress tensor referred to the updated ghost configuration. Similarly the Green–Lagrange strain can be decomposed into the sum of the strain in configuration t' and an increment. Using equation (7) it is observed that:

$$ {}^{t+\Delta t}_{t'}\varepsilon' = {}_{t'}\varepsilon' . \tag{12} $$

The Green–Lagrange strain increment ${}_{t'}\varepsilon'$ can be decomposed into the sum of a term linearly dependent in the displacement increments and in a nonlinear term, i.e.,

$$ {}_{t'}\varepsilon' = {}_{t'}\mathbf{e}' + {}_{t'}\eta' . \tag{13} $$

The linear and nonlinear terms are written respectively as:

$$ {}_{t'}\mathbf{e}'_{ij} = \frac{1}{2}\left(\frac{\partial u'_i}{\partial^{t'}b'_j} + \frac{\partial u'_j}{\partial^{t'}b'_i}\right) \tag{14} $$

$$ {}_{t'}\eta'_{ij} = \frac{1}{2}\,\frac{\partial u'_k}{\partial^{t'}b'_j}\,\frac{\partial u'_k}{\partial^{t'}b'_i} , \tag{15} $$

where the increment of displacement $\mathbf{u}'$, expressed in the body fixed coordinate system, is given by

$$ \mathbf{u}' = {}^{t+\Delta t}\mathbf{u}' - {}^{t'}\mathbf{u}' . \tag{16} $$

Now the strain increments are related with the stress increments by the constitutive equation:

$$ {}_{t'}\mathbf{S}' = {}_{t'}\mathbf{C}\; {}_{t'}\varepsilon' \,. \qquad (17) $$

Finally equations (10) through (17) can be substituted in equation (1) leading to

$$ \int_{{}^{t'}V} (\delta\, {}_{t'}\varepsilon')^T\, {}_{t'}\mathbf{C}\, {}_{t'}\varepsilon'\, {}^{t'}\mathrm{d}v + \int_{{}^{t'}V} (\delta\, {}_{t'}\eta')^{T}\, {}^{t'}\tau'\, {}^{t'}\mathrm{d}v = {}^{t+\Delta t}\mathbf{R} - \int_{{}^{t'}V} (\delta\, {}_{t'}\mathbf{e}')^{T}\, {}^{t'}\tau\, {}^{t'}\mathrm{d}v \,. \qquad (18) $$

In equation (18) the virtual work ${}^{t+\Delta t}\mathbf{R}$ must also be expressed in terms of the updated ghost configuration. For this purpose the integrands of the integrals on the right hand side of equation (2) are written as

$$ \begin{aligned} {}^{t+\Delta t}\rho\; {}^{t+\Delta t}_{t+\Delta t}\mathbf{f}_b\; {}^{t+\Delta t}\mathrm{d}v &= {}^{t'}\rho\; {}^{t+\Delta t}_{t'}\mathbf{f}_b\; {}^{t'}\mathrm{d}v \\ {}^{t+\Delta t}_{t+\Delta t}\mathbf{f}_s\; {}^{t+\Delta t}\mathrm{d}a &= {}^{t+\Delta t}_{t'}\mathbf{f}_s\; {}^{t'}\mathrm{d}a \\ {}^{t+\Delta t}\rho\; {}^{t+\Delta t}\ddot{\mathbf{h}}\; {}^{t+\Delta t}\mathrm{d}v &= {}^{t'}\rho\; {}^{t+\Delta t}\ddot{\mathbf{h}}\; {}^{t'}\mathrm{d}v \,, \end{aligned} \qquad (19) $$

where it is assumed that the surface and body forces are independent of the configuration in which they are represented, i.e., the loading conditions are independent of the deformation. Now the virtual work is written as

$$ {}^{t+\Delta t}\mathbf{R} = -\int_{{}^{t'}V} {}^{t'}\rho\, (\delta\mathbf{h})^T\, {}^{t+\Delta t}\ddot{\mathbf{h}}\; {}^{t'}\mathrm{d}v + \int_{{}^{t'}V} {}^{t'}\rho\, (\delta\mathbf{h})^T\, {}^{t+\Delta t}_{t'}\mathbf{f}_b\; {}^{t'}\mathrm{d}v + \int_{{}^{t'}A} (\delta\mathbf{h})^T\, {}^{t+\Delta t}_{t'}\mathbf{f}_s\; {}^{t'}\mathrm{d}a \,, \qquad (20) $$

where ${}^{t'}V$, ${}^{t'}A$ and ${}^{t'}\rho$ are respectively the volume, area and specific mass of the flexible body in the updated configuration.

The nonlinear equilibrium equation (18), resulting from the updated Lagrangian formulation, it is not linear in the increment of displacements. This implies that a direct solution of such equation is not possible. An approximate solution can be obtained by assuming $\delta_{t'}\varepsilon' = \delta_{t'}\mathbf{e}'$. In addition an approximate incremental constitutive equation must be used instead of equation (17), i.e.,

$$ {}_{t'}\mathbf{S}' = {}_{t'}\mathbf{C}\; {}_{t'}\mathbf{e}' \,. $$

Substituting these approximate relations into equation (18) results in:

$$ \int_{{}^{t'}V} (\delta\, {}_{t'}\mathbf{e}')^T\, {}_{t'}\mathbf{C}\, {}_{t'}\mathbf{e}'\, {}^{t'}\mathrm{d}v + \int_{{}^{t'}V} (\delta\, {}_{t'}\eta')^{T}\, {}^{t'}\tau'\, {}^{t'}\mathrm{d}v = {}^{t+\Delta t}\mathbf{R} - \int_{{}^{t'}V} (\delta\, {}_{t'}\mathbf{e}')^{T}\, {}^{t'}\tau\, {}^{t'}\mathrm{d}v \,, \qquad (21) $$

which is the linearized equation of motion of the flexible body for the updated Lagrangian formulation.

