RAYLEIGH SCATTERING IN PLANETARY ATMOSPHERES

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(Received 27 July, 1989)

Abstract. The vector equation of radiative transfer is solved both for conservative and non-conservative planetary atmospheres using the method of discrete ordinates. The atmosphere, bounded by a Lambert bottom, is considered plane-parallel and homogeneous. The scattering in the atmosphere obeys the Rayleigh or Rayleigh–Cabannes law. The compiled package of FORTRAN codes allows us to find the Stokes parameters for such an atmosphere at arbitrary optical depth.

1. Introduction

The brightness and polarization of the sunlit sky had been a major challenge for theoreticians for 76 years: the problem was first posed by Lord Rayleigh in 1871 and it was solved in an elegant way by Chandrasekhar in 1947.

The problem itself is clear enough – a parallel beam of the Sun's radiation (natural or in a given state of polarization) is incident on a plane-parallel homogeneous atmosphere of optical thickness τ_0 in some specified direction which is characterized by the cosine of the incident angle θ_0. It is required to find the external field of radiation, i.e., the distribution of intensity and polarization of the light, diffusely transmitted by the atmosphere at $\tau = \tau_0$ and of the light diffusely reflected by the atmosphere at $\tau = 0$.

A proper vector equation of transfer was formulated and solved by Chandrasekhar (1947) using the method of discrete ordinates.

For a conservatively-scattering atmosphere he succeeded in factorizing the characteristic equation which allowed to express the solution in an explicit form.

Applying this solution to the problem of determining the brightness and polarization of the sunlit sky, Chandrasekhar arrived at the theoretical results (Chandrasekhar and Elbert, 1954) which coincided very well with those obtained experimentally (Dorno, 1919).

Since then Chandrasekhar's solution has been used to explain many other features of radiative transfer and a voluminous set of tables has been compiled by Coulson *et al.* (1960) which is convenient for obtaining instant estimates of radiation field on Rayleigh scattering. Later on a lot of new methods to solve the vector equation of radiative transfer was elaborated which were powerful enough to cope with more general phase matrices than that of Rayleigh and in the same time allowing for the actual distribution of physical parameters in the atmosphere. Here we refer only to some of them:

(a) iteration methods,
(b) method of singular eigenfunctions,

Earth, Moon, and Planets **46**: 261–284, 1989.

(c) method of matrix operators,
(d) adding method,
(e) method of invariant imbedding etc.

The descriptions of these methods and the respective references are given by Hansen and Travis (1974) and van de Hulst (1980).

But, in elaborating new methods one has to rely on some firm benchmark results, which one's own results may be compared with. It is highly recommended that the accuracy of those benchmark results are known. For such a purpose the method of discrete ordinates is suited best.

Since the Rayleigh phase matrix is reducible with respect to the Stokes parameter V and since the various coefficients in the Fourier expansions of I_l, I_r and U have certain simple relations of proportionality, the solution of transfer equation reduces to the solution of one two-component vector equation and four scalar equations of pseudo-problem.

In the present paper the two-component vector equation was solved using the method of discrete ordinates for both conservative and non-conservative atmospheres (cf. Code, 1950). The attempts to use the Sobolev method of resolvent function for non-conservative atmospheres were unsuccessful as stated also by Domke (1971).

The remaining four scalar equations were solved using the approximation of the Sobolev resolvent function (Viik *et al.*, 1985).

As a result the explicit formulae were obtained to determine both the external and internal radiation field in case of illuminating the atmosphere with a parallel beam in a given state of polarization.

This solution has been generalized for the case of the Rayleigh–Cabannes phase matrix, i.e., when light is scattered by anisotropic particles or in a spectral line which bring along a certain component of non-polarized light.

The Lambert reflecting ground of the atmosphere is taken into account as well. Since the isotropy of the Lambert reflector it is clear that the ground does not affect the Stokes parameters U and V.

A package of FORTRAN subroutines has been compiled to solve the problem posed above (Viik, 1989) and a series of numerical experiments have been carried out to determine the limits and possibilities of the method described.

