Ann. Inst. Statist. Math.
Vol. 43, No. 2, 347–356 (1991)

ON THE ESTIMATION OF ORDERED MEANS OF TWO EXPONENTIAL POPULATIONS

AMARJOT KAUR AND HARSHINDER SINGH

Department of Statistics, Panjab University, Chandigarh 160014, India

(Received April 3, 1989; revised February 14, 1990)

Abstract. Let random samples of equal sizes be drawn from two exponential distributions with ordered means λ_i. The maximum likelihood estimator λ_i^* of λ_i is shown to have a smaller mean square error than that of the usual estimator $\bar{X}_i$, for each $i = 1, 2$. The asymptotic efficiency of λ_i^* relative to $\bar{X}_i$ has also been found.

Key words and phrases: Asymptotic efficiency, exponential distribution, isotonic regression, maximum likelihood estimation, mean square error.

1. Introduction

The maximum likelihood estimator (MLE) $\hat{\mu}$ of a nondecreasing regression function has been studied in detail (see Barlow *et al.* (1972), Robertson *et al.* (1988)). In terms of its coordinates, little was known about its quadratic loss until Lee (1981) showed that the mean square error (MSE) of $\hat{\mu}_i$ is less than the usual estimator $\bar{X}_i$ for each i, when $\bar{X}_1, \bar{X}_2, \dots, \bar{X}_k$ are independent normal variates. Robertson *et al.* (1988, p. 44) have mentioned that it is of interest to know if such a result is valid under other conditions, and in particular, whether it holds true for samples from other types of populations. In this article we give a partial answer to their question with regard to sampling from exponential distributions and show that the result holds for the case of samples of equal sizes from two exponential distributions having ordered means.

The results of this article will prove useful in reliability when one frequently comes across situations where the estimation of means of two ordered exponential life distributions is desired. For example, there are situations where one is interested in the estimation of the mean lives of two mechanical devices having exponential life distributions, of which one is an improvement of the other, and naturally the improved device should not have a mean life length less than that of the original device. In yet another situation, the interest may be in the estimation of the mean lives of two components having exponential life distributions, in which one is produced by a standard company whereas the other is manufactured by a local company, and where it is known, a priori, that the mean life of the component of the standard company is not less than that of the component produced by the local company. For more examples of statistical inferences arising from reliability, readers may refer to Marshal and Proschan (1965), Barlow *et al.* (1972), Hollander and Proschan (1984), Doksum and Yandell (1984) and Feltz and Dykstra (1985).

For other references on the estimation of ordered parameters, readers may also refer to Blumenthal and Cohen (1968), Cohen and Sackrowitz (1970), Sackrowitz (1970) and Kushary and Cohen (1989).

For $i = 1, 2$, let X_{ij}, $j = 1, 2, \ldots, n$ be a random sample from an exponential distribution with mean λ_i satisfying $\lambda_1 \leq \lambda_2$. The MLE λ_i^* of λ_i is the isotonic regression of $\bar{X}_i = \left(\sum_{j=1}^{n} X_{ij}\right) / n$ with equal weights; moreover,

$$\sum_{i=1}^{2} E[(\lambda_i^* - \lambda_i)^2] \leq \sum_{i=1}^{2} E[(\bar{X}_i - \lambda_i)^2]. \tag{1.1}$$

(see Robertson *et al.* (1988)).

In Section 2 of this paper, we show that

$$E[(\lambda_i^* - \lambda_i)^2] \leq E[(\bar{X}_i - \lambda_i)^2], \qquad i = 1,\ 2, \tag{1.2}$$

and study the asymptotic behaviour of the efficiency of λ_i^* relative to $\bar{X}_i$. The efficiency of λ_i^* relative to $\bar{X}_i$ has been calculated for some values of n and (λ_2/λ_1) and these values can be found in Tables 1 and 2.

Table 1. Efficiency $e(\lambda_1^*,\ \bar{X}_1)$ of λ_1^* relative to $\bar{X}_1$.