3. Finite Element Equations of Motion

In the formulation that follows, it is assumed that isoparametric finite elements are used. Furthermore, the matrices describing the flexible body are obtained from the assemblage of the

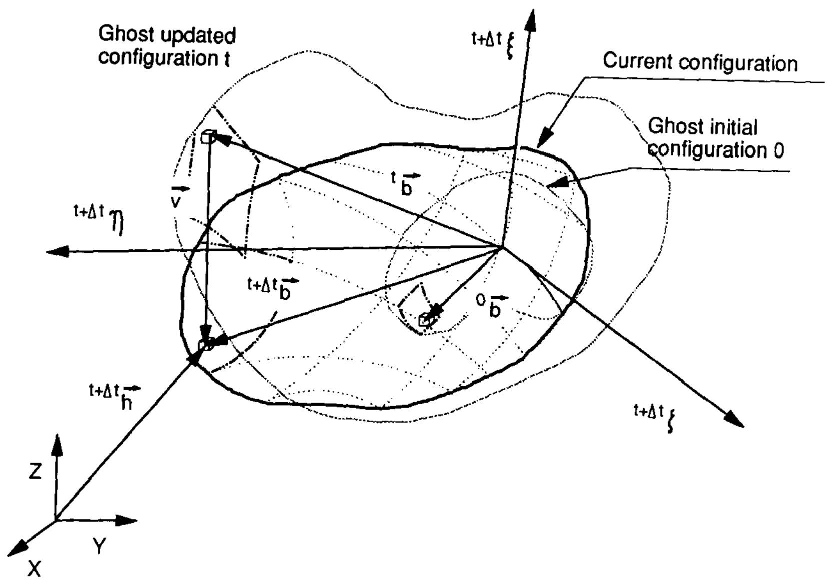

Fig. 4. Position of a point in a flexible body.

matrices of each finite element in the standard way. In this sense what follows is referred to each finite element.

Referring to Figure 4, the position of a point included in the domain of a finite element is written as:

$$ {}^{t+\Delta t}\mathbf{h} = \mathbf{r} + \mathbf{A}\,{}^{t+\Delta t}\mathbf{b}' , \tag{22}$$

where $\mathbf{r}$ is the position of the body fixed coordinate system ${}^{t+\Delta t}(\xi\eta\zeta)$ associated with the current configuration. In equation (22) $\mathbf{A}$ denotes the transformation matrix from the body fixed coordinate system to the inertial frame XYZ. Since $\mathbf{r}$ and $\mathbf{A}$ are always referred to the body fixed coordinate system associated with the current configuration, the left superscript $t + \Delta t$ is ignored for simplicity. Now the position of the point relative to the body fixed reference frame (denoted as ${}^{t+\Delta t}\mathbf{b}'$) can be decomposed in

$$ {}^{t+\Delta t}\mathbf{b}' = {}^{t}\mathbf{b}' + \mathbf{v}' , \tag{23}$$

where $\mathbf{v}'$ denotes the increment of the displacement of the point from the previously known ghost configuration to the current configuration.

In isoparametric finite elements, the coordinates and displacement fields are interpolated using the same shape functions. In this sense the position of each point in terms of nodal coordinates is written as:

$$ \begin{aligned} {}^{t}b_i' &= \sum_{k=1}^{M} N_k\,{}^{t}x_i^k \\ {}^{t+\Delta t}b_i' &= \sum_{k=1}^{M} N_k\,{}^{t+\Delta t}x_i^k , \end{aligned} \tag{24}$$

where ${}^{t'}x_i^k$ is the ith coordinate of the kth node for the updated ghost configuration t'. M is the number of nodes of the finite element, and the shape function for node k is denoted by N_k. The displacement field and the increment of displacements are given by

$$
\begin{aligned}
v_i' &= \sum_{k=1}^{M} N_k u_i^{k\prime} \\
{}^{t'}v_i' &= \sum_{k=1}^{M} N_k\, {}^{t'}u_i^{k\prime} \\
{}^{t+\Delta t}v_i' &= \sum_{k=1}^{M} N_k\, {}^{t+\Delta t}u_i^{k\prime} ,
\end{aligned}
\tag{25}
$$

where ${}^{t+\Delta t}\mathbf{u}'$ and ${}^{t'}\mathbf{u}'$ denote the vectors of nodal displacements in the current configuration and in the updated ghost configuration respectively. $\mathbf{u}'$ is the vector of nodal displacement increments calculated as:

$$
\mathbf{u}' = {}^{t+\Delta t}\mathbf{u}' - {}^{t'}\mathbf{u}' .
\tag{26}
$$

Now equations (23) through (26) can be substituted in equation (22) yielding:

$$
{}^{t+\Delta t}\mathbf{h} = \mathbf{r} + \mathbf{AN}({}^{t'}\mathbf{x} + \mathbf{u}') .
\tag{27}
$$

The virtual displacement of the point in terms of the nodal displacements is written as:

$$
\delta\, {}^{t+\Delta t}\mathbf{h} = [\mathbf{I} \quad -\mathbf{A}\tilde{\mathbf{b}}' \quad \mathbf{AN}] \begin{bmatrix} \delta\mathbf{R} \\ \delta\pi' \\ \delta\mathbf{u}' \end{bmatrix} ,
\tag{28}
$$

where π' is a small rotation of the body fixed coordinate system, and $\mathbf{b}'$ is evaluated for the current configuration. The left superscript $t + \Delta t$ is omitted for simplicity, and the overscore $\tilde{}$ represents a skewsymmetric 3×3 matrix made of the components of a 3-vector. The acceleration of the point is obtained by differentiating equation (27) twice with respect to time, i.e.,

$$
{}^{t+\Delta t}\ddot{\mathbf{h}} = [\mathbf{I} \quad -\mathbf{A}\tilde{\mathbf{b}}' \quad \mathbf{AN}] \begin{bmatrix} \ddot{\mathbf{r}} \\ \dot{\omega}' \\ \ddot{\mathbf{u}}' \end{bmatrix} + \mathbf{A}\tilde{\omega}'\tilde{\omega}'\mathbf{b}' + 2\mathbf{A}\tilde{\omega}'\mathbf{N}\dot{\mathbf{u}}'
\tag{29}
$$

where ω' and $\dot{\omega}'$ are the angular velocity and the angular acceleration of the body fixed coordinate system.

The virtual work due to the inertial forces can now be evaluated. Assume that the mass of the flexible body is conserved throughout the motion. In such a case ${}^0\rho = {}^t\rho$ and ${}^0V = {}^tV$, which implies that

$$
\int_{{}^0V} {}^0\rho(\delta\mathbf{h})^T\, {}^{t+\Delta t}\ddot{\mathbf{h}}\, {}^0\mathrm{d}v = \int_{{}^{t'}V} {}^{t'}\rho(\delta\mathbf{h})^T\, {}^{t+\Delta t}\ddot{\mathbf{h}}\, {}^{t'}\mathrm{d}v .
\tag{30}
$$