2. Formulation of the Problem

If the scattering in an atmosphere is described by a phase matrix $\mathbf{Z}$, the Stokes parameters of the radiation field $\mathbf{I} = (I_l, I_r, U, V)^{\mathrm{T}}$ satisfy the following equation of transfer

$$\mu \frac{\partial \mathbf{I}(\tau, \mu, \varphi, \tau_0)}{\partial \tau} = \mathbf{I}(\tau, \mu, \varphi, \tau_0) - \frac{1}{4} \lambda \, e^{-\tau/\mu_0} \mathbf{Z}(\mu, \varphi; -\mu_0, \varphi_0)\mathbf{F} -$$

$$-\frac{1}{4\pi}\lambda\int_0^{2\pi}\mathrm{d}\varphi'\int_{-1}^{+1}\mathrm{d}\mu'\mathbf{Z}(\mu,\varphi;\mu',\varphi')\mathbf{I}(\tau,\mu',\varphi',\tau_0)\,, \tag{1}$$

where

$$\mathbf{F}=(F_l,F_r,F_U,F_V)^{\mathrm{T}} \tag{2}$$

is the Stokes vector which represents the parallel beam of radiation incident on the atmosphere in the direction $(-\mu_0, \varphi_0)$. The net fluxes of the incident beam per unit area normal to the beam are $\pi\mathbf{F}$. In Equation (1) μ denotes the cosine of the angle to the outward normal and φ the azimuth angle. And, finally, λ is the albedo of single scattering $(0<\lambda\leqslant 1)$, τ – the optical depth, measured from the upper boundary of the atmosphere and τ_0 is the optical thickness of the atmosphere.

If the atmosphere is bounded by a Lambert reflector with albedo λ_0 and there is no incident diffuse radiation at $\tau=0$, the boundary conditions for Equation (1) are

$$\begin{aligned}&\mathbf{I}(0,-\mu,\varphi,\tau_0)=0\,,\\&\mathbf{I}(\tau_0,\mu,\varphi,\tau_0)=\lambda_0\mu_0\,\mathrm{e}^{-\tau_0/\mu_0}\mathbf{E}\mathbf{F}+\\&\qquad+\frac{1}{\pi}\lambda_0\mathbf{E}\int_0^{2\pi}\mathrm{d}\varphi'\int_0^{1}\mu'\,\mathrm{d}\mu'\mathbf{I}(\tau_0,-\mu',\varphi',\tau_0)\,,\end{aligned} \tag{3}$$

where

$$\mathbf{E}=\frac{1}{2}\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} \tag{4}$$

If the atmosphere scatters according to the Rayleigh or Rayleigh–Cabannes phase matrix the Stokes parameter V is uncoupled from parameter I_l, I_r and U.

Chandrasekhar (1960) has shown that the parameter V may be written in the form

$$\begin{aligned}V(\tau,\mu,\varphi,\tau_0)=&\tfrac{3}{2}F_V[-\mu\mu_0 f^{(3)}(\tau,\mu,\mu_0,\tau_0)+\\&+(1-\mu^2)^{1/2}(1-\mu_0^2)^{1/2}f^{(4)}(\tau,\mu,\mu_0,\tau_0)\cos(\varphi_0-\varphi)]\,.\end{aligned} \tag{5}$$

where the $f^{(i)}$ are the solutions of two scalar equations which will be dealt with further. Henceforth we observe the 3-component vector equation, the phase matrix of which is described as

$$\mathbf{Z}(\mu,\varphi;\mu_0,\varphi_0)=c\mathbf{Z}^*(\mu,\varphi;\mu_0,\varphi_0)+(1-c)\mathbf{E}\,, \tag{6}$$

where the coefficient c is expressed in terms of depolarization factor ρ_n in the form (cf. Penndorf, 1957)

$$c=\frac{2(1-\rho_n)}{2+\rho_n}. \tag{7}$$

The Rayleigh phase matrix $\mathbf{Z}^*$ may be decomposed in the form

$$\mathbf{Z}^*(\mu,\varphi;\mu_0,\varphi_0)=\mathbf{Z}^{(0)}(\mu,\mu_0)+(1-\mu^2)^{1/2}(1-\mu_0^2)^{1/2}\times \times\mathbf{Z}^{(1)}(\mu,\varphi;\mu_0,\varphi_0)+\mathbf{Z}^{(2)}(\mu,\varphi;\mu_0,\varphi_0), \tag{8}$$

