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Asymptotic Behavior of Morphological Filters

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Abstract. The connection between morphological and stack filters is used in the analysis of the statistical properties of morphological filters. Closed-form expressions for the output distributions of morphological filters are given, and their statistical symmetry properties are analyzed. Asytotically tight bounds on the expectations of two-dimensional morphological filters, and asymptotic formulas for the variances of one-dimensional morphological filters are derived. These results form the basis for analyzing general asymptotic properties of morphological filters.

Key words. morphological filters, stack filters, statistical properties, asymptotic analysis

1 Introduction

In most image-processing applications the performance of the filter depends on how well it can suppress the noise and retain the desired information. Obtaining quantitative information on how much the filters reduce noise and how biased they are requires statistical analysis. Since morphological filters are nonlinear and are based on geometrical concepts, standard statistical methods cannot be directly applied to their analysis. Thus it is not surprising that the statistical theory of morphological filters is still far from mature. Certain statistical properties have been analyzed in Stevenson and Arce [1], and the restoration and representation properties of noisy images with use of morphological filters have been studied, e.g., in Schonfeld and Goutsias [2]. The connection between stack and morphological filters, studied by Maragos and Schafer [3] and Koskinen et al. [4], [5], can be used in the derivation of the output-distribution formulas for morphological filters, and the purpose of this paper is to apply the stack-filter method both to the derivation of output distributions and to the asymptotic analysis of statistical properties of morphological filters. The asymptotic analysis is important because it shows us the general behavior of morphological filters.

This paper is organized as follows. In section 2 basic definitions and some basic properties of morphological and stack filters are given. To obtain an adequate understanding of the filtering of noisy signals, it is desirable to determine the output distribution of the filter in terms of the input distributions. In section 3 formulas for the output distributions of morphological filters are derived for the case of independent inputs. These formulas apply to any input distribution and can be used when the structuring set is small to moderately large. Certain symmetry properties stemming from the duality properties of morphological filters are also presented. In section 4 the asymptotic behavior of dilation and erosion is studied, and in section 5 bounds (that are asymptotically tight for Laplace distributions) on the expectations of a large class of two-dimensional morphological filters are derived. In section 6 analytical formulas for output distributions are derived for the case of one-dimensional morphological filters. The simulation results in Rustanius et al. [6] indicate that these formulas approximate the two-dimensional case well. The bias and the noise attenuation of the filter depend on the noise distribution. The analytical formulas of output distributions make it possible to derive asymptotic expressions for the output expectations and variances for uniform (short-tailed) and Laplace (heavy-tailed) input distributions.

2 Basic Definitions

In this section we recall the definitions of morphological filters that process discrete signals by sets (see, e.g., Dougherty and Ciardiana [7], Serra [8], Maragos and Schafer [3], [9], Matheron [10], and Chu and Del. [11] (one-dimensional application)) and the definitions of stack filters (see, e.g., Wendt et al. [12] and Yli-Harja et al. [13]).

In this paper the set of natural numbers is denoted by **N**, the set of integers by **Z**, and the set of real numbers by **R**.

The *structuring set* B is a finite subset of $\mathbf{Z}^{(m \in \mathbf{N})}$. If B is one dimensional, *i.e.*, if $B \subset \mathbf{Z}$, and if for all $k, n \in \mathbf{Z}(k > 0)$ the property $n, n+k \in B$ implies that $\{n+1, n+2, \ldots, n+k-1\} \subseteq B$, then the structuring set B is called *convex*. The *symmetric set* B^s of B is defined by $B^s = \{-x : x \in B\}$, the *translated set* B_x, where the set B is translated by $x \in \mathbf{Z}^m$, is defined by $B_x = \{x+y : y \in B\}$, and the *Minkowski sum* of structuring sets, $A, B \subset \mathbf{Z}^m$, is defined by $A + B = \{x_1 + x_2 : x_1 \in A, x_2 \in B\}$.

The operations dilation, erosion, closing, and opening by $B, B \subset \mathbf{Z}^m$, transform a signal $f, f : \mathbf{Z}^m \to \mathbf{R}$, to another signal by the following rules: The *dilation* of f by B is denoted by $f \oplus B^s$ and is defined by

$$(f \oplus B^s)(x) = \max_{y \in B_x}\{f(y)\}, \quad x \in \mathbf{Z}^m.$$

The *erosion* of f by B is denoted by $f \ominus B^s$ and is defined by

$$(f \ominus B^s)(x) = \min_{y \in B_x^s}\{f(y)\}, \quad x \in \mathbf{Z}^m.$$

The *closing* of f by B is denoted by f^B and is defined by

$$f^B(x) = [(f \oplus B^s) \ominus B](x), \quad x \in \mathbf{Z}^m.$$

The *opening* of f by B is denoted by f_B and is defined by

$$f_B(x) = [(f \ominus B^s) \oplus B](x), \quad x \in \mathbf{Z}^m.$$

In the same way that closing and opening were defined as dilation followed by erosion and erosion followed by dilation, clos-opening by the structuring set B is defined as closing by B followed by opening by B, and open-closing by B is defined as opening by B followed by closing by B. The *clos-opening* of f by B is denoted by $(f^B)_B$ and is defined by

$$(f^B)_B(x) = [(((f \oplus B^s) \ominus B) \ominus B^s) \oplus B](x), \quad x \in \mathbf{Z}^m.$$

The *open-closing* of f by B is denoted by $(f_B)^B$ and is defined by

$$(f_B)^B(x) = [(((f \ominus B^s) \oplus B) \oplus B^s) \ominus B](x), \quad x \in \mathbf{Z}^m.$$

In Boolean expressions we use $x \wedge y$ for "x AND y," $x \vee y$ for "x OR y," and $\overline{x}$ for "NOT x," where x and y are Boolean variables. In some formulas binary values are to be understood as being real 1's and 0's. The number of elements in a finite set A is denoted by $|A|$.

The relation "$\geq$" of binary vectors $\underline{x} = (x_1, x_2, \cdots, x_n)$ and $\underline{y} = (y_1, y_2, \cdots, y_n)$ is defined as $\underline{x} \geq \underline{y}$ if and only if $x_i \geq y_i$ for all $i \in \{1, 2, \cdots, n\}$. Because this relation is reflexive, antisymmetric, and transitive, it defines a partial ordering on the set of binary vectors. This order property is known as the *stacking property*, and it is said that $\underline{x}$ and $\underline{y}$ stack if $\underline{x} \geq \underline{y}$. There is also a natural ordering on Boolean functions. Let f and g be two Boolean functions. We write $f \geq g$ if and only if $f(\underline{x}) \geq g(\underline{x})$ for all $\underline{x}$. A Boolean function f is said to be *increasing* if the relation $\underline{x} \geq \underline{y}$ implies the relation $f(\underline{x}) \geq f(\underline{y})$. Filters that are defined by increasing Boolean functions are called *stack filters*. The Boolean function g_D is a *dual* of g if and only if $g_D(\underline{x}) = \overline{g(\overline{\underline{x}})}$ for all $\underline{x}$.

Let A be a finite subset of $\mathbf{Z}^m$. Then the *Boolean function* g *indexed by the set* A is a Boolean expression of variables $z_a, a \in A$, denoted by $g(\underline{z}) = g(z_a : a \in A)$.

Let $u(t)$ denote the real *unit-step function*

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then we use a Boolean function with its variables indexed by a set to define the corresponding continuous stack filter in the following way. Let A be a finite subset of $\mathbf{Z}^m$, let $g(\underline{z})$ be an increasing Boolean function indexed by A, and let $f : \mathbf{Z}^m \to \mathbf{R}$ be a signal. Then the *continuous stack filter* S corresponding to $g(\underline{z})$ is defined by

$$S(f)(x) = \max\{t \in \mathbf{R}| \\ g(u(f(x+a)-t) : a \in A) = 1\}.$$

3 General Statistical Properties

The objective of this section is to derive output distributions for morphological filters and to study certain symmetry properties of dual filters. Knowledge of the properties discussed in this section is important when we apply morphological filters to noisy signals since it will give us an idea of the noise-suppression capability and the biasing effects of the filters.

An attractive property of stack filters is that it is possible to derive analytical results for their statistical properties. For example, the output distribution of a continuous stack filter can be expressed by using the following proposition (Yli-Harja et al [13]).

PROPOSITION 1. Let the input values $X_b(b \in B)$ in the window B of a stack filter S be independent random variables having distribution functions $F_b(t)$, respectively. Then the output distribution function $G(t)$ of the stack filter S is

$$G(t) = \sum_{\underline{z} \in g^{-1}(0)} \prod_{b \in B} (1 - F_b(t))^{z_b} F_b(t)^{1-z_b}, \quad (1)$$

where $g(\underline{z})$ is an increasing Boolean function that corresponds to the stack filter S and $g^{-1}(0) = \{\underline{z} | g(\underline{z}) = 0\}$.