Consequently, the term of equation (20) equated with the inertial forces is written here as:

$$\int_{^0V} {}^0\rho(\delta\mathbf{h})^T\, {}^{t+\Delta t}\ddot{\mathbf{h}}\, {}^0\mathrm{d}v = \int_{^0V} {}^0\rho \begin{bmatrix} \delta\mathbf{r} \\ \delta\pi' \\ \delta\mathbf{u}' \end{bmatrix}^T \begin{bmatrix} \mathbf{I} \\ \tilde{\mathbf{b}}'\mathbf{A}^T \\ \mathbf{N}^T\mathbf{A}^T \end{bmatrix} [\mathbf{I} \quad -\mathbf{A}\tilde{\mathbf{b}}' \quad \mathbf{A}\mathbf{N}] \begin{bmatrix} \ddot{\mathbf{r}} \\ \dot{\omega}' \\ \ddot{\mathbf{u}}' \end{bmatrix} {}^0\mathrm{d}v$$
$$+ \int_{^0V} {}^0\rho \begin{bmatrix} \delta\mathbf{r} \\ \delta\pi' \\ \delta\mathbf{u}' \end{bmatrix}^T \begin{bmatrix} \mathbf{I} \\ \tilde{\mathbf{b}}'\mathbf{A}^T \\ \mathbf{N}^T\mathbf{A}^T \end{bmatrix} (\mathbf{A}\tilde{\omega}'\tilde{\omega}'\mathbf{b}' + 2\mathbf{A}\tilde{\omega}'\mathbf{N}\dot{\mathbf{u}}')\, {}^0\mathrm{d}v\,. \tag{31}$$

Now equation (31) can be written in a more compact form as:

$$\int_{^0V} {}^0\rho(\delta\mathbf{h})^T\, {}^{t+\Delta t}\ddot{\mathbf{h}}\, {}^0\mathrm{d}v = \begin{bmatrix} \delta\mathbf{r} \\ \delta\pi' \\ \delta\mathbf{u}' \end{bmatrix}^T \left(\mathbf{M} \begin{bmatrix} \ddot{\mathbf{r}} \\ \dot{\omega}' \\ \ddot{\mathbf{u}}' \end{bmatrix} + \mathbf{s} \right), \tag{32}$$

where the mass matrix $\mathbf{M}$ is symbolically written as:

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{rr} & \mathbf{M}_{r\phi} & \mathbf{m}_{rf} \\ \mathbf{M}_{\phi r} & \mathbf{M}_{\phi\phi} & \mathbf{M}_{\phi f} \\ \mathbf{M}_{fr} & \mathbf{m}_{f\phi} & \mathbf{M}_{ff} \end{bmatrix}. \tag{33}$$

The mass matrix is a symmetric matrix with its coefficients:

$$\mathbf{M}_{rr} = \mathbf{I} \int_{^0V} {}^0\rho\, {}^0\mathrm{d}v$$

$$\mathbf{M}_{r\phi} = -\mathbf{A} \int_{^0V} {}^0\rho\tilde{\mathbf{b}}'\, {}^0\mathrm{d}v$$

$$\mathbf{M}_{rf} = \mathbf{A} \int_{^0V} {}^0\rho\mathbf{N}\, {}^0\mathrm{d}v$$

$$\mathbf{M}_{\phi\phi} = -\int_{^0V} {}^0\rho\tilde{\mathbf{b}}'\tilde{\mathbf{b}}'\, {}^0\mathrm{d}v$$

$$\mathbf{M}_{\phi f} = \int_{^0V} {}^0\rho\tilde{\mathbf{b}}'\mathbf{N}\, {}^0\mathrm{d}v$$

$$\mathbf{M}_{ff} = \int_{^0V} {}^0\rho\mathbf{N}^T\mathbf{N}\, {}^0\mathrm{d}v\,.$$

The vector s in equation (32) denotes the terms that depend on the square of velocities (gyroscopic forces). In full, such vector is written as:

$$\mathbf{s} = \begin{bmatrix} \mathbf{A}\tilde{\omega}'\tilde{\omega}' \int_{^0V} {}^0\rho\mathbf{b}'\, {}^0\mathrm{d}v \\ \int_{^0V} {}^0\rho\tilde{\mathbf{b}}'\tilde{\omega}'\tilde{\omega}'\mathbf{b}'\, {}^0\mathrm{d}v \\ \int_{^0V} {}^0\rho\mathbf{N}^T\tilde{\omega}'\tilde{\omega}'\mathbf{b}'\, {}^0\mathrm{d}v \end{bmatrix} + 2 \begin{bmatrix} \mathbf{A}\tilde{\omega}' \int_{^0V} {}^0\rho\mathbf{N}\, {}^0\mathrm{d}v \\ \int_{^0V} {}^0\rho\tilde{\mathbf{b}}'\tilde{\omega}'\mathbf{N}\, {}^0\mathrm{d}v \\ \int_{^0V} {}^0\rho\mathbf{N}^T\tilde{\omega}'\mathbf{N}\, {}^0\mathrm{d}v \end{bmatrix} \dot{\mathbf{u}}'\,. \tag{34}$$

The rest of the terms of the virtual work are now evaluated for the updated Lagrangian formulation by substituting equation (28) into equation (20). The vector of the external applied forces is evaluated over the updated configuration as

$$
{}_{t'}\mathbf{g} = \begin{bmatrix} \int_{{}^{t'}A} {}^{t+\Delta t}_{t'}\mathbf{f}_s \, {}^{t'}da + \int_{{}^{t'}V} {}^{t'}\rho \, {}^{t+\Delta t}_{t'}\mathbf{f}_b \, {}^{t'}dv \\ \int_{{}^{t'}A} \tilde{\mathbf{b}}'\mathbf{A}^T \, {}^{t+\Delta t}_{t'}\mathbf{f}_s \, {}^{t'}da + \int_{{}^{t'}V} {}^{t'}\rho \tilde{\mathbf{b}}'\mathbf{A}^T \, {}^{t+\Delta t}_{t'}\mathbf{f}_b \, {}^{t'}dv \\ \int_{{}^{t'}A} \mathbf{N}^T\mathbf{A}^T \, {}^{t+\Delta t}_{t'}\mathbf{f}_s \, {}^{t'}da + \int_{{}^{t'}V} {}^{t'}\rho \mathbf{N}^T\mathbf{A}^T \, {}^{t+\Delta t}_{t'}\mathbf{f}_b \, {}^{t'}dv \end{bmatrix} . \tag{35}
$$

In order to form the flexible body equations of motion, the variation of the internal energy associated with the virtual displacement must be evaluated. For the updated Lagrangian formulation such terms are associated with the integrals of equation (21). For a finite element such integrals are written as:

$$
\int_{{}^{t'}V} (\delta_{t'}\mathbf{e}')^T \, {}_{t'}\mathbf{C} \, {}_{t'}\mathbf{e}' \, {}^{t'}dv = \begin{bmatrix} \delta\mathbf{r} \\ \delta\pi' \\ \delta\mathbf{u}' \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & {}^{t}_{t'}\mathbf{K}_L \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{u}' \end{bmatrix} \tag{36}
$$

$$
\int_{{}^{t'}V} ({}^{t}_{t'}\tau')^T \delta_{t'}\eta' \, {}^{t'}dv = \begin{bmatrix} \delta\mathbf{r} \\ \delta\pi' \\ \delta\mathbf{u}' \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & {}^{t}_{t'}\mathbf{K}_{NL} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{37}
$$

$$
\int_{{}^{t'}V} ({}^{t}_{t'}\tau')^T \delta_{t'}\mathbf{e}' \, {}^{t'}dv = \begin{bmatrix} \delta\mathbf{r} \\ \delta\pi' \\ \delta\mathbf{u}' \end{bmatrix}^T \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ {}^{t}_{t'}\mathbf{F} \end{bmatrix} . \tag{38}
$$