where

$$\mathbf{Z}^{(0)}(\mu,\mu_0)=\frac{3}{4}\begin{pmatrix}2(1-\mu^2)(1-\mu_0^2)+\mu^2\mu_0^2 & \mu^2 & 0\\ \mu_0^2 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{9}$$

$$\mathbf{Z}^{(1)}(\mu,\varphi;\mu_0,\varphi_0)=\frac{3}{2}\begin{pmatrix}2\mu\mu_0\cos(\varphi_0-\varphi) & 0 & \sin(\varphi_0-\varphi)\\ 0 & 0 & 0\\ -2\mu_0\sin(\varphi_0-\varphi) & 0 & \cos(\varphi_0-\varphi)\end{pmatrix},$$

$$\mathbf{Z}^{(2)}(\mu,\varphi;\mu_0,\varphi_0)= =\frac{3}{4}\begin{pmatrix}\mu^2\mu_0^2\cos 2(\varphi_0-\varphi) & -\mu^2\cos 2(\varphi_0-\varphi) & \mu^2\mu_0^2\sin 2(\varphi_0-\varphi)\\ -\mu_0^2\cos 2(\varphi_0-\varphi) & \cos 2(\varphi_0-\varphi) & -\mu_0\sin 2(\varphi_0-\varphi)\\ -2\mu\mu_0^2\sin 2(\varphi_0-\varphi) & 2\mu\sin 2(\varphi_0-\varphi) & 2\mu\mu_0\cos 2(\varphi_0-\varphi)\end{pmatrix} \tag{11}$$

The solution of Equation (1) is written in the form

$$\mathbf{I}(\tau,\mu,\varphi,\tau_0)=\mathbf{I}^{(0)}(\tau,\mu,\tau_0)+(1-\mu^2)^{1/2}(1-\mu_0^2)^{1/2}\mathbf{I}^{(1)}(\tau,\mu,\varphi,\tau_0)+ +\mathbf{I}^{(2)}(\tau,\mu,\varphi,\tau_0). \tag{12}$$

According to Chandrasekhar (1960) we may write

$$\mathbf{I}^{(1)}(\tau,\mu,\varphi,\tau_0)=\mathbf{Z}^{(1)}(\mu,\varphi;-\mu_0,\varphi_0)\mathbf{F}f^{(1)}(\tau,\mu,\mu_0,\tau_0), \tag{13}$$

$$\mathbf{I}^{(2)}(\tau,\mu,\varphi,\tau_0)=\mathbf{Z}^{(2)}(\mu,\varphi;-\mu_0,\varphi_0)\mathbf{F}f^{(2)}(\tau,\mu,\mu_0,\tau_0). \tag{14}$$

The functions $f^{(i)}$ in Equations (5), (13) and (14) satisfy the scalar equation of transfer for a pseudo-problem

$$\mu\frac{\partial f^{(i)}(\tau,\mu,\mu_0,\tau_0)}{\partial\tau}=f^{(i)}(\tau,\mu,\mu_0,\tau_0)-\frac{1}{4}\lambda\,\mathrm{e}^{-\tau/\mu_0}- -\lambda\int_{-1}^{+1}\Psi^{(i)}(\mu')f^{(i)}(\tau,\mu',\mu_0,\tau_0)\,\mathrm{d}\mu',\quad i=1,\ldots,4, \tag{15}$$

where

$$
\begin{aligned}
\Psi^{(1)}(\mu) &= \tfrac{3}{8}(1-\mu^2)(1+2\mu^2),\\
\Psi^{(2)}(\mu) &= \tfrac{3}{16}(1+\mu^2)^2\,,\\
\Psi^{(3)}(\mu) &= \tfrac{3}{4}\mu^2\,,\\
\Psi^{(4)}(\mu) &= \tfrac{3}{8}(1-\mu^2)\,.
\end{aligned}
\tag{16}
$$

Henceforth we shall use the Mullikin indices (1966).