λ_2/λ_1	n							
	2	5	10	20	30	50	100	1000
1	1.7777	1.5950	1.5107	1.4549	1.4312	1.4080	1.3853	1.3493
1.3	1.6358	1.4359	1.3225	1.2201	1.1632	1.0972	1.0320	1.0000
1.5	1.5547	1.3425	1.2172	1.1124	1.0644	1.0236	1.0023	1.0000
2.0	1.4025	1.1804	1.0785	1.0185	1.0049	1.0003	1.0000	1.0000
2.5	1.3034	1.1082	1.0296	1.0030	1.0003	1.0000	1.0000	1.0000
3.0	1.2367	1.0656	1.0118	1.0005	1.0000	1.0000	1.0000	1.0000
5.0	1.1105	1.0128	1.0005	1.0000	1.0000	1.0000	1.0000	1.0000
10.00	1.0351	1.0008	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 2. Efficiency $e(\lambda_2^*,\ \bar{X}_2)$ of λ_2^* relative to $\bar{X}_2$.

λ_2/λ_1	n							
	2	5	10	20	30	50	100	1000
1	1.0666	1.1450	1.1931	1.2304	1.2479	1.2660	1.2850	1.3176
1.3	1.1186	1.1627	1.1642	1.1360	1.1086	1.0693	1.0242	1.0000
1.5	1.1167	1.1319	1.1087	1.0656	1.0397	1.0152	1.0015	1.0000
2.0	1.0880	1.0670	1.0339	1.0088	1.0023	1.0001	1.0000	1.0000
3.0	1.0448	1.0182	1.0037	1.0001	1.0000	1.0000	1.0000	1.0000
5.0	1.0150	1.0023	1.0001	1.0000	1.0000	1.0000	1.0000	1.0000
10.00	1.0027	1.0001	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

2. Co-ordinatewise loss of MSE of λ_1^* and λ_2^*

By using the max-min formula of isotonic regression (see Barlow *et al.* (1972)), we have

$$\lambda_1^* = \min(\bar{X}_1,\ (\bar{X}_1 + \bar{X}_2)/2), \tag{2.1}$$

$$\lambda_2^* = \max(\bar{X}_2,\ (\bar{X}_1 + \bar{X}_2)/2). \tag{2.2}$$

We introduce the following notations

$$R(\lambda_i^*) = E[(\lambda_i^* - \lambda_i)^2], \qquad i = 1,\ 2. \tag{2.3}$$

$$R(\bar{X}_i) = E[(\bar{X}_i - \lambda_i)^2] = \lambda_i^2/n, \qquad i = 1,\ 2. \tag{2.4}$$

$$e(\lambda_i^*,\ \bar{X}_i) = R(\bar{X}_i)/R(\lambda_i^*), \qquad i = 1,\ 2. \tag{2.5}$$

We shall call $e(\lambda_i^*, \bar{X}_i)$ the efficiency of λ_i^* relative to $\bar{X}_i$, $i = 1,\ 2$. For convenience, we denote $\lambda_i/(\lambda_1+\lambda_2)$ by Z_i, $i = 1,\ 2$, (λ_2/λ_1) by y_1 and (λ_1/λ_2) by y_2. Since $\lambda_1 \le \lambda_2$, we have $y_1 \ge 1$, $0 < y_2 \le 1$, $0 < Z_1 \le (1/2)$ and $(1/2) \le Z_2 < 1$.

Since $\bar{X}_i$ has a gamma distribution with pdf

$$g_i(y) = (1/(n-1)!)(n/\lambda_i)^n y^{n-1} \exp(-ny/\lambda_i), \qquad y \ge 0 \tag{2.6}$$

for $i = 1,\ 2$ and the relationship

$$(1/(m-1)!)\alpha^m \int_x^\infty y^{m-1} \exp(-\alpha y)\, dy = \exp(-\alpha x) \sum_{i=0}^{m-1} (\alpha x)^i/i! \tag{2.7}$$

holds for any positive integer m and for any $\alpha > 0$, it easily follows that

$$
\begin{aligned}
(2.8)\qquad & E[(\lambda_1^* - \lambda_1)^2/\bar{X}_1 = x] \\
&= [(n+1)/4n]\lambda_2^2 + \lambda_1^2 - \lambda_1\lambda_2 + [(\lambda_2/2) - \lambda_1]x + (x^2/4) \\
&\quad + \exp(-nx/\lambda_2)\left\{ (3/4)x^2 \sum_{i=0}^{n-1} [(nx/\lambda_2)^i/i!] \right. \\
&\qquad - (\lambda_2/2)x \sum_{i=0}^{n} [(nx/\lambda_2)^i/i!] \\
&\qquad - [(n+1)/4n]\lambda_2^2 \sum_{i=0}^{n+1} [(nx/\lambda_2)^i/i!] \\
&\qquad - \lambda_1 x \sum_{i=0}^{n-1} [(nx/\lambda_2)^i/i!] \\
&\qquad \left. + \lambda_1\lambda_2 \sum_{i=0}^{n} [(nx/\lambda_2)^i/i!] \right\}.
\end{aligned}
$$

