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Alternative forms of the Shapley value and the Shapley-Shubik index*

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Abstract. We prove the validity of an alternative representation of the Shapley-Shubik (1954) index of voting power, based on the following model. Voting in an assembly consisting of n voters is conducted by roll-call. Every voter is assumed to vote "yea" or "nay" with equal probability, and all $n!$ possible orders in which the voters may be called are also assumed to be equiprobable. Thus there are altogether $2^n n!$ distinct roll-call patterns. Given a simple voting game (a decision rule), the *pivotal* voter in a roll-call is the one whose vote finally decides the outcome, so that the votes of all those called subsequently no longer make any difference. The main result, stated without proof by Mann and Shapley (1964), is that the Shapley-Shubik index of voter a in a simple voting game is equal to the probability of a being pivotal. We believe this representation of the index is much less artificial than the original one, which considers only the $n!$ roll-calls in which all voters say "yea" (or all say "nay"). The proof of this result proceeds by generalizing the representation so that one obtains a value for each player in any coalitional game, which is easily seen to satisfy Shapley's (1953) three axioms. Thus the generalization turns out to be an alternative representation of the Shapley value. This result implies a non-trivial combinatorial theorem.

Throughout this paper, U is an arbitrary finite non-empty set with $|U| = n$. A member of U is called a *player*, and a subset of U is called a *coalition*.

We denote by "s" and "d" (short for *sinister* and *dexter*) the canonical left-hand and right-hand projections from the cartesian product $\{1, 2, \ldots, n\} \times \{1, -1\}$; so $\mathrm{s}\langle i, j\rangle = i$ and $\mathrm{d}\langle i, j\rangle = j$.

By a *roll-call* we mean a map R from U to $\{1, 2, \ldots, n\} \times \{1, -1\}$ such that $\mathrm{s}R$ is a bijection from U to $\{1, 2, \ldots, n\}$. Thus $\mathrm{s}R$ induces a total order on U; we refer to it as the *order of the players in R*. For example, if $\mathrm{s}R(b) < \mathrm{s}R(a)$, we say that player b *precedes* player a in R. We say that a is *positive* or *negative* in R according as $\mathrm{d}R(a)$ is 1 or −1. Intuitively, we imagine that the players are called one by one, and each player declares "yea" or "nay". Thus $\mathrm{s}R(a) = i$ means that player a is the i-th to be called; and a declares "yea" or "nay" according as a is positive or negative in R.

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We denote by "$\mathcal{R}$" the set of all roll-calls and by "$\mathcal{R}^+$", the set of all *totally positive* roll-calls:

$$\mathcal{R}^+ = \{R \in \mathcal{R} : \mathrm{d}R(x) = 1 \text{ for all } x \in U\}.$$

Clearly, $|\mathcal{R}| = 2^n n!$ and $|\mathcal{R}^+| = n!$. It will be convenient to regard $\mathcal{R}$ and $\mathcal{R}^+$ as elementary probability spaces, in which all points are equiprobable.

For any $R \in \mathcal{R}$, we let $\mathrm{Y}(R)$ and $\mathrm{N}(R)$ be the set of positive and negative players, respectively, in R:

$$\mathrm{Y}(R) = \{x \in U : \mathrm{d}R(x) = 1\}, \qquad \mathrm{N}(R) = \{x \in U : \mathrm{d}R(x) = -1\}.$$

Further, for any player a we let $\mathrm{Y}(R, a)$ and $\mathrm{N}(R, a)$ be the set of positive and negative players, respectively, in R that precede or equal a in R:

$$\begin{aligned} \mathrm{Y}(R,a) &= \{x \in U : \mathrm{d}R(x) = 1 \;\&\; \mathrm{s}R(x) \leq \mathrm{s}R(a)\}, \\ \mathrm{N}(R,a) &= \{x \in U : \mathrm{d}R(x) = -1 \;\&\; \mathrm{s}R(x) \leq \mathrm{s}R(a)\}. \end{aligned}$$

A *coalitional game* (briefly, *CG*) is a real-valued function v defined on the set of coalitions (the power-set of U) such that $v(\oslash) = 0$. The *dual* v^* of a CG v is defined by putting $v^*(X) = v(U) - v(U - X)$ for each $X \subseteq U$.

Given any CG v, roll-call R and player a, we define a quantity $M(v, R, a)$ as follows:

$$M(v,\ R,\ a) = \begin{cases} v(\mathrm{Y}(R,a)) - v(\mathrm{Y}(R,a) - \{a\}) & \text{if } \mathrm{d}R(a) = 1, \\ v^*(\mathrm{N}(R,a)) - v^*(\mathrm{N}(R,a) - \{a\}) & \text{if } \mathrm{d}R(a) = -1. \end{cases}$$

Thus $M(v, R, a)$ may be regarded as the *marginal contribution* of a to $v(\mathrm{Y}(R))$ or to $v^*(\mathrm{N}(R))$, according as a is positive or negative in R.