It has been shown that the solution of equations (15) may be written (cf. Viik *et al.*, 1985) as

$$
\begin{aligned}
f^{(i)}(\tau,\mu,\mu_0,\tau_0) = \frac{\lambda\mu_0}{\mu+\mu_0}\{&X^{(i)}(\mu_0,\tau_0)[x^{(i)}(\tau,\mu,\tau_0)+y^{(i)}(\tau,\mu_0,\tau_0)-1]-\\
&-Y^{(i)}(\mu_0,\tau_0)[x^{(i)}(\tau_0-\tau,\mu_0,\tau_0)+y^{(i)}(\tau_0-\tau,\mu,\tau_0)-1]\}\,,
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
&f^{(i)}(\tau,-\mu,\mu_0,\tau_0)\\
&=\frac{\lambda\mu_0}{\mu-\mu_0}\{X^{(i)}(\mu_0,\tau_0)[y^{(i)}(\tau,\mu,\tau_0)-y^{(i)}(\tau,\mu_0,\tau_0)]-\\
&\quad -Y^{(i)}(\mu_0,\tau_0)[x^{(i)}(\tau_0-\tau,\mu,\tau_0)-x^{(i)}(\tau_0-\tau,\mu_0,\tau_0)]\}\,,
\end{aligned}
\tag{18}
$$

where

$$
\begin{aligned}
x^{(i)}(\tau,\mu,\tau_0) &= 1+\int_\tau^{\tau_0}\Phi^{(i)}(t,\tau_0)\,e^{-(t-\tau)/\mu}\,dt\,,\\
y^{(i)}(\tau,\mu,\tau_0) &= e^{-\tau/\mu}+\int_0^{\tau}\Phi^{(i)}(t,\tau_0)\,e^{-(\tau-t)/\mu}\,dt,\\
X^{(i)}(\mu,\tau_0) &= x^{(i)}(0,\mu,\tau_0)\,,\\
y^{(i)}(\mu,\tau_0) &= y^{(i)}(\tau_0,\mu,\tau_0)
\end{aligned}
\tag{19}
$$

and $\Phi^{(i)}$ is the resolovent function of Sobolev (1972) which satisfies an integral equation

$$
\Phi^{(i)}(\tau,\tau_0) = \int_0^{\tau_0} K^{(i)}(|t-\tau|)\Phi^{(i)}(t,\tau_0)\,dt + K^{(i)}(\tau)\,,
\tag{20}
$$

where

$$
K^{(i)}(\tau) = \lambda\int_0^1 \Psi^{(i)}(s)\,e^{-\tau/s}\,ds/s, \quad i=1,\ldots,4\,.
\tag{21}
$$

Applying the method of resolvent function approximation (Viik, 1986) we may easily find the functions (19) and the respective solutions, Equations (17) and (18).

If $\rho_n \neq 0$ then in Equation (15) we have to use the proper albedos of single scattering. According to Chandrasekhar they are

$$\lambda^{(1)} = \lambda^{(2)} = c\lambda,$$
$$\lambda^{(3)} = \lambda^{(4)} = \frac{2(1-2\rho_n)}{2+\rho_n}\lambda. \tag{22}$$

3. The Solution of the Vector Equation of Transfer

After the transformations described above we are left with a two-component vector equation of transfer

$$\begin{aligned}\mu \frac{\partial \mathbf{I}^{(0)}(\tau, \mu, \mu_0, \tau_0)}{\partial \tau} = \mathbf{I}^{(0)}(\tau, \mu, \mu_0, \tau_0) - &\\ -\frac{1}{2}\lambda c \int_{-1}^{+1} \mathbf{Z}^{(0)}(\mu, \mu')\mathbf{I}^{(0)}(\tau, \mu', \mu_0, \tau_0)\, \mathrm{d}\mu' - &\\ -\tfrac{1}{2}\lambda(1-c)\mathbf{E} \int_{-1}^{+1} \mathbf{I}^{(0)}(\tau, \mu', \mu_0, \tau_0)\, \mathrm{d}\mu' - &\\ -\left[\frac{1c}{4}\lambda \mathbf{Z}^{(0)}(\mu, \mu_0) + \frac{1}{4}(1-c)\lambda\mathbf{E}\right]\mathbf{F}\, \mathrm{e}^{-\tau/\mu_0}. &\end{aligned} \tag{23}$$