Example. The Boolean function $g(\underline{z}) = x_{-1}x_0 + x_{-1}x_1 + x_0x_1$ corresponds to a median filter whose window is $B = \{-1, 0, 1\}$. If the input at -1 is a random variable having a distribution function $F_1(t)$ and inputs at 0 and 1 are random variables having a common distribution function $F_2(t)$, then by Proposition 1 the output distribution $G_m(t)$ of the median filter is $G_m(t) = F_1(t)F_2(t)^2 + (1 - F_1(t))F_2(t)^2 + 2F_1(t)(1 - F_2(t))F_2(t)$.

In the following we derive the output distributions of morphological filters when the values of the input signal are independent random variables. The following simple proposition gives explicit stack-filter expressions for morphological filters (Koskinen *et al.* [4], [14]). These expressions make it possible to calculate the output distributions of morphological filters by using Proposition 1.

PROPOSITION 2. Let B be a structuring set. Then the positive Boolean function that corresponds to stack-filter expression of

(a) dilation by B is $g_d(\underline{z}) = \vee_{b \in B^{z_b}}$,
(b) erosion by B is $g_e(\underline{z}) = \wedge_{b \in B^{z_b}}$,
(c) closing by B is $g_c(\underline{z}) = \wedge_{a \in B^s}(\vee_{b \in B_a z_b})$,
(d) opening by B is $g_o(\underline{z}) = \vee_{a \in B^s}(\wedge_{b \in B_a z_b})$,
(e) clos-opening by B is $g_{co}(\underline{z}) = \vee_{a \in B^s}(\wedge_{b \in (B+B^s)_a}(\vee_{c \in B_b z_c}))$,
(f) open-closing by B is $g_{oc}(\underline{z}) = \wedge_{a \in B^s}(\vee_{b \in (B+B^s)_a}(\wedge_{c \in B_b z_c}))$.

Proof. We will prove only case (e); the other cases can be proved in a similar way. Let f denote the signal to be clos-opened. We obtain the equations

$$\begin{aligned}(f^B)_B(x) &= [(((f \oplus B^s) \ominus B) \ominus B^s) \oplus B](x) \\ &= \max_{a \in B^s}(\min_{b \in B_a}(\min_{c \in B_b^s}(\max_{d \in B_c} f(x+d)))) \\ &= \max_{a \in B^s}(\min_{b \in (B+B^s)_a}(\max_{c \in B_b} f(x+c))) \\ &= \max\{t \in \mathbf{R}| \vee_{a \in B^s}(\wedge_{b \in (B+B^s)_a} \\ &\quad (\vee_{c \in B_b} u(f(x+c)-t))) = 1\}. \quad (2)\end{aligned}$$

So, by the definition of the continuous stack filter, the Boolean function that corresponds to clos-opening by B is $g_{co}(\underline{z}) = \vee_{a \in B^s}(\wedge_{b \in (B+B^s)_a}(\vee_{c \in B_b} z_c))$.

Because dilations and erosions are, in fact, local extremes, their statistical properties are extensively studied in the theory of order statistics; see e.g., Castillo [15] and Galambos [16]. Proposition 3 is a simple result of order statistics and is here formulated in terms of morphology

(Koskinen et al. [4, 14]). It shows that the statistical properties of dilation and erosion can be analyzed easily and do not depend on the shape of the structuring set.

PROPOSITION 3. Consider a discrete signal f and dilation and erosion by structuring set B of size n at point x_0. Let the values $f(x_0 + b)(b \in B)$ be independent random variables having the distribution functions $F_1(t), F_2(t), \cdots, F_n(t)$. Then the distribution function G_d of the value $(f \oplus B^s)(x_0)$ is

$$G_d(t) = \prod_{i=1}^{n} F_i(t), \quad (3)$$

and the distribution function G_e of the value $(f \ominus B^s)(x_0)$ is

$$G_e(t) = 1 - \prod_{i=1}^{n}(1 - F_i(t)). \quad (4)$$

Proof. Let the structuring set of the dilation and the erosion be B, where the size of B is n. Then by Proposition 2 the positive Boolean function that corresponds to the stack-filter expression is $g(\underline{z}) = \vee_{b \in B} z_b$ for dilation and is $p(\underline{z}) = \wedge_{b \in B} z_b$ for erosion. Since $g^{-1}(0) = \{\underline{0}\}$, Proposition 1 implies that

$$G_d(t) = \sum_{\underline{z}=\underline{0}} \prod_{i=1}^{n}(1 - F_i(t))^{z_i} F_i(t)^{1-z_i} = \prod_{i=1}^{n} F_i(t). \quad (5)$$

Similarly, since $p^{-1}(0) = \{0,1\}^n - \{\underline{1}\}$ (all components of 1's are 1's), by using Proposition 1 we obtain the result

$$G_e(t) = \sum_{\underline{z} \neq \underline{1}} \prod_{i=1}^{n}(1 - F_i(t))^{z_i} F_i(t)^{1-z_i} = 1 - \prod_{i=1}^{n}(1 - F_i(t)). \quad (6)$$

Propositions 4 and 5 (Koskinen et al. [4, 14]) are direct consequences of Propositions 1 and 2.

PROPOSITION 4. Consider a discrete signal f and the closing of f by a structuring set B at point x_0. Let the values $f(x_0 + b)(b \in B + B^s)$ be independent random variables having the distribution functions $F_b(t)$, respectively. Then the distribution function $G_c(t)$ of the value $f^B(x_0)$ of the closed signal is

$$G_c(t) = \sum_{\underline{z} \in g^{-1}(0)} \prod_{b \in B + B^s} (1 - F_b(t))^{z_b} F_b(t)^{1-z_b}, \quad (7)$$

where $g(\underline{z}) = \wedge_{a \in B^s}(\vee_{b \in B_a} z_b)$.

PROPOSITION 5. Consider a discrete signal f and the clos-opening of f by a structuring set B at point x_0. Let the values $f(x_0 + b)(b \in (B + B^s) + (B + B^s))$ be independent random variables having the distribution functions $F_b(t)$, respectively. Then the distribution function $G_{co}(t)$ of the value $(f^B)_B(x_0)$ of the clos-opened signal is

$$G_{co}(t) = \sum_{\underline{z} \in g^{-1}(0)} \prod_{b \in (B+B^s)+(B+B^s)} (1 - F_b(t))^{z_b} F_b(t)^{1-z_b}, \quad (8)$$

where $g(\underline{z}) = \vee_{a \in B^s}(\wedge_{b \in (B+B^s)_a}(\vee_{c \in B_b} z_c))$.

Later we will have to know exactly what is meant by the term "same shape." The mathematical definition is given by the following equivalence relation.

DEFINITION Let A and B be structuring sets. We say that A and B are *congruent* if A can be transformed to B by applying translations, reflections, and rotations by $\pi/2$. If A and B are congruent, then we write $A \sim B$.

If $A \sim B$, then Proposition 4 shows that closing by A and closing by B have the same statistical behavior in the case of independently and identically distributed inputs since the same Boolean function corresponds to the stack-filter expressions of closing by A and closing by B. Obviously, this conclusion is also valid for clos-opening.

Propositions 4 and 5 offer a straightforward method for calculating output distributions for closing and clos-opening. On the other hand, if we know the distribution function of closing by B or clos-opening by B, Proposition 6 (Koskinen et al. [5]) gives us an easy way to find the output distributions of opening by B or open-closing by B. The reason for this is that the Boolean functions corresponding to the stack-filter expressions of opening and open-closing are the duals of those that correspond to closing and clos-opening.