By taking the expectation of (2.8) with respect to the distribution of $\bar{X}_1$, we have, after simplification,

$$
\begin{aligned}
(2.9)\qquad R(\lambda_1^*) = {} & (\lambda_1^2 + \lambda_2^2)[(n+1)/4n] - (\lambda_1\lambda_2)/2 \\
& + (1 - Z_1)^n \left\{ [3(n+1)/4n]\lambda_2^2 Z_1^2 \sum_{i=0}^{n-1} \binom{n+i+1}{i} Z_1^i \right. \\
& \quad - (1/2)\lambda_2^2 Z_1 \sum_{i=0}^{n} \binom{n+1}{i} Z_1^i \\
& \quad - [(n+1)/4n]\lambda_2^2 \sum_{i=0}^{n+1} \binom{n+i-1}{i} Z_1^i \\
& \quad - \lambda_1\lambda_2 Z_1 \sum_{i=0}^{n-1} \binom{n+i}{i} Z_1^i \\
& \quad \left. + \lambda_1\lambda_2 \sum_{i=0}^{n} \binom{n+i-1}{i} Z_1^i \right\}.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
(2.10)\qquad R(\lambda_2^*) &= (\lambda_2^2/n) \\
&\quad + (1-z_2)^n \Bigg\{ -[3(n+1)/4n]\lambda_1^2 Z_2^2 \sum_{i=0}^{n-1} \binom{n+i+1}{i} Z_2^i \\
&\qquad + (\lambda_1^2/2) Z_2 \sum_{i=0}^{n} \binom{n+i}{i} Z_2^i \\
&\qquad + [(n+1)/4n]\lambda_1^2 \sum_{i=0}^{n+1} \binom{n+i-1}{i} Z_2^i \\
&\qquad + \lambda_1\lambda_2 Z_2 \sum_{i=0}^{n-1} \binom{n+i}{i} Z_2^i \\
&\qquad - \lambda_1\lambda_2 \sum_{i=0}^{n} \binom{n+i-1}{i} Z_2^i \Bigg\}.
\end{aligned}
$$

THEOREM 2.1.
(a) $R(\lambda_1^*) < R(\bar{X}_1)$,
(b) $\lim_{(\lambda_2/\lambda_1)\to\infty} e(\lambda_1^*,\ \bar{X}_1) = 1$,
(c) $\lim_{n\to\infty} e(\lambda_1^*,\ \bar{X}_1) = \begin{cases} 1 & \text{if} \quad \lambda_2 > \lambda_1, \\ 4/3 & \text{if} \quad \lambda_2 = \lambda_1. \end{cases}$

PROOF. Since $(1-Z_1)^{-m} = \sum_{k=0}^{\infty} \binom{m+k-1}{k} Z_1^k$, for any positive integer m, it follows from (2.9) that

$$
\begin{aligned}
(2.11)\qquad & [R(\lambda_1^*) - R(\bar{X}_1)]/R(\bar{X}_1) \\
&= n\Bigg\{ -[3(n+1)/4n] \sum_{k=n}^{\infty} \binom{n+k+1}{k} Z_1^k (1-Z_1)^{n+2} \\
&\quad + (y_1/2) \sum_{k=n+1}^{\infty} \binom{n+k}{k} Z_1^k (1-Z_1)^{n+1} \\
&\quad + [(n+1)/4n] y_1^2 \sum_{k=n+2}^{\infty} \binom{n+k-1}{k} Z_1^k (1-Z_1)^n \\
&\quad + \sum_{k=n}^{\infty} \binom{n+k}{k} Z_1^k (1-Z_1)^{n+1} \\
&\quad - y_1 \sum_{k=n+1}^{\infty} \binom{n+k-1}{k} Z_1^k (1-Z_1)^n \Bigg\}.
\end{aligned}
$$