Following the Chandrasekhar method of replacing integrals by the corresponding Gauss sums of order of N in the interval $(0, 1)$ we obtain the solution of Equation (23) in the form (Viik, 1989)

$$\begin{aligned}I_l(\tau, \pm\mu_i, \mu_0, \tau_0) = \sum_{\beta=1}^{N} a_\beta Q_i^{\pm}(\kappa_\beta)\, \mathrm{e}^{-\kappa_\beta \tau} + &\\ + \sum_{\beta=1}^{N} b_\beta Q_i^{\mp}(\kappa_\beta)\, \mathrm{e}^{-\kappa_\beta(\tau_0 - \tau)} + &\\ + \sum_{\alpha=1}^{N} c_\alpha Q_i^{\pm}(k_\alpha)\, \mathrm{e}^{-k_\alpha \tau} + &\\ + \sum_{\alpha=1}^{N} d_\alpha Q_i^{\mp}(k_\alpha)\, \mathrm{e}^{-k_\alpha(\tau_0 - \tau)} + \frac{A\mu_i^2 + B}{1 \pm \mu_i/\mu_0}\mathrm{e}^{-\tau/\mu_0}, &\end{aligned} \tag{24}$$

$$I_r(\tau, \pm\mu_i, \mu_0, \tau_0) = \sum_{\beta=1}^{N} a_\beta R_i^{\pm}(\kappa_\beta)\, e^{-\kappa_\beta \tau} + \sum_{\beta=1}^{N} b_\beta R_i^{\mp}(\kappa_\beta)\, e^{-\kappa_\beta(\tau_0-\tau)} + \sum_{\alpha=1}^{N} c_\alpha R_i^{\pm}(k_\alpha)\, e^{-k_\alpha \tau} + \sum_{\alpha=1}^{N} d_\alpha R_i^{\mp}(k_\alpha)\, e^{-k_\alpha(\tau_0-\tau)} + \frac{C}{1 \pm \mu_i/\mu_0}\, e^{-\tau/\mu_0}, \tag{25}$$

where

$$Q_i^{\pm}(x) = \frac{\mu_i^2 + z(x)}{1 \pm \mu_i x} \quad \text{and} \quad R_i^{\pm}(x) = \frac{v(x)}{1 \pm \mu_i x}. \tag{26}$$

This solution is valid only for the non-conservative Rayleigh–Cabannes atmosphere ($\lambda \neq 1$, $\rho_n \neq 0$). For a conservative Rayleigh–Cabannes atmosphere ($\lambda = 1$, $\rho_n \neq 0$) there are minor differences in Equations (24) and (25); namely, the first two summations begin from $\beta = 2$ and we have to add to both formulae the following term

$$a_i + b_1(\tau \pm \mu_i).$$

If we have the conservative Rayleigh atmosphere ($\lambda = 1$, $\rho_n = 0$) then the solution of Equation (23) is (cf. Chandrasekhar, 1960) as

$$I_l(\tau, \pm\mu_i, \mu_0, \tau_0) = a_1 + \sum_{\beta=2}^{N} a_\beta Q_i^{\pm}(\kappa_\beta)\, e^{-\kappa_\beta \tau} + b_1(\tau \pm \mu_i) + \sum_{\beta=2}^{N} b_\beta Q_i^{\mp}(\kappa_\beta)\, e^{-\kappa_\beta(\tau_0-\tau)} + \sum_{\alpha=1}^{N} c_\alpha Q_i^{\pm}(k_\alpha)\, e^{-k_\alpha \tau} + \sum_{\alpha=1}^{N} d_\alpha Q_i^{\mp}(k_\alpha)\, e^{-k_\alpha(\tau_0-\tau)} + \frac{A\mu_i^2 + B}{1 \pm \mu_i/\mu_0}\, e^{-\tau/\mu_0},$$

$$I_r(\tau. \pm\mu_i, \mu_0, \tau_0) = a_1 + b_1(\tau \pm \mu_i) - \sum_{\alpha=1}^{N} c_\alpha R_i^{\pm}(k_\alpha)\, e^{-k_\alpha \tau} - \sum_{\alpha=1}^{N} d_\alpha R_i^{\mp}(k_\alpha)\, e^{-k_\alpha(\tau_0-\tau)} + \frac{c}{1 \pm \mu_i/\mu_0}\, e^{-\tau/\mu_0}, \tag{28}$$

where

$$Q_i^{\pm}(\kappa_\beta) = \frac{1-\mu_i^2}{1 \pm \mu_i \kappa_\beta}, \quad Q^{\pm}(k_\alpha) = 1 \pm k_\alpha \mu_i ,$$

and (29)