PROPOSITION 6. Let g be a positive Boolean function, let g_D be the dual of g, let S be the stack filter defined by g, and let S_D be the stack filter defined by g_D. Consider the filtering of a discrete signal f by the stack filter S at point x_0 where the values in the moving window $B(|B| = n)$ of the stack filter S are independent random variables having the distribution functions $F_1(t), F_2(t), \cdots, F_n(t)$. If the distribution function of the value $S(f)(x_0)$ is $G(F_1(t), F_2(t), \cdots, F_n(t))$, then the distribution function of the value $S_D(f)(x_0)$ is

$$G_D(F_1(t), F_2(t), \cdots, F_n(t)) = 1 - G((1 - F_1(t)), (1 - F_2(t)), \cdots, (1 - F_n(t))). \tag{9}$$

Proof. Because the filtering by S corresponds to the positive Boolean function $g(\underline{z})$, Proposition 1 implies that the distribution function of the value $S(f)(x_0)$ is

$$G(F_1(t), F_2(t), \cdots, F_n(t)) = \sum_{\underline{z} \in G^{-1}(0)} \prod_{i=1}^{n} (1 - F_i(t))^{z_i} F(t)^{1-z_i}, \tag{10}$$

where n is the size of the moving window A that corresponds to the stack filter S. Since g_D is the dual of g, then $g_D(\underline{z}) = 0$ if and only if $g(\overline{\underline{z}}) = 1$. Thus Proposition 1 implies that

$$\begin{aligned} G_D(F_1(t), F_2(t), \cdots, F_n(t)) &= \sum_{\underline{z} \in g_D^{-1}(0)} \prod_{i=1}^{n} (1 - F_i(t))^{z_i} F_i(t)^{1-z_i} \\ &= 1 - \sum_{\underline{z} \in g^{-1}(0)} \prod_{i=1}^{n} (1 - F_i(t))^{1-z_i} F_i(t)^{z_i} \\ &= 1 - G((1 - F_1(t)), (1 - F_2(t)), \cdots, (1 - F_n(t))). \end{aligned} \tag{11}$$

Example. Consider an image f where the gray-level values $f(i,j)$ of the pixels are independent random variables having a common distribution function $F(t)$. Then by Proposition 4 the distribution function $G_c(t)$ of the pixel values after closing by $B = \{(0,0),(0,1),(1,0)\}$ is $G_c(t) = F(t)^7 + 6(1 - F(t))F(t)^6 + 15(1 - F(t))^2F(t)^5 + 12(1 - F(t))^3F(t)^4 + 3(1 - F(t))^4F(t)^3$. In addition, by Proposition 6, after opening by B the distribution function is $G_o(t) = 1 - (1 - F(t))^7 - 6(1 - F(t))^6F(t) - 15(1 - F(t))^5F(t)^2 - 12(1 - F(t))^4F(t)^3 - 3(1 - F(t))^3F(t)^4$.

Proposition 2 shows that the basic morphological filters that are duals (in the morphological sense) of each other are duals also in a stack-filter sense. This implies the following statistical symmetry properties (Koskinen et al. [5]).

PROPOSITION 7. Consider the filtering of a discrete signal f whose values are independent, identically and symmetrically distributed random variables having a common distribution function $F(t)$ and the expectation μ. Let g be a positive Boolean function, let g_D be the dual of g, let S be the stack filter defined by g, and let S_D be the stack filter defined by g_D. If the expectation of the values of f after filtering by S is $E\{S(f)\} = \mu + \xi$, then the expectation of the values of f after filtering by S_D is $E\{S_D(f)\} = \mu - \xi$. Moreover, the output variances of S and S_D are equal.

Proof. Proposition 1 implies that the distribution function of the values of the signal after filtering by S is ($w(z)$ is the Hamming weight of z)

$$G(F(t)) = \sum_{\underline{z} \in g^{-1}(0)} (1 - F(t))^{w(\underline{z})} F(t)^{n-w(\underline{z})} \tag{12}$$

and that the distribution function of the values of the signal after filtering by S_D is

$$G_D(F(t)) = \sum_{\underline{z} \in g_D^{-1}(0)} (1-F(t))^{w(\underline{z})} F(t)^{n-w(\underline{z})}. \tag{13}$$

Now, Proposition 6 implies that

$$\sum_{\underline{z}\in g^{-1}(0)} (1 - F(t))^{w(\underline{z})} F(t)^{n-w(\underline{z})}$$
$$= 1 - \sum_{\underline{z}\in g_D^{-1}(0)} (1 - F(t))^{n-w(\underline{z})} F(t)^{w(\underline{z})}. \quad (14)$$

Since $F(t)$ is symmetric, $F(\mu+\xi) = 1-F(\mu-\xi)$, implying

$$\begin{aligned} G(F(\mu + \xi)) &= \sum_{\underline{z}\in g^{-1}(0)} (1 - F(\mu + \xi))^{w(\underline{z})} F(\mu + \xi)^{n-w(\underline{z})} \\ &= 1 - \sum_{\underline{z}\in g_D^{-1}(0)} (1 - F(\mu + \xi))^{n-w(\underline{z})} F(\mu + \xi)^{w(\underline{z})} \\ &= 1 - \sum_{\underline{z}\in g_D^{-1}(0)} (1 - F(\mu - \xi))^{w(\underline{z})} F(\mu - \xi)^{n-w(\underline{z})} \\ &= 1 - G_D(F(\mu - \xi)). \end{aligned} \quad (15)$$

Thus the output distributions of S and S_D are mirror images with respect to μ. This proves Proposition 7.

COROLLARY 7.1. Consider the filtering of a discrete signal f whose values are independent, identically and symmetrically distributed random variables having the expectation μ. Let the expectation of the output after dilation by B be $E\{f \oplus B\} = \mu + \xi_1$, let the expectation of the output after closing by B be $E\{f^B\} = \mu + \xi_2$, and let the expectation of the output after clos-opening by B be $E\{(F^B)_B\} = \mu + \xi_3$. Then

(a) after erosion by B the expectation is $E\{f \ominus B\} = \mu - \xi_1$,
(b) after opening by B the expectation is $E\{f_B\} = \mu - \xi_2$,
(c) after open-closing by B the expectation is $E\{(f_B)^B\} = \mu - \xi_3$.

COROLLARY 7.2. Consider the filtering of a discrete signal f whose values are independent, identically and symmetrically distributed random variables. Then

(a) the variances of the dilated signal $f \oplus B$ and the eroded signal $f \ominus B$ are equal;
(b) the variances of the closed signal f^B and the opened signal f_B are equal;
(c) the variances of the clos-opened signal $(f^B)_B$ and the open-closed signal $(f_B)^B$ are equal.

4 Asymptotic Behavior of Dilation and Erosion

In this section we analyze the asymptotic behavior of dilation and erosion in the cases of uniform and Laplace distributions. The results that we derive here are important because they are useful in the analysis of compound morphological operations. First, we recall one concept. Let $f : \mathbf{N} \to \mathbf{R}$ and $g : \mathbf{N} \to \mathbf{R}$ be two functions. Then f is *asymptotically dominated* by g if and only if there exist real numbers A and n_0 such that $|f(n)| \le A|g(n)|$ for all $n > n_0$. If f is asymptotically dominated by g, we write $f(n) = O(g(n))$. The following proposition is a direct consequence of Proposition 3.

PROPOSITION 8. Consider the filtering of a discrete signal f whose values are independent and identically uniformly distributed on [0, 1]. Let B be a structuring set of size n. Then after dilation by B, for the expectation

$$E\{f \oplus B\} = 1 - \frac{1}{n+1} \quad (16)$$

and for the variance

$$V\{f \oplus B\} = \frac{1}{n^2} + O\left(\frac{1}{n^3}\right), \quad (17)$$

and after erosion by B, for the expectation

$$E\{f \ominus B\} = \frac{1}{n+1} \quad (18)$$

and for the variance

$$V\{f \ominus B\} = \frac{1}{n^2} + O\left(\frac{1}{n^3}\right). \quad (19)$$

Henceforth, we denote by $L(\alpha)$ the *Laplace* *distribution* whose distribution function $F(t)$ is

$$F(t) = \int_{-\infty}^{t} \frac{\alpha}{2} e^{-\alpha|x|} dx. \qquad (20)$$

When we study the asymptotic behavior of morphological filters in the case of the Laplace distribution, the following two lemmas are very useful.