The *Shapley value* is a function φ that assigns to each CG v and each player a a real number $\varphi_a[v]$, called the *value of v for a*. Shapley (1953) defined this function and showed that it is uniquely characterized by three conditions (his Axioms 1–3). In Section 6 of that paper he showed, in effect, that $\varphi_a[v]$ is the expected value of $M(v, R, a)$ in the probability space $\mathcal{R}^+$:

$$\varphi_a[v] = \sum\{M(v,R,a) : R \in \mathcal{R}^+\}/n!. \tag{1}$$

We now claim:

THEOREM. $\varphi_a[v]$ is the expected value of $M(v, R, a)$ in the probability space $\mathcal{R}$:

$$\varphi_a[v] = \sum\{M(v,R,a) : R \in \mathcal{R}\}/2^n n!. \tag{2}$$

Proof. It is straightforward to verify that the right-hand side of (2) satisfies Shapley's Axioms 1–3. QED

A *simple voting game* (briefly, *SVG*) is a CG v that assumes only the values 0 and 1, is monotone in the sense that

$$X \subseteq Y \subseteq U \Rightarrow v(X) \leq v(Y),$$

and does not vanish everywhere, so that $v(U) = 1$. If v is an SVG and $X \subseteq U$, we say that X *wins* or *loses* in v according as $v(X)$ is 1 or 0. Clearly, the dual v^* of an SVG v is itself an SVG; and a coalition X wins in v^* iff $U - X$ loses in v.

The players of an SVG are called *voters*. The intuitive idea behind the notion of SVG is that U is a committee or some decision-making assembly, and v provides its decision rule: a bill is passed iff the set X of voters supporting it is a winning coalition in v.

Let v be an SVG. It is not difficult to see that for each $R \in \mathcal{R}$ there is a unique voter a such that $M(v, R, a) = 1$, whereas $M(v, R, x) = 0$ for every voter $x \neq a$. We call this unique a the *pivot* of R in v and put $a = \mathrm{Piv}(v, R)$. $\mathrm{Piv}(v, R)$ can be characterized as the first voter a in R such that if $\mathrm{Y}(R)$ wins in v then $\mathrm{Y}(R, a)$ wins in v; and if $\mathrm{Y}(R)$ loses in v then $U - \mathrm{N}(R, a)$ loses in v. Put another way, $\mathrm{Piv}(v, R)$ is the first voter a in R for which the following holds: if R' is any roll-call such that $\mathrm{s}R' = \mathrm{s}R$ and such that $\mathrm{d}R'(x) = \mathrm{d}R(x)$ for all voters x that precede or equals a in R, then $\mathrm{Y}(R')$ wins in v iff $\mathrm{Y}(R)$ wins in v. Thus, in the roll-call R the outcome of the voting is decided when the pivotal voter's declaration is made, and the declarations of subsequent voters in R make no difference.

The *Shapley-Shubik index* is the restriction of the Shapley value φ to the class of SVGs. Shapley and Shubik (1954) argue that $\varphi_a[v]$ measures the relative power of voter a in the SVG v. In view of what has just been said, we obtain from (1), for any SVG v:

$$\varphi_a[v] = | \{R \in \mathcal{R}^+ : a = \mathrm{Piv}(v, R)\} | / n!. \tag{3}$$

However, from the Theorem we obtain:

COROLLARY. *If v is any SVG then*

$$\varphi_a[v] = | \{R \in \mathcal{R} : a = \mathrm{Piv}(v, R)\} | / 2^n n!. \tag{4}$$

COMMENTS. Formulas (2) and (4) are not useful from a practical computational point of view. Their value, if any, is conceptual. We make no great claim for formula (2); in our view the bargaining model embodied in it is scarcely more realistic than Shapley's original model embodied in (1). Our own interest in (2) is due to the fact that it allows us to deduce (4), which we do regard as conceptually more attractive than (3). In the model underlying (3), all voters are assumed to vote "yea", and only the *order* in which they cast their votes is allowed to vary. (Shapley and Shubik (1954: 788) interpret the order of voting "as an indication of the relative degrees of support by the different members, with the most enthusiastic members 'voting' first, etc.") This has been criticized by several authors – rightly, in our opinion – as highly artificial (see, for example, Luce and Raiffa, 1957: 255–256; Brams, 1975: 168–169). In contrast, (4) is based on the assumption that voters act independently of each other, each voting "yea" or "nay" with equal prior probability. The result can easily be generalized, with virtually the same proof, to the case where all voters vote "yea" with probability p, which need not be 1/2. This result was stated, without proof, by Mann and Shapley (1964).

Finally, let us point out that our easy proof of (2) – and hence of (4) – is indirect: it makes use of Shapley's theorem on the unique characterization of φ. If one attempts to prove (4) by showing *directly* that the right-hand side of (4) is equal to that of (3), one encounters rather formidable combinatorial difficulties. This suggests that our Theorem and Corollary are a disguised form of a combinatorial fact that is certainly non-trivial, and may be of some independent interest.

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