$$R_i^{\pm}(k_\alpha) = \frac{k_\alpha^2 - 1}{1 \pm \mu_i k_\alpha}.$$

The two functions z and v are to be found from the coupled algebraic system

$$z[1 - 2\xi(D_0 - D_2) - \zeta D_0] - v\zeta D_0 = 2\xi(D_2 - D_4) + \zeta D_2 ,$$

$$- z(\xi D_2 + \zeta D_0) + v[1 - D_0(\xi + \zeta)] = \xi D_4 + \zeta D_2 , \quad (30)$$

where

$$\xi = \tfrac{3}{8}\lambda c, \qquad \zeta = \tfrac{1}{4}\lambda(1 - c)$$

and (31)

$$D_m(x) = 2 \sum_{i=1}^{N} \frac{w_i \mu_i^m}{1 - \mu_i^2 x^2}.$$

In Equation (31) μ_i and w_i are the points and weights of the gaussian quadrature of order N in the interval $(0, 1)$.

The coefficients A, B and C of the particular solutions we find from the following algebraic system

$$A[1 - \xi(3D_4 - 2D_2)] - \xi B(3D_2 - 2D_0) - \xi C D_0 =$$
$$= \tfrac{1}{2}\xi[(3\mu_0^2 - 2)F_l + F_r] ,$$
$$- A[2\xi(D_2 - D_4) + \zeta D_2] + B[1 - 2\xi(D_0 - D_2) - \zeta D_0] - \zeta C D_0 =$$
$$= \xi(1 - \mu_0^2)F_l + \tfrac{1}{2}\zeta(F_l + F_r) ,$$
$$- A(\xi D_4 + \zeta D_2) - B(\xi D_2 + \zeta D_0) + C[1 - D_0(\xi + \zeta)] =$$
$$= \tfrac{1}{2}\xi(\mu_0^2 F_l + F_r) + \tfrac{1}{2}\zeta(F_l + F_r) , \quad (32)$$

where

$$D_m = D_m(\mu_0^{-1}).$$

The constants κ_β and k_α we find from the condition that the determinant of the system of homogeneous linear algebraic equations obtained from the general solution of Equation (23) is equal to zero

$$[3\lambda(D_0 - D_2) - 4][3\lambda c(D_0 - D_2) - 8] - 36\lambda c(1 - \lambda)D_4 +$$
$$+ 12\lambda D_2[c(3 - \lambda) - 2] + 8\lambda(1 - c)D_0 = 0, \quad (33)$$

where D_m is given by Equation (31).

This equation has exactly N pairs of solutions $\pm x$ if $\lambda \neq 1$. In the conservative case the number of pairs of solutions is $N-1$ since $x^2=0$ is a root of Equation (33).

We note that if $\lambda=1$, Equation (33) does not coincide with Equation (246) given by Chandrasekhar (1960, ch. X, §74).

The roots of characteristic equation (33) satisfy the following inequalities:

$$0 \leq |x_1| < \mu_N^{-1} < |x_2| < \mu_{N-1}^{-1} < \cdots < |x_N| < \mu_1^{-1} ,$$

which make solution of Equation (33) very easy. Though Equation (30) cannot be factorized as in the case of conservative Rayleigh atmosphere, we relate the smaller roots in intervals $(\mu_n^{-1}, \mu_{n-1}^{-1})$ to the l component and denote them by $\kappa_\beta(\beta=1, \ldots, N)$, and the larger roots to the r component and denote them by $k_\alpha(\alpha=1, \ldots, N)$.

4. The Boundary Conditions of Equation (23)

The only coefficients in Equations (24) and (25) which are still undetermined are the coefficeints a_β, b_β, c_α and d_α. They are to be found from the boundary conditions (3) which for the two-component problem may be written in the form

$$\begin{aligned} &I_l(0, -\mu_i, \mu_0, \tau_0) = I_r(0, -\mu_i, \mu_0, \tau_0) = 0 , \\ &I_l(\tau_0, \mu_i, \mu_0, \tau_0) = I_r(\tau_0, \mu_i, \mu_0, \tau_0) = \\ &\qquad = \tfrac{1}{2}\lambda_0\mu_0\, \mathrm{e}^{-\tau_0/\mu_0}(F_l + F_r) + \lambda_0 \int_0^1 [I_l(\tau_0, -\mu', \mu_0, \tau_0) + \\ &\qquad + I_r(\tau_0, -\mu', \mu_0, \tau_0)]\mu'\, \mathrm{d}\mu' . \end{aligned} \tag{34}$$

Here we have to distinguish between three cases:

1. $\lambda \neq 1$, $\rho_n \neq 0$,
2. $\lambda = 1$, $\rho_n \neq 0$,
3. $\lambda = 1$, $\rho_n = 0$.