In equations (36) and (37) the matrices ${}^{t}_{t'}\mathbf{K}_L$ and ${}^{t}_{t'}\mathbf{K}_{NL}$ denote respectively the linear stiffness matrix and the nonlinear (or geometric) stiffness matrix. In full, the stiffness matrices are written as:

$$
{}^{t}_{t'}\mathbf{K}_L = \int_{{}^{t'}V} {}^{t}_{t'}\mathbf{B}_L^T \, {}_{t'}\mathbf{C} \, {}^{t}_{t'}\mathbf{B}_L \, {}^{t'}dv \tag{39}
$$

$$
{}^{t}_{t'}\mathbf{K}_{NL} = \int_{{}^{t'}V} {}^{t}_{t'}\mathbf{B}_{NL}^T \, {}^{t}_{t'}\tau' \, {}^{t}_{t'}\mathbf{B}_{NL} \, {}^{t'}dv . \tag{40}
$$

In equation (38) ${}^{t}_{t'}\mathbf{F}$ denotes the nodal forces due to the actual state of stress, i.e.,

$$
{}^{t}_{t'}\mathbf{F} = \int_{{}^{t'}V} {}^{t}_{t'}\mathbf{B}_L^T \, {}^{t}_{t'}\hat{\tau}' \, {}^{t'}dv . \tag{41}
$$

The finite element equations of motion for the flexible body can now be assembled. For such purpose equations (36) through (38) are substituted into equation (21) yielding:

$$
\begin{bmatrix} \mathbf{M}_{rr} & \mathbf{M}_{r\phi} & \mathbf{M}_{rf} \\ \mathbf{M}_{\phi r} & \mathbf{M}_{\phi\phi} & \mathbf{M}_{\phi f} \\ \mathbf{M}_{fr} & \mathbf{M}_{f\phi} & \mathbf{M}_{ff} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \dot{\omega}' \\ \ddot{\mathbf{u}}' \end{bmatrix} = \begin{bmatrix} \mathbf{g}_r \\ \mathbf{g}_\phi \\ \mathbf{g}'_f \end{bmatrix} - \begin{bmatrix} \mathbf{s}_r \\ \mathbf{s}'_\phi \\ \mathbf{s}'_f \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ {}^{t}_{t'}\mathbf{F} \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & {}^{t}_{t'}\mathbf{K}_L + {}^{t}_{t'}\mathbf{K}_{NL} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{u}' \end{bmatrix} . \tag{42}
$$

The two first rows in equation (42) are related to the motion of the body fixed coordinate frame. Note that such motion does not depend explicitly on the deformation of the flexible body.

However the motion of the body fixed coordinate frame and the deformation of the body are effectively coupled due to the form of the mass matrix and the gyroscopic terms.

The form of equation (42) can be substantially simplified if: (a) a lumped mass formulation is used to calculate equations (33) and (34), and (b) the vector nodal accelerations with respect to the body fixed coordinate system (vector $\ddot{\mathbf{u}}'$) is replaced by a vector of nodal accelerations with respect to the inertial frame (vector $\ddot{\mathbf{q}}_f'$). Such procedure is used in this paper.

The vectors of nodal accelerations can be partitioned in translational and angular accelerations as

$$\ddot{\mathbf{u}}' = \begin{bmatrix} \ddot{\delta}' \\ \ddot{\theta}' \end{bmatrix}; \qquad \ddot{\mathbf{q}}_f' = \begin{bmatrix} \ddot{\mathbf{d}}' \\ \alpha' \end{bmatrix}.$$

The relation between the relative nodal accelerations $\ddot{\mathbf{u}}'$ and the absolute nodal accelerations $\ddot{\mathbf{q}}_f'$ for a node k is given by

$$\begin{bmatrix} \ddot{\delta}' \\ \ddot{\theta}' \end{bmatrix}_k = \begin{bmatrix} \ddot{\mathbf{d}}' \\ \alpha' \end{bmatrix}_k - \begin{bmatrix} \mathbf{A}^T & -(\tilde{\mathbf{x}}^k + \tilde{\delta}^k)' \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \dot{\omega}' \end{bmatrix} - \begin{bmatrix} \tilde{\omega}'\tilde{\omega}'(x^k + \delta^k)' + 2\tilde{\omega}'(\dot{\delta}^k)' \\ \tilde{\omega}'(\dot{\theta}^k)' \end{bmatrix}. \tag{43}$$

Equation (43) is evaluated for every node of the finite element mesh (n nodes) and the result is substituted into equation (42). Rearranging the outcome of such substitution leads to

$$\sum_{k=1}^{n} (m\ddot{\mathbf{d}}')_k = \mathbf{g}_r' \tag{44}$$

$$\sum_{k=1}^{n} [m(\tilde{\mathbf{x}}' + \tilde{\delta}')\ddot{\mathbf{d}}']_k = \mathbf{g}_\theta' \tag{45}$$

$$\mathbf{M}_{ff}\ddot{\mathbf{q}}_f' = \mathbf{g}_f' - {}_{t'}^{t}\mathbf{F} - ({}_{t'}^{t}\mathbf{K}_L + {}_{t'}^{t}\mathbf{K}_{NL})\mathbf{u}'. \tag{46}$$

Equation (44) is the equation of motion for a system of particles. It can be shown [23] that

$$\sum_{k=1}^{n} (m\ddot{\mathbf{d}}')_k = \left(\sum_{k=1}^{n} m_k\right)\ddot{\mathbf{r}}_c',$$

where $\ddot{\mathbf{r}}_c'$ is the acceleration of the center of mass of the system of particles. Equation (45) indicates that the time rate of change of the angular momentum about the origin of the body fixed coordinate system is equal to the total moment, of the external forces acting over the flexible body, about the origin of the body fixed coordinates. Then, assuming that the body fixed coordinate system is fixed to the center of mass of the flexible body, equations (44) and (45) are simply the translational and rotational equations of motion of the body fixed coordinate frame. Equation (46) is the equation of motion of the nodes of the flexible body in reference to the body fixed coordinate frame. Note that due to the use of a lumped mass formulation the mass matrix $\mathbf{M}_{ff}$ can be written as

$$\mathbf{M}_{ff} = Diag(m_1\mathbf{I}, \mathbf{0}, \ldots, m_n\mathbf{I}, \mathbf{0}),$$

where m_k is the lumped mass of node k, and $\mathbf{I}$ and $\mathbf{0}$ are 3×3 identity and null matrices associated with the translational and rotational accelerations respectively. If the finite elements used in the discretization of the flexible body do not have rotational degrees of freedom, then no null matrices will appear in the mass matrix $\mathbf{M}_{ff}$.