Using the relationship between binomial and negative binomial distributions (see Meyer (1970), p. 174) we have from (2.11)

$$
\begin{aligned}
(2.12)\qquad & [R(\lambda_1^*) - R(\bar{X}_1)]/R(\bar{X}_1) \\
&= n\{-[3(n+1)/4n]B(n+1,\ 2n+1,\ 1-Z_1) \\
&\quad + (y_1/2)B(n,\ 2n+1,\ 1-Z_1) \\
&\quad + [(n+1)/4n]y_1^2 B(n-1,\ 2n+1,\ 1-Z_1) \\
&\quad + B(n,\ 2n,\ 1-Z_1) - y_1 B(n-1,\ 2n,\ 1-Z_1)\},
\end{aligned}
$$

where

$$
(2.13)\qquad B(k,\ n,\ p) = \sum_{r=0}^{k} \binom{n}{r} p^r (1-p)^{n-r}
$$

$$
(2.14)\qquad = (n-k)\binom{n}{k} \int_0^{1-p} t^{n-k-1}(1-t)^k\,dt, \qquad \text{for } 0 \le k < n
$$

(see Feller (1957), p. 163 for (2.14)). Using (2.14) in (2.12), we obtain

$$
\begin{aligned}
(2.15)\quad & [R(\lambda_1^*) - R(\bar{X}_1)]/R(\bar{X}_1) \\
&= n\binom{2n}{n} \int_0^{1/(1+y_1)} t^{n-1}(1-t)^{n-1}[A(y_1)t^2 + B(y_1)t + C]\,dt,
\end{aligned}
$$

where

$$
\begin{aligned}
A(y_1) &= [(2n+1)/4](y_1^2 - 2y_1 - 3), \\
B(y_1) &= (1/2)(y_1 + 4n + 3), \\
C &= -(2n+3)/4.
\end{aligned}
$$

It can be shown that

$$
A(y_1)t^2 + B(y_1)t + C < 0 \qquad \text{for} \quad t \in [0,\ 1/(1+y_1)).
$$

By (2.15), part (a) of Theorem 2.1 holds.

Since $4t(1-t) \le 1$, we have from (2.15)

$$
\begin{aligned}
|[R(\lambda_1^*)/R(\bar{X}_1)] - 1| &\le n\binom{2n}{n}(1/2)^{2n-2} \int_0^{1/(1+y_1)} |A(y_1)t^2 + B(y_1)t + C|\,dt \\
&\le n\binom{2n}{n}(1/2)^{2n-2}\{[|A(y_1)|/3][1/(1+y_1)]^3 \\
&\qquad + [|B(y_1)|/2][1/(1+y_1)]^2 \\
&\qquad + |C|[1/(1+y_1)]\}.
\end{aligned}
$$

The right-hand side of the above inequality tends to zero as $y_1 \to \infty$ and part (b) of Theorem 2.1 has been established.

Consider the case when $y_1 > 1$. Then

$$nB(n+1,\ 2n+1,\ 1-Z_1) = n^2\binom{2n+1}{n+1}\int_0^{Z_1} t^{n-1}(1-t)^{n+1}\,dt \tag{2.16}$$
$$\leq n^2\binom{2n+1}{n+1}[Z_1(1-Z_1)]^{n-1}\int_0^1 (1-t)^2\,dt.$$

Using Stirling's approximation formula, the right-hand side of (2.16) is asymptotically equivalent to

$$n^{(3/2)}[4Z_1(1-Z_1)]^{n-1}(8/3\sqrt{\pi})$$

and it tends to zero as $n \to \infty$. It follows that the first term of (2.12) tends to zero as $n \to \infty$, so are the remaining terms of (2.12). Part (c) of Theorem 2.1 for the case $\lambda_2 > \lambda_1$ has been established.

For $y_1 = 1$, we have from (2.15)

$$\begin{aligned}
&[R(\lambda_1^*) - R(\bar{X}_1)] - 1 \\
&= n\binom{2n}{n}\int_0^{1/2}[-(2n+1)t^2 + 2(n+1)t - (2n+3)/4]t^{n-1}(1-t)^{n-1}\,dt \\
&= n\binom{2n}{n}\left\{(2n+2)\int_0^{1/2} t^n(1-t)^n\,dt - [(2n+3)/4]\int_0^{1/2} t^{n-1}(1-t)^{n-1}\,dt \right. \\
&\qquad \left. + \int_0^{1/2} t^{n+1}(1-t)^{n-1}\,dt\right\} \\
&= n\binom{2n}{n}\left\{(n+1)\int_0^1 t^n(1-t)^n\,dt - [(2n+3)/8]\int_0^1 t^{n-1}(1-t)^{n-1}\,dt \right. \\
&\qquad \left. + [(n+1)/2n]\int_0^1 t^n(1-t)^n\,dt - (1/n)(1/2)^{2n+1}\right\}.
\end{aligned}$$