LEMMA 1. Let $F(t)$ be the distribution function of the Laplace distribution $L(\alpha)$. Then it holds

$$\int_{-\infty}^{\infty} x \left(\frac{d}{dx} F(x)^n \right) dx = -\frac{1}{\alpha n 2^n} + \frac{1}{\alpha} \sum_{k=1}^{n} \frac{1-2^{-k}}{k}. \qquad (21)$$

Proof. First we show that

$$\int_{0}^{\infty} x \left(\frac{d}{dx} F(x)^n \right) dx = \frac{1}{\alpha} \sum_{k=1}^{n} \frac{1-2^{-k}}{k}. \qquad (22)$$

Substituting $e^{-\alpha x} = 2t$, we obtain

$$\begin{aligned}\int_{0}^{\infty} x \left(\frac{d}{dx} F(x)^n \right) dx &= \frac{n}{2} \int_{0}^{\infty} x \left(1 - \frac{e^{-\alpha x}}{2} \right)^{n-1} \alpha e^{-\alpha x} dx \\ &= \frac{n}{2} \int_{1/2}^{0} \left(-\frac{\ln 2t}{\alpha} \right) (1-t)^{n-1} (-2) dt \\ &= \frac{n}{\alpha} \int_{0}^{1/2} (-\ln 2t)(1-t)^{n-1} dt. \qquad (23)\end{aligned}$$

We write

$$A(n) = \int_{0}^{1/2} (-\ln 2t)(1-t)^{n-1} dt, \qquad (24)$$

and so

$$\begin{aligned}A(n) &= \int_{0}^{1/2} (-\ln 2t)(1-t)^{n-2} - t(-\ln 2t)(1-t)^{n-2} dt \\ &= A(n-1) + \int_{0}^{1/2} t \ln 2t (1-t)^{n-2} dt \\ &= A(n-1) + \int_{0}^{1/2} \ln 2t \frac{(1-t)^{n-1}}{n-1} dt + \int_{0}^{1/2} \frac{(1-t)^{n-1}}{n-1} dt \\ &= A(n-1) - \frac{1}{n-1} A(n) - \left(\frac{(1-t)^n}{n(n-1)} \right) \Bigg|_{0}^{1/2} \\ &= A(n-1) - \frac{1}{n-1} A(n) + \frac{1-2^{-n}}{n(n-1)}. \qquad (25)\end{aligned}$$

Writing (25) then in form

$$nA(n) = (n-1)A(n-1) + \frac{1-2^{-n}}{n} \qquad (26)$$

and solving $nA(n)$ gives

$$A(n) = \frac{1}{n} \sum_{k=1}^{n} \frac{1-2^{-k}}{k}, \qquad (27)$$

implying equation (22). Since

$$\int_{-\infty}^{0} x \left(\frac{d}{dx} F(x)^n \right) dx = -\frac{1}{\alpha n 2^n}, \qquad (28)$$

Lemma 1 follows.

LEMMA 2. Let $F(t)$ be the distribution function of the Laplace distribution $L(\alpha)$. Then it holds that

$$\int_{-\infty}^{\infty} x^2 \left(\frac{d}{dx} F(x)^n \right) dx = \frac{1}{\alpha^2 n^2 2^{n-1}} + \frac{1}{\alpha^2} \left(\sum_{k=1}^{n} \frac{2}{k} \sum_{l=1}^{k} \frac{1-2^{-l}}{l} \right). \qquad (29)$$

Proof. First we show that

$$\int_{0}^{\infty} x^2 \left(\frac{d}{dx} F(x)^n \right) dx = \frac{1}{\alpha^2} \left(\sum_{k=1}^{n} \frac{2}{k} \sum_{l=1}^{k} \frac{1-2^{-l}}{l} \right). \qquad (30)$$

Substituting $e^{-\alpha x} = 2t$, we obtain

$$\begin{aligned}
&\int_0^{\infty} x^2 \left(\frac{d}{dx}F(x)^n\right) dx \\
&= \frac{n}{2}\int_0^{\infty} x^2 \left(1 - \frac{e^{-\alpha x}}{2}\right)^{n-1} \alpha e^{-\alpha x} dx \\
&= \frac{n}{2}\int_{1/2}^{0} \left(-\frac{\ln 2t}{\alpha}\right)^2 (1-t)^{n-1}(-2)dt \\
&= \frac{n}{\alpha^2}\int_0^{1/2} (\ln 2t)^2(1-t)^{n-1}dt. \qquad (31)
\end{aligned}$$

We write

$$B(n) = \int_0^{1/2} (\ln 2t)^2(1-t)^{n-1}dt, \qquad (32)$$

and so

$$\begin{aligned}
B(n) &= \int_0^{1/2} (\ln 2t)^2(1-t)^{n-2} \\
&\qquad -t(\ln 2t)^2(1-t)^{n-2}dt \\
&= B(n-1) \\
&\qquad - \int_0^{1/2} t(\ln 2t)^2(1-t)^{n-2}dt \\
&= B(n-1) \\
&\qquad +2\int_0^{1/2} (-\ln 2t)\frac{(1-t)^{n-1}}{n-1}dt \\
&\qquad - \int_0^{1/2} (\ln 2t)^2\frac{(1-t)^{n-1}}{n-1}dt \\
&= B(n-1) + \frac{1}{n-1}B(n) \\
&\qquad +2\int_0^{1/2} (-\ln 2t)\frac{(1-t)^{n-1}}{n-1}dt. \qquad (33)
\end{aligned}$$

Since by equation (27)

$$\begin{aligned}
2&\int_0^{1/2} (-\ln 2t)\frac{(1-t)^{n-1}}{n-1}dt \\
&= \frac{2}{n^2-n}\sum_{k=1}^{n}\frac{1-2^{-k}}{k}, \qquad (34)
\end{aligned}$$

we obtain

$$\begin{aligned}
B(n) &= B(n-1) - \frac{1}{n-1}B(n) \\
&\qquad + \frac{2}{n^2-n}\sum_{k=1}^{n}\frac{1-2^{-k}}{k}, \qquad (34)
\end{aligned}$$

and solving

$$B(n) = \frac{1}{n}\left(\sum_{k=1}^{n}\frac{2}{k}\sum_{l=1}^{k}\frac{1-2^{-l}}{l}\right) \qquad (36)$$

as before implies equation (29). Since

$$\int_{-\infty}^{0} x^2 \left(\frac{d}{dx}F(x)^n\right) dx = \frac{1}{\alpha^2 n^2 2^{n-1}}, \qquad (37)$$

Lemma 2 follows.

LEMMA 3. Let $F(t)$ be the distribution function of the Laplace distribution $L(\alpha)$, and let the distribution function of a random variable Y be $F(t)^n$ Then for the expectation of Y

$$E\{Y\} = \frac{1}{\alpha}\ln n + O(1), \qquad (38)$$

and for the variance of Y

$$V\{Y\} = \eta \ln n + O(1), \qquad (39)$$

where

$$\eta = \frac{1}{\alpha^2}\sum_{k=1}^{\infty}\frac{2^{-k}}{k}. \qquad (40)$$

Proof. Since the output distribution function of Y is $F(t)^n$, by using Lemma 1 we obtain the asymptotic formula for the expectation

$$\begin{aligned}
E\{Y\} &= -\frac{1}{\alpha n 2^n} + \frac{1}{\alpha}\sum_{k=1}^{n}\frac{1-2^{-k}}{k} \\
&= \frac{1}{\alpha}\ln n + O(1). \qquad (41)
\end{aligned}$$

By Lemma 2 we also obtain the second moment

$$\begin{aligned}
E\{Y^2\} &= \frac{1}{\alpha^2 n^2 2^{n-1}} \\
&\qquad + \frac{1}{\alpha^2}\left(\sum_{k=1}^{n}\frac{2}{k}\sum_{l=1}^{k}\frac{1-2^{-l}}{l}\right). \qquad (42)
\end{aligned}$$

Now, using equations (41) and (42), we obtain the asymptotic formula for the variance

$$\begin{aligned}
V\{Y\} &= E\{Y^2\} - (E\{Y\})^2 \\
&= \frac{1}{\alpha^2 n^2 2^{n-1}} + \frac{1}{\alpha^2}\sum_{k=1}^{n}\frac{2}{k}\sum_{l=1}^{k}\frac{1-2^{-l}}{l} \\
&\quad - \left(-\frac{1}{\alpha n 2^n} + \frac{1}{\alpha}\sum_{k=1}^{n}\frac{1-2^{-k}}{k}\right)^2 \\
&= \frac{1}{\alpha^2}\sum_{k=1}^{n}\frac{2}{k}\sum_{l=1}^{k}\frac{1-2^{-l}}{l} \\
&\quad - \left(\frac{1}{\alpha}\sum_{k=1}^{n}\frac{1-2^{-k}}{k}\right)^2 + O(1) \\
&= \frac{1}{\alpha^2}\sum_{k=1}^{n}\frac{1}{k}\sum_{l=1}^{k}\frac{1-2^{-l}}{l} \\
&\quad - \frac{1}{\alpha^2}\sum_{k=1}^{n}\frac{1}{k}\sum_{l=k+1}^{n}\frac{1-2^{-l}}{l} \\
&\quad - \frac{1}{\alpha^2}\sum_{k=1}^{n}\frac{2^{-k}}{k}\sum_{l=1}^{n}\frac{1-2^{-l}}{l} + O(1) \\
&= \frac{1}{\alpha^2}\sum_{k=1}^{n}\frac{1}{k}\sum_{l=1}^{k}\frac{1-2^{-l}}{l} \\
&\quad - \frac{1}{\alpha^2}\sum_{k=1}^{n}\frac{1}{k}\sum_{l=1}^{k-1}\frac{1-2^{-l}}{l} \\
&\quad + \frac{1}{\alpha^2}\sum_{k=1}^{n}\frac{2^{-k}}{k}\sum_{l=1}^{n}\frac{1-2^{-l}}{l} + O(1) \\
&= \frac{1}{\alpha^2}\sum_{k=1}^{n}\frac{2^{-k}}{k}\sum_{l=1}^{n}\frac{1}{l} + O(1).
\end{aligned} \tag{43}$$