In the derivation of equations (44) through (46) no assumption was made about the location of the body fixed coordinate frame on the flexible body. However it was observed that if the body fixed coordinate system is fixed to the center of mass of the flexible body then equations (44) and (45) have a simple interpretation. Very often it is desirable to locate the origin of the body fixed frame in a point different from the center of mass. For this purpose assume that there is a region of the flexible body (as small or large as desired) that is rigid and to which the body fixed frame is attached, as illustrated in Figure 5. Let the mass of the rigid part be denoted by m and its tensor of inertia by $\mathbf{J}$. If the rigid part is separated from the rest of the flexible body, then the Newton–Euler equations of motion for such region are written as

$$m\ddot{\mathbf{r}} = \mathbf{f} \tag{47}$$

$$\mathbf{J}'\dot{\omega}' = \mathbf{n}' - \tilde{\omega}'\mathbf{J}'\omega' , \tag{48}$$

where $\mathbf{f}$ and $\mathbf{n}$ are the external forces and moments applied over the rigid part.

Equations (47) and (48) can now be used instead of equations (44) and (45) provided that the proper kinematic constraints between the flexible and rigid parts of the body are enforced. Such kinematic constraints affect the nodes in the boundary between the rigid and flexible parts and can be written as:

$$\delta' = \dot{\delta}' = \ddot{\delta}' = \theta' = \dot{\theta}' = \ddot{\theta}' = \mathbf{0} \tag{49}$$

Consequently the relation between the absolute accelerations of the boundary nodes and the acceleration of the body fixed coordinate frame is found by substituting equation (49) into

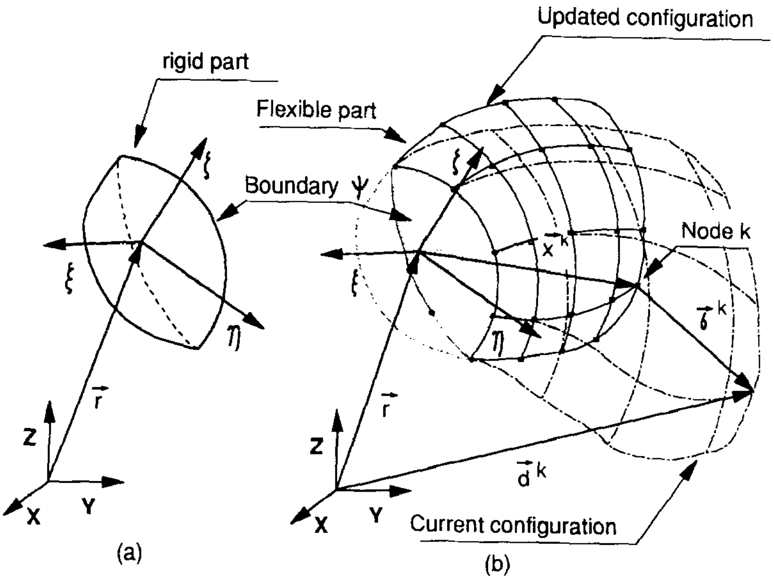

Fig. 5. Flexible body with an undeformable region.

equation (43) to get

$$\begin{bmatrix} \bar{\mathbf{A}}^T & -\mathbf{S} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{I}} & \mathbf{0} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \dot{\omega}' \\ \ddot{\mathbf{d}}' \\ \alpha' \end{bmatrix} = \begin{bmatrix} -\mathbf{s}_\omega \\ \mathbf{0} \end{bmatrix}, \tag{50}$$

where

$$\bar{\mathbf{A}}^T = \begin{bmatrix} \mathbf{A}^T \\ \mathbf{A}^T \\ \\ \mathbf{A}^T \end{bmatrix}; \quad \mathbf{S} = \begin{bmatrix} (\tilde{\mathbf{x}}^1)' \\ (\tilde{\mathbf{x}}^2)' \\ \vdots \\ (\tilde{\mathbf{x}}^n)' \end{bmatrix}; \quad \bar{\mathbf{I}} = \begin{bmatrix} \mathbf{I} \\ \mathbf{I} \\ \vdots \\ \mathbf{I} \end{bmatrix}; \quad \mathbf{S}_\omega = \begin{bmatrix} \tilde{\omega}'\tilde{\omega}'(\mathbf{x}^1)' \\ \tilde{\omega}'\tilde{\omega}'(\mathbf{x}^2)' \\ \vdots \\ \tilde{\omega}'\tilde{\omega}'(\mathbf{x}^n)' \end{bmatrix}.$$

The dimension of these matrices is $(3\underline{n} \times 3)$, where $\underline{n}$ is the number of boundary nodes. Now equation (49) can be directly used in equation (46), and through the use of the Lagrange multiplier technique the constraints represented by equation (50) are appended to equations (46), (47) and (48). The outcome of such substitution is

$$\begin{bmatrix} \mathbf{m} + \bar{\mathbf{A}}\underline{\mathbf{M}}^*\bar{\mathbf{A}}^T & -\bar{\mathbf{A}}\underline{\mathbf{M}}^*\mathbf{S} & \mathbf{0} \\ -(\bar{\mathbf{A}}\underline{\mathbf{M}}^*\mathbf{S})^T & \mathbf{J}' + \mathbf{S}^T\underline{\mathbf{M}}^*\mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}_{ff} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \dot{\omega}' \\ \ddot{\mathbf{q}}_f' \end{bmatrix} = \begin{bmatrix} \mathbf{f} + \bar{\mathbf{A}}\mathbf{C}_\delta' \\ \mathbf{n}' - \tilde{\omega}'\mathbf{J}'\omega' - \mathbf{S}^T\mathbf{C}_\delta' - \bar{\mathbf{I}}^T\mathbf{C}_\theta' \\ \mathbf{g}_f' - {}_{t'}^{t}\mathbf{F} - ({}_{t'}^{t}\mathbf{K}_L + {}_{t'}^{t}\mathbf{K}_{NL})\mathbf{u}' \end{bmatrix}, \tag{51}$$