After simplification, the above gives

$$[R(\lambda_1^*)/R(\bar{X}_1)] - 1 = -\binom{2n}{n}(1/2)^{2n+1} - (1/4).$$

As $n \to \infty$, the first term on the right-hand side tends to zero by using Stirling's approximation formula. Thus for $\lambda_2 = \lambda_1$, $\lim_{n\to\infty}[R(\lambda_1^*)/R(\bar{X}_1)] = 3/4$, which proves part (c) of Theorem 2.1.

THEOREM 2.2.

(a) $R(\lambda_2^*) < R(\bar{X}_2)$, *for* $n \geq 2$, $\lambda_1 \leq \lambda_2$ *and for* $n = 1$, $\lambda_1 < \lambda_2$. $R(\lambda_2^*) = R(\bar{X}_2)$, *for* $n = 1$, $\lambda_1 = \lambda_2$.

(b) $\lim_{(\lambda_2/\lambda_1)\to\infty} e(\lambda_2^*,\ \bar{X}_2) = 1$.

(c) $\lim_{n\to\infty} e(\lambda_2^*,\ \bar{X}_2) = \begin{cases} 1 & \textit{if} \quad \lambda_2 > \lambda_1, \\ (4/3) & \textit{if} \quad \lambda_2 = \lambda_1. \end{cases}$

PROOF. Proceeding on similar lines as those for Theorem 2.1 and by using the relationship

$$1 - B(k,\ n,\ p) = n\binom{n-1}{k}\int_0^p t^k(1-t)^{n-k-1}\,dt, \qquad 0 \leq k < n. \tag{2.17}$$

(see Feller (1957), p. 163), we get

$$(2.18)\quad [R(\lambda_2^*) - R(\bar{X}_2)]/R(\bar{X}_2)$$
$$= n\binom{2n}{n}\int_0^{y_2/(1+y_2)} [A_1(y_2)t^2 + B_1(y_2)t + C_1(y_2)]t^{n-1}(1-t)^{n-1}\,dt,$$

where

$$A_1(y_2) = -[(2n+1)/4](3 + 2y_2 - y_2^2),$$
$$B_1(y_2) = [(2n+1)/2](y_2 - y_2^2) + n(1+y_2),$$
$$C_1(y_2) = [(2n+1)/4]y_2^2 - ny_2.$$

We consider $[R(\lambda_2^*) - R(\bar{X}_2)]/R(\bar{X}_2)$ given by (2.18) as a function of y_2 and denote it by $\phi(y_2)$. Thus

$$(2.19)\quad \phi(y_2) = n\binom{2n}{n}\int_0^{y_2/(1+y_2)} [A_1(y_2)t^2 + B_1(y_2)t + C_1(y_2)]t^{n-1}(1-t)^{n-1}\,dt.$$

It can be shown that if $0 < y_2 \le n/(n+1)$, then $y_2/(1+y_2)$ is the smallest positive root of $A_1(y_2)t^2 + B_1(y_2)t + C_1(y_2) = 0$ and $A_1(y_2)t^2 + B_1(y_2)t + C_1(y_2) < 0$ for $t \in [0,\ y_2/(1+y_2))$. Thus, from (2.19) it follows that

$$(2.20)\qquad \phi(y_2) < 0 \quad \text{for} \quad y_2 \in (0,\ n/(n+1)].$$

Differentiating (2.19) twice with respect to y_2 gives

$$(2.21)\quad \phi''(y_2) = n\binom{2n}{n}\Big\{[(2n+1)/2]\int_0^{y_2/(1+y_2)} t^{n-1}(1-t)^{n+1}\,dt$$
$$+ [y_2/(1+y_2)]^{n-1}[1/(1+y_2)]^{n+1}[(n+1)y_2 - n]\Big\}.$$