Thus

$$V\{Y\} = \frac{1}{\alpha^2}\sum_{k=1}^{n}\frac{2^{-k}}{k}\ln n + O(1). \tag{44}$$

PROPOSITION 9. Consider the filtering of a discrete signal f whose values are independent and identically distributed random variables having the Laplace distribution $L(\alpha)$. Let B be a structuring set of size n, and let

$$\eta = \frac{1}{\alpha^2}\sum_{k=1}^{\infty}\frac{2^{-k}}{k}. \tag{45}$$

Then after dilation by B, for the expectation

$$E\{f \oplus B\} = \frac{1}{\alpha}\ln n + O(1) \tag{46}$$

and for the variance

$$V\{f \oplus B\} = \eta \ln n + O(1), \tag{47}$$

and after erosion by B, for the expectation

$$E\{f \ominus B\} = -\frac{1}{\alpha}\ln n + O(1) \tag{48}$$

and for the variance

$$V\{f \ominus B\} = \eta \ln n + O(1). \tag{49}$$

Proof. Let $F(t)$ be the distribution function of the Laplace distribution $L(\alpha)$. Since by Proposition 3 the distribution function of a random variable $f \oplus B$ is $F(t)^n$, Lemma 3 implies that

$$E\{f \oplus B\} = \frac{1}{\alpha}\ln n + O(1) \tag{50}$$

and

$$V\{f \oplus B\} = \eta \ln n + O(1). \tag{51}$$

Now, Corollary 7.1 implies equation (48), and Corollary 7.2 implies equation (49).

5 Bounds on the Expectations of Two-Dimensional Closing and Opening

The nonlinear nature of closings and openings makes it difficult to carry out an accurate analysis of the effects of noise in the case of two-dimensional closing and opening since their statistical properties depend on both the shape and the size of the structuring set. However, we will show that by using the stack-filter method one can also derive bounds on the output expectations for two-dimensional morphological filters when the values of the signal to be filtered are independent and identically distributed.

In the following we derive a lower bound, which is asymptotically tight for the Laplace distribution, on the expectation of closing by an $n \times n$ square structuring set. This bound can then be used for more general structuring sets containing a square. Moreover, the method of this section can also be used for other types of structuring sets whose shapes are "symmetric enough." The following lemma is a central tool in what follows.

LEMMA 4. Consider a random variable Y whose distribution function is $u(F(t))$, where $F(t)$ is a strictly monotonous and piecewise differentiable distribution function and $u : [0,1] \to [0,1]$ is a differentiable function. Let the integrals $\int_{-\infty}^{\infty} tF'(t)dt$ and $\int_{-\infty}^{\infty} tu'(F(t))F'(t)dt$ be finite, and let $v(t) : [0,1] \to [0,\infty)$ be a differentiable function that satisfies $v(0) = 0, v(1) = 1$ and $v(t) \geq u(t)$ for all $0 < t < 1$. Then for the expectation of Y

$$E\{Y\} \geq \int_{-\infty}^{\infty} tv'(F(t))F'(t)dt. \tag{52}$$

Proof. Because $\int_{-\infty}^{\infty} tu'(F(t))F'(t)dt$ is finite, the expectation of Y exists and satisfies

$$\begin{aligned} E\{Y\} &= \int_{-\infty}^{\infty} tu'(F(t))F'(t)dt \\ &= \int_{-\infty}^{0} tu'(F(t))F'(t)dt \\ &\quad + \int_{0}^{\infty} tu'(F(t))F'(t)dt \\ &= \left(tu(F(t))\right)\Big|_{-\infty}^{0} \\ &\quad - \int_{-\infty}^{0} u(F(t))dt \\ &\quad - \left(t(1-u(F(t)))\right)\Big|_{0}^{\infty} \\ &\quad + \int_{0}^{\infty} (1-u(F(t)))dt \\ &= -\int_{-\infty}^{0} u(F(t))dt \\ &\quad + \int_{0}^{\infty} (1-u(F(t)))dt. \end{aligned} \tag{53}$$

Because $\int_{-\infty}^{\infty} tF'(t)dt$ is finite, we have

$$\left(tF(t)\right)\Big|_{-\infty}^{0} = \left(t(1-F(t))\right)\Big|_{0}^{\infty} = 0 \tag{54}$$

Thus the assumption that $v(t) : [0,1] \to [0,\infty)$ is a differentiable function and satisfies $v(0) = 0, v(1) = 1$ implies that

$$\left(tv(F(t))\right)\Big|_{-\infty}^{0} = \left(t(1-v(F(t)))\right)\Big|_{0}^{\infty} = 0. \tag{55}$$

Now, a manipulation similar to the one above leads to the equation

$$\begin{aligned} &\int_{-\infty}^{\infty} tv'(F(t))F'(t)dt \\ &= -\int_{-\infty}^{0} v(F(t))dt \\ &\quad + \int_{0}^{\infty} (1-v(F(t)))dt, \end{aligned} \tag{56}$$

from which Lemma 4 follows since $v(t) \geq u(t)$ for all $t \in [0,1]$.

To bound the distribution function of closing we need the following combinatorial lemma (Koskinen and Astola [17]).

LEMMA 5. Consider an $m \times m$ square $\mathrm{Sq}(m) = \{(i,j) \in \mathbf{Z}^2 | -n < i < n, -n < j < n\} (m = 2n-1)$, and let $\mathrm{Sq}_n(i,j)$ denote the $n \times n$ subsquare having its northeast corner at $(i,j), 1 \leq i \leq n, 1 \leq j \leq n$. For $l = 0, 1, \ldots, m^2$ let $\nu(l)$ denote the number of subsets $S \subseteq \mathrm{Sq}(m)$, with $|S| = l$, such that $S \cap \mathrm{Sq}_n(i,j) = \emptyset$ for some $i = 1, \ldots, n$ and $j = 1, \ldots, n$, i.e.,

$$\begin{aligned} \nu(l) = |\{S \subseteq \mathrm{Sq}(m) \mid |S| &= l, S \cap \mathrm{Sq}_n(i,j) = \emptyset \\ &\text{for some } i,j = 1,\cdots,n\}|. \end{aligned} \tag{57}$$

Then

$$\begin{aligned} \nu(l) \leq \binom{m-n^2}{l} &+ (2n-2)\left(\binom{m-n^2}{l} - \binom{m-n^2-n}{l}\right) \\ &+ (n-1)^2\left(\binom{m-n^2}{l} - 2\binom{m-n^2-n}{l}\right. \end{aligned}$$

$$+\binom{m-n^2-2n}{l}\Bigg). \quad (58)$$

Proof. Let $1 \le l \le m - n^2$. If we fix (i,j), the number of S, $(S \subseteq \mathrm{Sq}(m)), |S| = l$ and $S \cap \mathrm{Sq}_n(i,j) = \emptyset$, is obviously

$$\binom{m-n^2}{l}.$$

If we just let (i,j) run through points $1 \le i,j \le n$, we obtain

$$\nu(l) \le n^2\binom{m-n^2}{l}, \quad (59)$$

which is too crude. To obtain a sharper bound, we let (i,j) run through the points in the following order:

$$\begin{array}{l}(1,1),(1,2),\cdots,(1,k),\\ (2,1),(3,1),\cdots,(k,1),\\ (2,2),(2,3),\cdots,(2,k),\\ (3,2),(3,3),\cdots,(3,k),\\ \vdots\\ (k,2),(k,3),\cdots,(k,k)\end{array}$$

(top border from left to right, left border from top to bottom, and the rest row-wise from left to right).

Let us now count for each $\mathrm{Sq}_n(i,j)$ the sets $S, |S| = l$ and $S \cap \mathrm{Sq}_n(i,j) = \emptyset$, so that the sets that were counted for $\mathrm{Sq}_n(i-1,j)$ or $\mathrm{Sq}_n(i,j-1)$ are not counted (they have been counted before). This gives

$$\begin{aligned}\nu(l) \le &\binom{m-n^2}{l}\\ &+(2n-2)\left(\binom{m-n^2}{l} - \binom{m-n^2-n}{l}\right)\\ &+(n-1)^2\left(\binom{m-n^2}{l} - 2\binom{m-n^2-n}{l} + \binom{m-n^2-2n}{l}\right). \quad (60)\end{aligned}$$

The last term in (60) results from the fact that the contribution of both points $(i-1,j)$ and $(i,j+1)$ would otherwise be subtracted twice.