where $\underline{\mathbf{M}}^*$ is a diagonal mass matrix containing the mass of the boundary nodes. Vectors $\mathbf{C}_\delta'$ and $\mathbf{C}_\theta'$ represent respectively the reaction force and reaction moment of the flexible part of the body over the rigid part. These reaction force/moments are written as

$$\mathbf{C}_\delta' = \mathbf{g}_{\underline{\delta}}' - \underline{\mathbf{M}}^*\mathbf{s}_\omega - {}_{t'}^{t}\mathbf{F}_{\underline{\delta}} - ({}_{t'}^{t}\mathbf{K}_L + {}_{t'}^{t}\mathbf{K}_{NL})_{\underline{\delta}\delta}\delta' - ({}_{t'}^{t}\mathbf{K}_L + {}_{t'}^{t}\mathbf{K}_{NL})_{\underline{\delta}\theta}\theta' \tag{52}$$

$$\mathbf{C}_\theta' = -\mathbf{g}_{\underline{\theta}}' + {}_{t'}^{t}\mathbf{F}_{\underline{\theta}} + ({}_{t'}^{t}K_L + {}_{t'}^{t}\mathbf{K}_{NL})_{\underline{\theta}\delta}\delta' + ({}_{t'}^{t}\mathbf{K}_L + {}_{t'}^{t}\mathbf{K}_{NL})_{\underline{\theta}\theta}\theta'. \tag{53}$$

In these equations the subscripts δ and θ refer to the partition of the vectors and matrices with respect to the translational and rotational nodal degrees of freedom. The underlined subscripts are referred to the boundary nodes between the rigid and flexible parts. For more details on the derivation of equations (51) through (53) refer to reference [24].

Equation (51) is the finite element equation of motion for the flexible body. However when finite elements with rotational degrees of freedom are used to discretize the flexible body some null elements appear in the diagonal of the submatrix $\mathbf{M}_{ff}$. Therefore, equation (51) cannot be solved explicitly for the accelerations. Two approaches may be used to solve this problem. In the first approach an alternate formulation for the submatrix $\mathbf{M}_{ff}$ is used. The rotational inertias of the nodes are obtained by lumping the off-diagonal terms of the consistent mass matrix $\mathbf{M}_{ff}$ according to the methodology of Surana [25]. The price paid for this approach is that the contribution of the rotational inertias of the nodes for the inertial coupling between the nodal coordinates and the reference coordinates must be neglected, otherwise the mass matrix of equation (51) will no longer be diagonal. In the second approach, a static condensation of the nodal rotational degrees of freedom is used. Let the last row of equation (51) be written in the following partitioned form:

$$\begin{bmatrix} \mathbf{M}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{d}}' \\ \alpha' \end{bmatrix} = \begin{bmatrix} \mathbf{g}'_\delta \\ \mathbf{g}'_\theta \end{bmatrix} - \begin{bmatrix} {}^{t}_{t'}\mathbf{F}'_\delta \\ {}^{t}_{t'}\mathbf{F}'_\theta \end{bmatrix} - \begin{bmatrix} {}^{t}_{t'}\mathbf{K}_{\delta\delta} & {}^{t}_{t'}\mathbf{K}_{\delta\theta} \\ {}^{t}_{t'}\mathbf{K}_{\theta\delta} & {}^{t}_{t'}\mathbf{K}_{\theta\theta} \end{bmatrix} \begin{bmatrix} \delta' \\ \theta' \end{bmatrix}, \tag{54}$$

where ${}^{t}_{t'}\mathbf{K} = {}^{t}_{t'}\mathbf{K}_L + {}^{t}_{t'}\mathbf{K}_{NL}$. The relation between the translational degrees of freedom and the rotational coordinates is given by

$$\theta' = {}^{t}_{t'}\mathbf{K}_{\theta\theta}^{-1}(\mathbf{g}'_\theta - {}^{t}_{t'}\mathbf{F}'_\theta - {}^{t}_{t'}\mathbf{K}_{\theta\delta}\delta') . \tag{55}$$

Finally the equations of motion for the flexible body is written as:

$$\begin{bmatrix} \mathbf{m} + \bar{\mathbf{A}}\underline{\mathbf{M}}^*\bar{\mathbf{A}}^T & -\bar{\mathbf{A}}\underline{\mathbf{M}}^*\mathbf{S} & \mathbf{0} \\ -(\bar{\mathbf{A}}\underline{\mathbf{M}}^*\mathbf{S})^T & \mathbf{J}' + \mathbf{S}^T\underline{\mathbf{M}}^*\mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}_{ff} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \dot{\omega}' \\ \ddot{\mathbf{d}}' \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{f} + \mathbf{A}\mathbf{C}'_\delta \\ \mathbf{n}' - \tilde{\omega}'\mathbf{J}'\omega' - \mathbf{S}^T\mathbf{C}'_\delta - \bar{\mathbf{I}}^T\mathbf{C}'_\theta \\ \mathbf{g}'_\delta - {}^{t}_{t'}\mathbf{F}'_\delta - {}^{t}_{t'}\mathbf{K}_{\delta\theta}\, {}^{t}_{t'}\mathbf{K}_{\theta\theta}^{-1}(\mathbf{g}'_\theta - {}^{t}_{t'}\mathbf{F}'_\theta) - ({}^{t}_{t'}\mathbf{K}_{\delta\delta} - {}^{t}_{t'}\mathbf{K}_{\delta\theta}\, {}^{t}_{t'}\mathbf{K}_{\theta\theta}^{-1}\, {}^{t}_{t'}\mathbf{K}_{\theta\delta})\delta' \end{bmatrix} . \tag{56}$$

By a proper choice for the location and orientation of the body fixed coordinate frame in the rigid region, the mass matrix in equation (56) can be turned into an invariant diagonal matrix. In this sense the position of the origin of the body fixed coordinate system must be coincident with the center of mass of the rigid part plus the boundary nodes of the flexible part. Furthermore such coordinate system must be aligned with the principal inertia directions of the rigid part plus the boundary nodes.

4. Elasto-Plastic Deformations

A multibody system may experience elasto-plastic deformations of one or more of its flexible components. An example of this type of behavior is the structural impact and crash analysis of vehicles. For these problems an elasto-plastic constitutive tensor ${}_{t'}\mathbf{C}$ must be used to evaluate the linear stiffness matrix in equation (39).