Firstly, using Equations (24) and (25) and Equations (34) we arrive at the following results

$$\begin{aligned} &\sum_{\beta=1}^{N} a_\beta Q_i^-(\kappa_\beta) + \sum_{\beta=1}^{N} b_\beta Q_i^+(\kappa_\beta)\, \mathrm{e}^{-\kappa_\beta\tau_0} + \\ &\quad + \sum_{\alpha=1}^{N} c_\alpha Q_i^-(k_\alpha) + \sum_{\alpha=1}^{N} d_\alpha Q_i^+(k_\alpha)\, \mathrm{e}^{-k_\alpha\tau_0} = -\frac{A\mu_i^2 + B}{1-\mu_i/\mu_0}, \end{aligned} \tag{35}$$

$$\begin{aligned} &\sum_{\beta=1}^{N} a_\beta R_i^-(\kappa_\beta) + \sum_{\beta=1}^{N} b_\beta R_i^+(\kappa_\beta)\, \mathrm{e}^{-\kappa_\beta\tau_0} + \sum_{\alpha=1}^{N} c_\alpha R_i^-(k_\alpha) + \sum_{\alpha=1}^{N} d_\alpha R_i^+(k_\alpha) \times \\ &\quad \times \mathrm{e}^{-k_\alpha\tau_0} = -\frac{C}{1-\mu_i/\mu_0}, \end{aligned} \tag{36}$$

$$\sum_{\beta=1}^{N} a_\beta \Pi_i^+(\kappa_\beta)\,e^{-\kappa_\beta\tau_0} + \sum_{\beta=1}^{N} b_\beta \Pi_i^-(\kappa_\beta) + \sum_{\alpha=1}^{N} c_\alpha \Pi_i^+(k_\alpha)\,e^{-k_\alpha\tau_0} + \\ + \sum_{\alpha=1}^{N} d_\alpha \Pi_i^-(k_\alpha) = \Lambda^-(\mu_0) - \frac{A\mu_i^2 + B}{1+\mu_i/\mu_0}\,e^{-\tau_0/\mu_0}, \tag{37}$$

$$\sum_{\beta=1}^{N} a_\beta P_i^+(\kappa_\beta)\,e^{-\kappa_\beta\tau_0} + \sum_{\beta=1}^{N} b_\beta P_i^-(\kappa_\beta) + \sum_{\alpha=1}^{N} c_\alpha P_i^+(k_\alpha)\,e^{-k_\alpha\tau_0} + \\ + \sum_{\alpha=1}^{N} d_\alpha P_i^-(k_\alpha) = \Lambda^-(\mu_0) - \frac{C}{1+\mu_i/\mu_0}\,e^{-\tau_0/\mu_0}, \tag{38}$$

where

$$\Pi_i^\pm(x) = Q_i^\pm(x) - \lambda_0 T_i^\mp(x), \\ P_i^\pm(x) = R_i^\pm(x) - \lambda_0 T_i^\mp(x), \\ \Lambda^-(\mu_0) = \lambda_0[A D_3^-(\mu_0^{-1}) + D_1^-(\mu_0^{-1})(B+C)] \times \\ \times e^{-\tau_0/\mu_0} + \tfrac{1}{2}\lambda_0\mu_0\,e^{-\tau_0/\mu_0}(F_l + F_r) \tag{39}$$

and

$$T_i^\pm(x) = D_3^\pm(x) + D_1^\pm(x)[z(x) + v(x)].$$

For the second case the respective Equations differ from Equations (35)–(38), namely, the summation with respect to index β starts from $\beta = 2$. Besides, in Equations (35) and (38) there appear the following supplementary terms

$$a_1 - b_1\mu_i,$$

and in Equations (37)–(38)

$$a_1(1-\lambda_0) + b_1[\tau_0(1-\lambda_0) + \mu_i + \tfrac{2}{3}\lambda_0].$$

For the third case – pure Rayleigh scattering – we prefer to write down the respective boundary conditions explicitly