Astola and Neuvo [18] have shown that whenever the inputs are independent and identically distributed random variables that can attain arbitrarily large values, the expectation of closing (opening) tends to infinity as the size of the structuring set increases. The following proposition (Koskinen and Astola [7]) gives an asymptotically tight bound on the expectation in the case of the Laplace distribution.

PROPOSITION 10. Let $f : \mathbf{Z}^2 \to \mathbf{R}$ be a signal whose values are independent and identically distributed random variables having a common strictly monotonous and piecewise-differentiable distribution function $F(t)$. Consider the closing of f by a square structuring set

$$\mathrm{Sq}(n) = \{(i,j) \in \mathbf{Z}^2 \mid -k < i < k, -k < j < k\}, \quad (61)$$

where $n = 2k - 1$. Then after closing by $\mathrm{Sq}(n)$, for the expectation

$$E\{f^{\mathrm{Sq}(n)}\} \ge \int_{-\infty}^{\infty} t\left(\frac{\partial}{\partial t}G(F(t),n)\right)dt, \quad (62)$$

where

$$G(x,n) = n^2x^{n^2} - 2(n^2-n)x^{n^2+n} + (n^2-2n+1)x^{n^2+2n}. \quad (63)$$

Proof. Let $M = \mathrm{Sq}(n) + \mathrm{Sq}(n)$, and consider the characteristic function of each subset $S \subseteq M$ as a Boolean vector of dimension $|M| = m = n^2$. For $S \subseteq M$, denote by $f^{\mathrm{Sq}(n)}(S)$ the Boolean function determined by the output of the closing of S by $\mathrm{Sq}(n)$. For $l = 0, 1, \cdots, m$ we write

$$\nu(l) = |\{S \subseteq M \mid |S| = l, f^{\mathrm{Sq}(n)}(S) = 0\}|. \quad (64)$$

Now Proposition 1 implies that the distribution function $H(t)$ of the values of the closed signal f^B is

$$H(t) = u(F(t)), \quad (65)$$

where

$$u(x) = \sum_{l=0}^{m} \nu(l)(1-x)^l x^{m-1}. \quad (66)$$

Substituting the bound of Lemma 5 in (66) and performing the summations, we get

$$u(x) \leq x^{n^2} + (2n-2)(x^{n^2} - x^{n^2+n}) + (n-1)^2(x^{n^2} - 2x^{n^2+n} + x^{n^2+2n}), \quad (67)$$

and the result now follows from Lemma 4.

COROLLARY 10.1. Consider a gray-level image whose pixel values are white noise uniformly distributed in [0, 1], and perform closing by an $n \times n$ square structuring set. Then the expectation E of the gray-level of the closed image satisfies

$$E \geq 1 - 5/n^2 + O(1). \quad (68)$$

This shows that on the constant regions of a noisy image the output will follow the upper tail of the noise distribution, giving a smooth but biased output.

The following proposition (Koskinen and Astola [7]) shows that for the Laplace distribution the above bound (62) is asymptotically tight. First we need the following lemma.

LEMMA 6. Let $F(t)$ be the distribution function of the Laplace distribution $L(\alpha)$, and let $G(x, n)$ be defined by (63). Then

$$\int_{-\infty}^{\infty} t\left(\frac{\partial}{\partial t} G(F(t), n)\right) dt = \frac{2}{\alpha} \ln n + O(1). \quad (69)$$

Proof. Applying Lemma 3 to each term of the right-hand side of (62), we obtain

$$\int_{-\infty}^{\infty} t\left(\frac{\partial}{\partial t} G(F(t), n)\right) dt = \frac{n^2}{\alpha}\Big(\ln(n^2) - 2\ln(n^2+n) + \ln(n^2+2n)\Big) + \frac{2n}{\alpha}\Big(\ln(n^2+n) - \ln(n^2+2n)\Big) + \frac{1}{\alpha}\ln(n^2+2n) + O(1)$$
$$= \frac{2}{\alpha}\ln n + O(1). \quad (70)$$

PROPOSITION 11. Let $f : \mathbf{Z}^2 \rightarrow \mathbf{R}$ be a signal whose values are independent and identically distributed random variables having the Laplace distribution $L(\alpha)$. Consider the closing of f by an $n \times n$ square structuring set $\mathrm{Sq}(n)$. Then after closing by $\mathrm{Sq}(n)$, for the expectation

$$E\{f^{\mathrm{Sq}(n)}\} = \frac{2}{\alpha}\ln n + O(1). \quad (71)$$

Proof. From Lemma 6 we obtain an asymptotic lower bound for $E\{f^{\mathrm{Sq}(n)}\}$, and, surprisingly, the same upper bound results from comparing closing by $\mathrm{Sq}(n)$ to dilation by $\mathrm{Sq}(n)$. The relation

$$f^{\mathrm{Sq}(n)} \leq f \oplus \mathrm{Sq}(n) \quad (72)$$

implies that

$$E\{f^{\mathrm{Sq}(n)}\} \leq E\{f \oplus \mathrm{Sq}(n)\}, \quad (73)$$

and Proposition 9 gives

$$E\{f \oplus \mathrm{Sq}(n)\} = \frac{2}{\alpha}\ln n + O(1). \quad (74)$$

Proposition 11 now follows from Proposition 10 and Lemma 6.

Propositions 10 and 11 were formulated for square structuring sets, but they imply similar results for a large class of structuring sets that can be suitably approximated by a square structuring set.

DEFINITION Let A and B be structuring sets. Then A *is shrouded* by B if there exist C such that $C \sim B$ and for all $c \in C^s$ there exists an $a \in A^s$ such that $A_a \subseteq C_c$.

LEMMA 7. Consider the closing of a signal f whose values are independent and identically distributed random variables. Let A and B be such structuring sets that A is shrouded by B. Then

$$E\{f^A\} \leq E\{f^B\}. \quad (75)$$

Proof. Since B shrouds A, there exists C such that $C \sim B$, and for all $c \in C^s$ there exists an $a \in A^s$ such that $A_a \subseteq C_c$. Then we obtain the relation $f^A \leq f^C$, which implies that $E\{f^A\} \leq E\{f^C\}$. Since closing by B and closing by C have the same statistical behavior in the case of independently and identically distributed inputs, we obtain Lemma 7.

Now we can state the asymptotic results in Proposition 12 (Koskinen and Astola [17]).

PROPOSITION 12. Let $f : \mathbf{Z}^2 \to \mathbf{R}$ be a signal whose values are independent and identically distributed random variables having the Laplace distribution $L(\alpha)$, and let ρ be a real number. Consider a sequence $B_n, n = 1, 2, \ldots,$ of structuring sets such that $\mathrm{Sq}(n)$ is shrouded by B_n and $n^2 \leq |B_n| \leq \rho n^2$ for $n = 1, 2, \ldots$. Then for the expectations it holds that

$$E\{f^{B_n}\} = \frac{2}{\alpha} \ln n + O(1) \tag{76}$$

and

$$E\{f_{B_n}\} = -\frac{2}{\alpha} \ln n + O(1). \tag{77}$$

Proof. By Lemma 7 the assumption that $\mathrm{Sq}(n)$ is shrouded by B_n implies that

$$E\{f^{B_n}\} \geq E\{f^{\mathrm{Sq}(n)}\}, \tag{78}$$

and the relation

$$f^{B_n} \leq f \oplus B_n \tag{79}$$

implies that

$$E\{f^{B_n}\} \leq E\{f \oplus B_n\}. \tag{80}$$

Now by using Propositions 9 and 10 and the relation $|B| \leq \rho n^2$ we obtain equation (76), and by using Corollary 7.1 we obtain equation (77).

6 Asymptotic Behavior of One-Dimensional Morphological Filters

In general, the statistical properties of morphological filters depend on both the shape and the size of the structuring set. As a result, it is difficult to derive analytical expressions for these properties. However, in the case of a one-dimensional convex structuring set we can derive analytical expressions for the output distributions, and we can study the second-order statistical properties of closing, opening, clos-opening, and open-closing when the values of the input signal are independent and identically distributed random variables (Koskinen *et al.* [5]). These results can also be used to approximate the statistical properties of two-dimensional morphological filters since the simulation results in Rustanius et al. [6] indicate that the statistical properties of morphological filters are mainly determined by the size of the structuring set.