Thus, $\phi''(y_2) > 0$ for $y_2 \ge n/(n+1)$, which implies that $\phi(y_2)$ is convex in $[n/(n+1),\ 1]$. From (2.19) we also have

$$\phi(1) = n\binom{2n}{n}\int_0^{1/2} [-(2n+1)t^2 + 2nt - (2n-1)/4]t^{n-1}(1-t)^{n-1}\,dt$$
$$= n\binom{2n}{n}\Big\{n\int_0^1 t^n(1-t)^n\,dt - [(2n-1)/8]\int_0^1 t^{n-1}(1-t)^{n-1}\,dt$$
$$- [(n+1)/2n]\int_0^1 t^n(1-t)^n\,dt + (1/n)(1/2)^{2n+1}\Big\}.$$

After simplification, the above gives

$$(2.22)\qquad \phi(1) = (1/2)\Big[\binom{2n-1}{k}(1/2)^{2n-1} - (1/2)\Big].$$

It is easily seen from (2.17) that

$$\sum_{k=n}^{2n-1} \binom{2n-1}{k} (1/2)^{2n-1} = (1/2). \tag{2.23}$$

Thus, from (2.22) and (2.23) we have

$$\begin{cases} \phi(1) < 0 & \text{for } \ n \geq 2, \\ \phi(1) = 0 & \text{for } \ n = 1. \end{cases} \tag{2.24}$$

Since $\phi(y_2)$ is convex in $[n/(n+1),\ 1]$, it follows that

$$\phi(y_2) < 0, \qquad y_2 \in (n/(n+1),\ 1). \tag{2.25}$$

Part (a) of Theorem 2.2 follows from (2.20), (2.24) and (2.25). The proofs of part (b) and (c) of Theorem 2.2 similarly follow from their counterparts in Theorem 2.1.

From the expressions (2.9) and (2.10), it immediately follows that $e(\lambda_i^*,\ \bar{X}_i)$ is a function of n and λ_2/λ_1. By using these expressions, the values of $e(\lambda_i^*,\ \bar{X}_i)$ for $i = 1,\ 2$ have been calculated by computer for some values of n and λ_2/λ_1. These have been tabulated in Tables 1 and 2.

Acknowledgements

The authors sincerely wish to thank Dr. J. C. Parnami, Department of Mathematics, Panjab University, for helpful discussions. The authors also thank the referees for suggestions leading to several improvements in the presentation.

References

Barlow, R. E., Bartholomew, D. J., Bremner, J. M. and Brunk, H. D. (1972). *Statistical Inference Under Order Restrictions*, Wiley, New York.

Blumenthal, S. and Cohen, A. (1968). Estimation of two ordered translation parameters, *Ann. Math. Statist.*, **39**, 517–530.

Cohen, A. and Sackrowitz, H. B. (1970). Estimation of last mean of monotone sequence, *Ann. Math. Statist.*, **41**, 2021–2034.

Doksum, K. and Yandell, B. S. (1984). Tests for exponentiality, *Handbook of Statistics*, Vol. 4, (eds. P. R. Krishnaiah and P. K. Sen), p. 579, North-Holland, Amsterdam.

Feller, W. (1957). *An Introduction to Probability Theory and Its Applications*, Vol. I, Second ed., Wiley, New York.

Feltz, C. J. and Dykstra, R. L. (1985). Maximum likelihood estimation of the survival functions of N stochastically ordered random variables, *J. Amer. Statist. Assoc.*, **80**, 1012–1019.

Hollander, M. and Proschan, F. (1984). Nonparametric concepts and methods in reliability, *Handbook of Statistics*, Vol. 4, (eds. P. R. Krishnaiah and P. K. Sen), p. 613, North-Holland, Amsterdam.

Kushary, D. and Cohen, A. (1989). Estimation of ordered Poisson parameters, Preprint, Rutgers University, New Brunswick, New Jersey.

Lee, C. I. C. (1981). The quadratic loss of isotonic regression under normality, *Ann. Statist.*, **9**, 686–688.

Marshal, A. W. and Proschan, F. (1965). Maximum likelihood estimation for distributions with monotone failure rate, *Ann. Math. Statist.*, **36**, 69–77.

Meyer, P. L. (1970). *Introductory Probability and Statistical Applications*, 2nd ed., Oxford and IBH Publishing, New Delhi.

Robertson, T., Wright, F. T. and Dykstra, R. L. (1988). *Order Restricted Statistical Inference*, Wiley, New York.

Sackrowitz, H. B. (1970). Estimation of monotone parameter sequences. The discrete case, *Ann. Math. Statist.*, **41**, 609–620.