If isotropic hardening and isothermal conditions are assumed, the material yield condition is given as

$$f({}^{t'}\tau, {}^{t'}\kappa) = 0 \tag{57}$$

where ${}^{t'}\tau$ is the Cauchy stress tensor and ${}^{t'}\kappa$ is the hardening parameter (which is a function of the state of strain). Yielding occurs when equation (57) is satisfied. In such case the strain increment ${}_{t'}\mathbf{e}$ is partially elastic and partially plastic. Assume that the strain increment can be decomposed into the sum of a plastic part and of an elastic part, which is written in terms of differential strain increments as

$$\mathrm{d}\,{}_{t'}\mathbf{e} = \mathrm{d}\,{}_{t'}\mathbf{e}^P + \mathrm{d}\,{}_{t'}\mathbf{e}^E . \tag{58}$$

Furthermore, assuming associated plasticity, the differential plastic strain increment is related to the yield surface by

$$\mathrm{d}\,{}_{t'}\mathbf{e}^{P} = \lambda \frac{\partial f}{\partial\, {}^{t'}\tau}\,, \tag{59}$$

where λ is the plastic multiplier (still unknown). The relation between the differential elastic strain increment and the differential stress increment is

$$\mathrm{d}\,{}_{t'}\mathbf{e}^{E} = ({}_{t'}\mathbf{C}^{E})^{-1}\,\mathrm{d}_{t'}\mathbf{S}\,, \tag{60}$$

where ${}_{t'}\mathbf{C}^{E}$ is the elastic constitutive tensor. Now equations (59) and (60) can be substituted in equation (58) to obtain

$$\mathrm{d}\,{}_{t'}\mathbf{e} = ({}_{t'}\mathbf{C}^{E})^{-1}\,\mathrm{d}_{t'}\mathbf{S} + \lambda \frac{\partial f}{\partial\, {}^{t'}\tau}\,. \tag{61}$$

During the plastic deformation $f = 0$ and consequently

$$\mathrm{d}f \equiv \left(\frac{\partial f}{\partial\, {}^{t'}\tau}\right)^{T} \mathrm{d}\,{}_{t'}\mathbf{S} + \frac{\partial f}{\partial\, {}^{t'}\kappa}\,\mathrm{d}\,{}^{t'}\kappa = 0\,,$$

or after rearranging

$$\left(\frac{\partial f}{\partial\, {}^{t'}\tau}\right)^{T} \mathrm{d}\,{}_{t'}\mathbf{S} - H\lambda = 0\,, \tag{62}$$

where $H = -\lambda^{-1}(\partial f/\partial\, {}^{t'}\kappa)\,\mathrm{d}\,{}^{t'}\kappa$. The plastic multiplier can now be eliminated between equation (61) and equation (62) (being careful not to divide or multiply anything by H), which yields

$$\mathrm{d}\,{}_{t'}\mathbf{S} = {}_{t'}\mathbf{C}\ \mathrm{d}\,{}_{t'}\mathbf{e}\,, \tag{63}$$

where the elasto-plastic constitutive tensor is written as

$${}_{t'}\mathbf{C} = -{}_{t'}\mathbf{C}^{E} - {}_{t'}\mathbf{C}^{E} \frac{\partial f}{\partial\, {}^{t'}\tau} \left(\frac{\partial f}{\partial\, {}^{t'}\tau}\right)^{T} {}_{t'}\mathbf{C}^{E}\left[H + \left(\frac{\partial f}{\partial\, {}^{t'}\tau}\right)^{T} {}_{t'}\mathbf{C}^{E} \frac{\partial f}{\partial\, {}^{t'}\tau}\right]^{-1}. \tag{64}$$

The parameter H remains unknown and needs to be solved for. Zienckiewicz [26] shows that if the Huber–Van Mises yield surface is used in equation (57), and if the simple work hardening rule is assumed, then H is the slope of the plot of the stress versus plastic strain for a uniaxial test.

Note that the formulation of the constitutive relations for elasto-plastic analysis is very similar to the small displacement analysis. This feature turns the updated Lagrangian formulation quite attractive for studying this type of problem.

5. Equations of Motion for a Multibody System

A multibody system is a collection of rigid and flexible bodies joined together by kinematic joints and force elements as depicted in Figure 6. For the ith body in the system $\mathbf{q}_i$ denotes a vector of coordinates which contains the Cartesian translational coordinates $\mathbf{r}_i$, a set of rotational coordinates, and a set of nodal coordinates $\mathbf{q}_f^{\prime}$ or $\delta^{\prime}$ (if body i is flexible). A vector of velocities for a

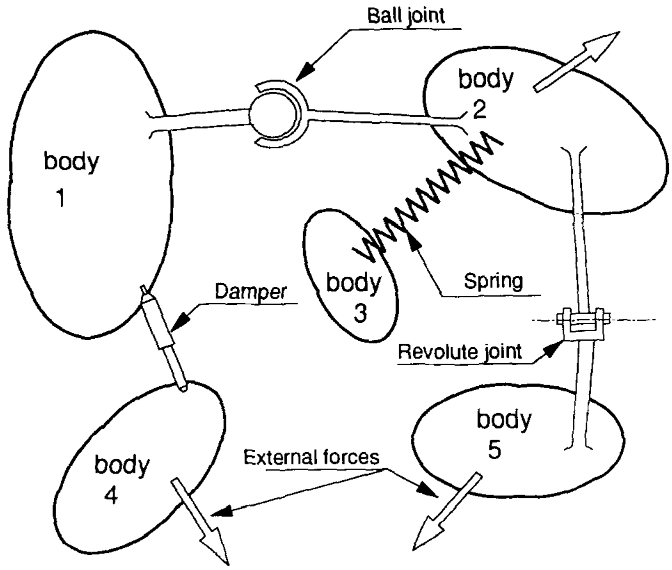

Fig. 6. Schematic representation of a multibody system.

rigid body i is defined as $\mathbf{v}_i$, which contains a 3-vector of translational velocities $\dot{\mathbf{r}}_i$ and a 3-vector of angular velocities ω_i (defined in the XYZ coordinate system). If body i is flexible then the vector of velocities $\mathbf{v}_i$ contains $\dot{\mathbf{r}}_i$, ω'_i (defined in the $\xi\eta\zeta_i$ coordinate system) and a vector of nodal velocities $\dot{\mathbf{q}}'_f$ or $\dot{\delta}'$. The vector of accelerations for the body is denoted by $\dot{\mathbf{v}}_i$ and it is simply the time derivative of $\mathbf{v}_i$. For a multibody system containing b bodies, the vectors of coordinates, velocities, and accelerations are $\mathbf{q}$, $\mathbf{v}$ and $\dot{\mathbf{v}}$ which contain the elements of $\mathbf{q}_i$, $\mathbf{v}_i$ and $\dot{\mathbf{v}}_i$, respectively, for $i = 1, \ldots, b$.