PROPOSITION 13. Consider a discrete signal f whose values are independent and identically distributed random variables having a common distribution function $F(t)$. Let f be closed by using a convex one-dimensional structuring set B of length n. Then the distribution function $G_c(t)$ of the values of the closed signal f^B is

$$G_c(t) = nF(t)^n - (n-1)F(t)^{n+1}. \tag{81}$$

Proof. Consider the closing by a convex one-dimensional structuring set B of length n. By Proposition 4 we need to compute the number of the vectors $\underline{z} = (z_1, z_2, \cdots, z_{2n-1})$ of each weight in primage $g^{-1}(0)$ of the Boolean function

$$g(\underline{z}) = \wedge_{a \in B^s}(\wedge_{b \in B_a} z_b), \tag{82}$$

which corresponds to the stack-filter expression of closing.

Now the number of the vectors of weight $2n - 1 - s$ in $g^{-1}(0)$ equals the number of the vectors of weight $2n - 1 - s$ containing at least n consecutive 0's. Divide these vectors into n distinct classes in the following way. The class C_0 contains all vectors

$$b^{(0)} = (\overbrace{0, 0, \cdots, 0}^{n \text{ times}}, *, *, \cdots, *),$$

and the class C_l contains all vectors such that

$$b^{(l)} = (\overbrace{*, *, \cdots, *}^{l \text{ times}}, 1, \overbrace{0, 0, \cdots, 0}^{n \text{ times}}, *, *, \cdots, *),$$

where $*$ denotes 0 or 1. It is easy to see that C_0 consists of

$$\binom{n-1}{s-n}$$

vectors of weight $2n-1-s$ and C_l consists of

$$\binom{n-2}{s-n}$$

vectors of weight $2n-1-s$ for all $l = 1, 2, \cdots, n-1$. So there are

$$\binom{n-1}{s-n} + (n-1)\binom{n-2}{s-n}$$

vectors of weight $2n-1-s$ in $g^{-1}(0)$. Now Proposition 4 implies that the distribution function $G_c(t)$ of the closed signal is

$$\begin{aligned} G_c(t) &= \sum_{i=n}^{2n-1}\left(\binom{n-1}{i-n} + (n-1)\binom{n-2}{i-n}\right) \\ &\quad (1-F(t))^{2n-1-i}F(t)^i \\ &= \sum_{i=0}^{n-1}\left(\binom{n-1}{i} + (n-1)\binom{n-2}{i}\right) \\ &\quad (1-F(t))^{n-1-i}F(t)^{i+n} \\ &= F(t)^n \sum_{i=0}^{n-1}\binom{n-1}{i} \\ &\qquad (1-F(t))^{n-1-i}F(t)^i \\ &\qquad +(n-1)F(t)^n(1-F(t)) \\ &\qquad \sum_{i=0}^{n-1}\binom{n-2}{i} \\ &\qquad (1-F(t))^{n-2-i}F(t)^i \\ &= nF(t)^n - (n-1)F(t)^{n+1}. \end{aligned} \tag{83}$$

Using Proposition 6, we obtain the following corollary.

COROLLARY 13.1. Consider a discrete signal f whose values are independent and identically distributed random variables having a common distribution function $F(t)$. Let f be opened by using a convex one-dimensional structuring set B of length n. Then the distribution function $G_o(t)$ of the values of the opened signal f_B is

$$\begin{aligned} G_o(t) &= 1 - n(1-F(t))^n \\ &\quad +(n-1)(1-F(t))^{n+1}. \end{aligned} \tag{84}$$

Let f be a discrete signal whose values are independent and identically distributed random variables having the Laplace distribution $L(\alpha)$, and let f be closed by a convex one-dimensional structuring set B of length n. Then Proposition 13 and Lemmas 1 and 2 imply that for random variable $y = f^B$

$$\begin{aligned} E\{y\} &= -\frac{n+3}{\alpha(n+1)2^{n+1}} \\ &\quad +\frac{1}{\alpha}\left(\sum_{k=1}^{n}\frac{1-2^{-k}}{k} - \frac{n-1}{n+1}\right. \\ &\qquad \left. +\frac{n-1}{n+1}2^{-n-1}\right) \end{aligned} \tag{85}$$

and

$$\begin{aligned} E\{y^2\} &= \frac{n^3+5n^2+2n}{\alpha^2 n^2(n+1)^2 2^n} \\ &\quad +\frac{1}{\alpha^2}\left(\sum_{k=1}^{n}\frac{2}{k}\sum_{l=1}^{k}\frac{1-2^{-l}}{l}\right. \\ &\qquad \left. -\frac{2n-2}{n+1}\sum_{l=1}^{n+1}\frac{1-2^{-l}}{l}\right). \end{aligned} \tag{86}$$

In the same way, using Proposition 1 and Lemmas 1 and 2, one can derive exact output expectations and variances for all stack filters whose windows are of a moderate size and whose inputs are independent random variables having a Laplace distribution.

PROPOSITION 14. Consider a discrete signal f whose values are independent and identically distributed random variables having a common distribution function $F(t)$. Let f be clos-opened by using a convex one-dimensional structuring set B of length n, where $n > 2$. Then the distribution function $G_{co}(t)$ of the values of the clos-opened signal $(f^B)_B$ is

$$G_{co}(t) = \frac{n^2-n-2}{2}F(t)^{2n+2}$$

$$+(-n^2+n+1)F(t)^{2n+1} + \frac{n^2-n}{2}F(t)^{2n} - (n-1)F(t)^{n+1} + nF(t)^n. \quad (87)$$

Proof. Consider the clos-opening by a convex one-dimensional structuring set B of length $n > 2$. By Proposition 5 we need to compute the number of the vectors $\underline{z} = (z_1, z_2, \cdots, z_{4n-3})$ of each weight in the preimage $g^{-1}(0)$ of the Boolean function

$$g(\underline{z}) = \vee_{a\in B^s}(\wedge_{b\in(B+B^s)_a}(\vee_{c\in B_b} z_c)), \quad (88)$$

which corresponds to the stack-filter expression of clos-opening.

The number of the vectors of weight $4n-3-s$ in $g^{-1}(0)$ equals the number of the vectors $\underline{z} = (z_1, z_2, \cdots, z_{4n-3})$ of weight $4n-3-s$ such that each subvector $(z_i, z_{1+i}, \cdots, z_{3n-3+i})$ contains at least n consecutive 0's for all $i \in \{1, 2, \cdots, n\}$.

First we divide $g^{-1}(0)$ into three distinct classes $A, B,$ and C: $A = \{\underline{z} \in \{0,1\}^{4n-3} |$ for some $i \in \{0, \ldots, n-1\}, z_{n+i} = 0, \ldots, z_{2n-1+i} = 0\}, B = \{\underline{z} \in \{0,1\}^{4n-3} | z_{2n-1} = 0$ and for some nonnegative i, j such that $i + j \leq n-2, z_{n-1-i} = 0, \ldots, z_{2n-2-i} = 0$ and $z_{2n+j} = 0, \ldots, z_{3n-1+j} = 0$ and for some nonnegative l, m such that $l + m \leq n-4, z_{2n-2-l} = z_{2n+m} = 1\}$, and $C = \{\underline{z} \in \{0,1\}^{4n-3} | z_{2n-1} = 1$ and for some nonnegative i, j such that $i + j \leq n-2, z_{n-1-i} = 0, \ldots, z_{2n-2-i} = 0$ and $z_{2n+j} = 0, \ldots, z_{3n-1+j} = 0\}$.

Then we divide C again into three distinct classes $C1, C2$ and $C3$: $C1 = \{\underline{z} \in C | z_{n-1} = 0, \ldots, z_{2n-2} = 0\}, C2 = \{\underline{z} \in C | z_{2n} = 0, \ldots, z_{3n-1} = 0$ and for some i such that $0 \leq i \leq n-3, z_{2n-2-i} = 1\}$, and $C3 = \{\underline{z} \in C |$ for some nonegative i, j such that $i, j \leq n-4, z_{2n-2-i} = z_{2n+j} = 1\}$.

Using the same method as in the proof of Proposition 13, we see that class A consists of

$$\binom{3n-3}{s-n} + (n-1)\binom{3n-4}{s-n}$$

vectors of weight $4n-3-s$. We can calculate directly that class B consists of

$$\sum_{i=1}^{n-3} i \binom{2n-6}{s-2n-1}$$

$$= \frac{n^2-5n+6}{2}\binom{2n-6}{s-2n-1} \quad (89)$$

and that class $C3$ consists of

$$\sum_{i=1}^{n-3} i \binom{2n-6}{s-2n} = \frac{n^2-5n+6}{2}\binom{2n-6}{s-2n} \quad (90)$$

vectors of weight $4n-3-s$.

It is easy to see that class $C1$ consists of

$$\binom{2n-4}{s-2n} + (n-2)\binom{2n-5}{s-2n},$$

and that class $C2$ consists of

$$(n-2)\binom{2n-5}{s-2n}$$

vectors of weight $4n-3-s$.

So the number $t(s)$ of the vectors of weight $4n-3-s$ in $g^{-1}(0)$ is

$$t(s) = \binom{3n-3}{s-n} + (n-1)\binom{3n-4}{s-n} + \frac{n^2-5n+6}{2}\binom{2n-6}{s-2n-1} + \frac{n^2-5n+6}{2}\binom{2n-6}{s-2n} + \binom{2n-4}{s-2n} + (2n-4)\binom{2n-5}{s-2n}. \quad (91)$$

Now the output distribution $G_{co}(t)$ for clos-opening is given by

$$G_{co}(t) = \sum_{i=0}^{4n-3} t(i)(1-F(t))^{4n-3-i}F(t)^i$$
$$= (2n-4)(1-F(t))^2F(t)^{2n} + (1-F(t))F(t)^{2n} + \frac{n^2-5n+6}{2}(1-F(t))^3F(t)^{2n} + \frac{n^2-5n+6}{2}(1-F(t))^2F(t)^{2n+1} + F(t)^n + (n-1)(1-F(t))F(t)^n$$
$$= \frac{n^2-n-2}{2}F(t)^{2n+2}$$

$$+(-n^2+n+1)F(t)^{2n+1} + \frac{n^2-n}{2}F(t)^{2n} - (n-1)F(t)^{n+1} + nF(t)^n. \quad (92)$$

Using Proposition 6, we obtain the following corollary.

COROLLARY 14.1. Consider a discrete signal f whose values are independent and identically distributed random variables having a common distribution function $F(t)$. Let f be open-closed by using a convex structuring set B of length n, where $n > 2$. Then the distribution function $G_{oc}(t)$ of the values of the open-closed signal $(f_B)^B$ is

$$G_{oc}(t) = 1 - \frac{n^2-n-2}{2}(1-F(t))^{2n+2} - (-n^2+n+1)(1-F(t))^{2n+1} - \frac{n^2-n}{2}(1-F(t))^{2n} + (n-1)(1-F(t))^{n+1} - n(1-F(t))^n. \quad (93)$$

Formula (87) defines the output distribution of clos-opening in terms of the input distribution. Another formula for the output distribution of clos-opening is given in Stevenson and Arce [1], but this formula does not define the correct output distribution in terms of the input distribution.

When the inputs are independent and identically distributed random variables, the analytical forms of the output distribution functions also allow us to compute the asymptotic approximations for the expectation and the variance in the cases of uniform or Laplace distributions. The tails of the uniform density function are zero-value, and morphological filters attenuate this kind of noise well. On the other hand, the density function of Laplace distribution is heavy-tailed, and morphological filtering increases the variance of this kind of noise.

Using Propositions 13 and 14, we obtain after lengthy computations (performed by using a symbolic program) the following result (Koskinen et al. [5]), which shows that the output variance of compound morphological operations decreases very fast as the length of the structuring set increases in the case of the uniform distribution.

PROPOSITION 15. Consider the filtering of a discrete signal f whose values are independent and identically uniformly distributed on [0, 1]. Let B be a convex one-dimensional structuring set of length n. Then

(a) after closing by B, for the expectation

$$E\{f^B\} = 1 - \frac{2}{n} + O\left(\frac{1}{n^2}\right) \quad (94)$$

and for the variance

$$V\{f_B\} = \frac{2}{n^2} + O\left(\frac{1}{n^3}\right); \quad (95)$$

(b) after opening by B, for the expectation

$$E\{f_B\} = \frac{2}{n} + O\left(\frac{1}{n^2}\right) \quad (96)$$

and for the variance

$$V\{f_B\} = \frac{2}{n^2} + O\left(\frac{1}{n^3}\right); \quad (97)$$

(c) after clos-opening by B, for the expectation

$$E\{(f^B)_B\} = 1 - \frac{17}{8n} + O\left(\frac{1}{n^2}\right) \quad (98)$$

and for the variance

$$V\{(f^B)_B\} = \frac{119}{64n^2} + O\left(\frac{1}{n^3}\right); \quad (99)$$

(d) after open-closing by B, for the expectation

$$E\{(f_B)^B\} = \frac{17}{8n} + O\left(\frac{1}{n^2}\right) \quad (100)$$

and for the variance

$$V\{(f_B)^B\} = \frac{119}{64n^2} + O\left(\frac{1}{n^3}\right). \quad (101)$$

The following asymptotic formulas show that the output expectations and the output variances of compound morphological operations increase logarithmically with the length of the structuring set.

PROPOSITION 16. Consider the filtering of a discrete signal f whose values are independent and identically distributed having the Laplace distribution $L(\alpha)$. Let B be a convex one-dimensional structuring set of length n, and let

$$\eta = \frac{1}{\alpha^2} \sum_{k=1}^{\infty} \frac{2^{-k}}{k}. \tag{102}$$

Then

(a) after closing or clos-opening by B, for the expectation

$$E\{f^B\} = E\{(f^B)_B\} = \frac{1}{\alpha} \ln n + O(1) \tag{103}$$

and for the variance

$$V\{f^B\} = V\{(f^B)_B\} = \eta \ln n + O(1), \tag{104}$$

(b) after opening or open-closing by B, for the expectation

$$\begin{aligned} E\{f_B\} &= E\{(f_B)^B\} \\ &= -\frac{1}{\alpha} \ln n + O(1) \end{aligned} \tag{105}$$

and for the variance

$$V\{f_B\} = V\{(f_B)^B\} = \eta \ln n + O(1). \tag{106}$$

Proof. Lemma 3 and Proposition 13 imply that

$$\begin{aligned} E\{f^B\} &= \frac{1}{\alpha}\Big(n \ln n - (n-1)\ln(n+1)\Big) \\ &\quad + O(1) \\ &= \frac{1}{\alpha}\Big(\ln n + (n-1)\ln\Big(\frac{n}{n+1}\Big)\Big) \\ &\quad + O(1) \\ &= \frac{1}{\alpha} \ln n + O(1) \end{aligned} \tag{107}$$

and

$$\begin{aligned} V\{f^B\} &= \eta\Big(n \ln n - (n-1)\ln(n+1)\Big) \\ &\quad + O(1) \\ &= \eta\Big(\ln n + (n-1)\ln\Big(\frac{n}{n+1}\Big)\Big) \\ &\quad + O(1) \\ &= \eta \ln n + O(1). \end{aligned} \tag{108}$$

Similarly, Lemma 3 and Proposition 14 imply that

$$E\{(f^B)_B\} = \frac{1}{\alpha} \ln n + O(1) \tag{109}$$

and

$$V\{(f^B)_B\} = \eta \ln n + O(1). \tag{110}$$

If the length of the structuring set is n, then the output variance of linear and median filters is $O(1/n)$ for both the uniform and the Laplace distributions. On the contrary, Propositions 8, 9, 15, and 16 show that the output variances of morphological filters are $O(1/n^2)$ for uniform distribution and the output variances of morphological filters increase logarithmically with the size of the structuring set for the Laplace distribution. Thus morphological operations are statistically unstable operations whose behavior is extremely sensitive to a change in the type of noise distribution.

Propositions 9, 12, and 16 show that the expectations of one- and two-dimensional morphological operations have the same asymptotic increasing rate in the case of the Laplace distribution. This is evidence of the major role of the size of the structuring set when we compare the effect of the size and the shape of the structuring set on the statistical properties.

The morphological filters and median filters have many similar deterministic and impulsive noise-attenuation properties; see, e.g., Maragos and Schafer [3] and Justusson [19]. However, the above results show that the statistical behaviors of these filters differ greatly: morphological filters are unstable and median filters are stable in a statistical sense, and in the case of short-tailed and heavy-tailed noise, the behaviors of median and morphological filters are quite opposite.

7 Conclusions

When the output distribution function of a filter is known, all other statistical properties of this filter can be derived from this distribution. In this paper a closed-form expression for the output distribution in terms of the input distribution has been given for morphological filters. Certain statistical symmetry properties stemming from the duality of morphological filters have also been presented.

By using the expressions of output distributions, asymptotic formulas for the output variances and for the expectations of morphological filters have been derived for the cases of uniform and Laplace distributions. These formulas show that the asymptotic behavior of morphological filters is extremely sensitive to a change in the type of noise distribution, leading to the conclusion that morphological filters may be unstable under noisy conditions.

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