Let the kinematic joints between rigid bodies be described by mr independent constraints as

$$\Phi(\mathbf{q}) = \mathbf{0} . \tag{65}$$

The first and second derivatives of the constraints yield the kinematic velocity and acceleration equations.

$$\dot{\Phi} \equiv \mathbf{D}\mathbf{v} = \mathbf{0} \tag{66}$$

$$\ddot{\Phi} \equiv \mathbf{D}\dot{\mathbf{v}} + \dot{\mathbf{D}}\mathbf{v} = \mathbf{0} , \tag{67}$$

where $\mathbf{D}$ is the Jacobian matrix of the constraints. The equation of motion for the system of rigid bodies are written (see reference [27])

$$\mathbf{M}\dot{\mathbf{v}} = \mathbf{D}^T\lambda = \mathbf{g} , \tag{68}$$

where $\mathbf{M}$ is the inertia matrix, λ is a vector of Lagrange multipliers, and $\mathbf{g} = \mathbf{g}(\mathbf{q}, \mathbf{v})$ contains the gyroscopic terms, and the forces and moments that act on the rigid bodies.

The constrained equations of motion expressed by equations (65) to (68) can be converted to a smaller set of equations in terms of a set of coordinates known as joint coordinates. Such transformation is briefly discussed here, (for more detail refer to Nikravesh and Gim [28, 29]). The relative configurations of two adjacent bodies are described by a set of relative coordinates, known as joint coordinates, equal to the number of relative degrees of freedom between the bodies. The vector of joint coordinates for a system of rigid bodies is denoted by β and it contains all the joint coordinates and the absolute coordinates of the floating base bodies. The vector of joint velocities is defined as $\dot{\beta}$, which is the time derivative of β and its relation with $\mathbf{v}$ is given by [28]

$$\mathbf{v} = \mathbf{B}\dot{\beta} . \tag{69}$$

Matrix $\mathbf{B}$ is the velocity transformation matrix and it is orthogonal to the Jacobian matrix $\mathbf{D}$. The time derivative of equation (69) provides the formula for the transformation of the accelerations,

$$\dot{\mathbf{v}} = \mathbf{B}\ddot{\beta} + \dot{\mathbf{B}}\dot{\beta} . \tag{70}$$

Substituting equation (70) into equation (68), premultiplying by $\mathbf{B}^T$, and using the orthogonality between $\mathbf{B}$ and $\mathbf{D}$ yield

$$\mathcal{M}\dot{\beta} = f , \tag{71}$$

where

$$\mathcal{M} = \mathbf{B}^T\mathbf{M}\mathbf{B} \tag{72}$$

$$f = \mathbf{B}^T(\mathbf{g} - \mathbf{M}\dot{\mathbf{B}}\dot{\beta}) . \tag{73}$$

Equation (71) represents the generalized equation of motion for an open-loop system of rigid bodies. Equation (71), which contains the minimum number of second-order differential equations, can be used instead of the mixed set of differential-algebraic equations given by equations (65) through (68). For the equations of motion of a system containing closed kinematic loops, the interested reader may refer to reference [29] where more detail and further discussion is provided.

If there are rigid and rigid-flexible bodies in the multibody system, and the kinematic joints are only between the rigid parts, then equations (71) can be used to describe the motion of the system. The only difference is that now the vector of velocities $\mathbf{v}$ and the vector of joint velocities $\dot{\beta}$ also contain the nodal velocities $\dot{\mathbf{q}}_f$. Consequently a part of the velocity transformation matrix $\mathbf{B}$ that relates the nodal velocities in vector $\mathbf{v}$ with the nodal velocities in vector $\dot{\beta}$ in equation (69) must be an identity submatrix. Furthermore, it is convenient to rewrite equations (42), (51) or (56) in terms of $\dot{\omega}$ rather than $\dot{\omega}'$.

It is possible that some of the kinematic joints are between flexible bodies or between flexible and rigid bodies. In order to derive the equations of motion for such a system, these particular kinematic joints may temporarily be removed. For this reduced system, the equations of motion are written as in equation (71). A vector β^* is then defined as a new vector of joint coordinates containing the vector of joint coordinates β (these are the joint coordinates between rigid bodies/parts) and the nodal coordinates $\mathbf{q}_f$. Now if the removed kinematic constraints involving the flexible bodies are reintroduced to the system, then mf independent holonomic constraint equations in the form

$$\Psi(\beta^*) = \mathbf{0} \tag{74}$$

can be defined. The time derivative of these constraints yield

$$\dot{\Psi} \equiv \mathbf{C}\dot{\beta}^* = \mathbf{0}\,, \tag{75}$$

where $\mathbf{C}$ is the Jacobian for the constraints. Similarly the time derivative of these equations yields the acceleration constraints for the flexible bodies as

$$\ddot{\Psi} \equiv \mathbf{C}\ddot{\beta}^* + \dot{\mathbf{C}}\dot{\beta}^* = \mathbf{0}\,. \tag{76}$$

Now the equations of motion for this multibody system are obtained by introducing the mf kinematic constraints in equation (71) through the use of a set of mf Lagrange multipliers ν as

$$\mathcal{M}^*\ddot{\beta}^* - \mathbf{C}^T\nu = f^*\,, \tag{77}$$

where the superscript * denotes the fact the generalized mass matrix and the generalized force vector are augmented by appending the nodal equations of motion of the flexible bodies to equation (71). Now equation (76) and (77) can be solved for the joint accelerations $\ddot{\beta}^*$ and for the Lagrange multipliers ν.

6. Discussion and Conclusions

In this paper the equations of motion for a flexible body experiencing nonlinear deformations as a result of large strain and nonlinear material law are derived based on a finite element procedure. A modified updated Lagrangian formulation is used to provide a simpler form of the elasto-plastic constitutive relations. The difference between the traditional updated Lagrangian formulation and the one described here is that a corotated ghost configuration, coincident with the updated configuration, is used. The advantage of this procedure is that the need for objectivity in the material law is lowered once the average large rotation of the finite elements is extracted from the stress and strain measures.

In order to simplify the form of the flexible body equations of motion, a lumped mass formulation for the evaluation of the mass matrix and the gyroscopic force vector is used. Such formulation, combined with the reference of the nodal accelerations to the inertial coordinate frame, greatly simplifies the form of the equations of motion. The point of attachment of the body fixed coordinate frame is defined in the rigid part of the rigid/flexible body. The result of this methodology is a form of the equations of motion where the inertial coupling between the gross rigid body motion and the deformation of the body is preserved while the mass matrix is diagonal and constant. When finite elements with rotational degrees of freedom are used in the discretization of the flexible body, the explicit use of the nodal angular accelerations is eliminated by a static condensation process.

Finally the equations of motion for a multibody system with flexible components are presented. A set of joint coordinates is used to describe the kinematic relation between rigid bodies and the rigid parts of the flexible bodies. Constraints involving the nodal coordinates of the flexible bodies are introduced in the system through the use of Lagrange multipliers.

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