$$a_1 + \sum_{\beta=2}^{N} a_\beta Q_i^-(\kappa_\beta) - b_1\mu_i + \sum_{\beta=2}^{N} b_\beta Q_i^+(\kappa_\beta)\,e^{-\kappa_\beta\tau_0} + \sum_{\alpha=1}^{N} c_\alpha Q_i^-(k_\alpha) + \\ + \sum_{\alpha=1}^{N} d_\alpha Q_i^+(k_\alpha)\,e^{-k_\alpha\tau_0} = -\frac{A\mu_i^2 + B}{1-\mu_i/\mu_0}, \tag{40}$$

$$a_1 + \sum_{\beta=2}^{N} a_\beta R_i^-(\kappa_\beta) - b_1\mu_i + \sum_{\beta=2}^{N} b_\beta R_i^+(\kappa_\beta)\,e^{-\kappa_\beta\tau_0} + \sum_{\alpha=1}^{N} c_\alpha R_i^-(k_\alpha) + \\ + \sum_{\alpha=2}^{N} d_\alpha R_i^+(k_\alpha)\,e^{-k_\alpha\tau_0} = -\frac{C}{1-\mu_i/\mu_0}, \tag{41}$$

$$a_1(1-\lambda_0) + \sum_{\beta=2}^{N} a_\beta \Pi_i^+(\kappa_\beta)\,\mathrm{e}^{-\kappa_\beta\tau_0} + b_1[\tau_0(1-\lambda_0) + \mu_i + \tfrac{2}{3}\lambda_0] +$$
$$+ \sum_{\beta=2}^{N} b_\beta \Pi_i^-(\kappa_\beta) + \sum_{\alpha=1}^{N} c_\alpha \Pi_i^+(k_\alpha)\,\mathrm{e}^{-k_\alpha\tau_0} +$$
$$+ \sum_{\alpha=1}^{N} d_\alpha \Pi_i^-(k_\alpha) = \Lambda^-(\mu_0) - \frac{A\mu_i^2 + B}{1+\mu_i/\mu_0}\,\mathrm{e}^{-\tau_0/\mu_0}\,, \tag{42}$$

$$a_1(1-\lambda_0) + b_1[\tau_0(1-\lambda_0) + \mu_i + \tfrac{2}{3}\lambda_0] +$$
$$+ \sum_{\alpha=1}^{N} c_\alpha P_i^+(k_\alpha)\,\mathrm{e}^{-k_\alpha\tau_0} + \sum_{\alpha=1}^{N} d_\alpha P_i^-(k_\alpha)$$
$$= \Lambda^-(\mu_0) - \frac{C}{1+\mu_i/\mu_0}\,\mathrm{e}^{-\tau_0/\mu_0}\,, \tag{43}$$

where we have used Equations (27)–(29) and (39), only

$$T_i^\pm(\kappa_\beta) = D_1^\pm(\kappa_\beta) - D_3^\pm(\kappa_\beta)$$

and

$$T_i^\pm(k_\alpha) = \tfrac{1}{2} \pm \tfrac{1}{3}k_\alpha + (1-k_\alpha^2)D_1^\mp(k_\alpha)\,. \tag{44}$$

The boundary conditions (35)–(38) or (40)–(43) give us a set of linear algebraic equations of order $4N$ which are easy to solve using any of the well-known methods (Press *et al.*, 1986).

Thus, we have completed the solution of the two-component vector transfer equation.

5. Numerical Results

A package of subroutines has been written in FORTRAN to compute the Stokes parameters in a plane-parallel homogeneous atmosphere illuminated by a parallel beam (Viik, 1989). This package has been extensively used to find out how the non-conservativeness of the atmosphere or the anisotropy of its particles influence the radiative field.

Following van de Hulst (1980), we determined the intensity of the solar radiation in the zenith and nadir when the sun is overhead or 30° above the horizon, the optical thickness of the atmosphere varying from zero to infinity. The zenith intensity displays a clear maximum in the range of optical thicknesses $\tau_0 = 0.5 - 2.5$ (Figure 1). Decrease in λ naturally decreases also the zenith intensity shifting the maximum to optically thinner atmospheres.

In thick absorbing atmospheres the nadir intensity shows the saturation effect the level of which depends on the albedo of single scattering, and which is more pronounced for the lower Sun (Figure 2).

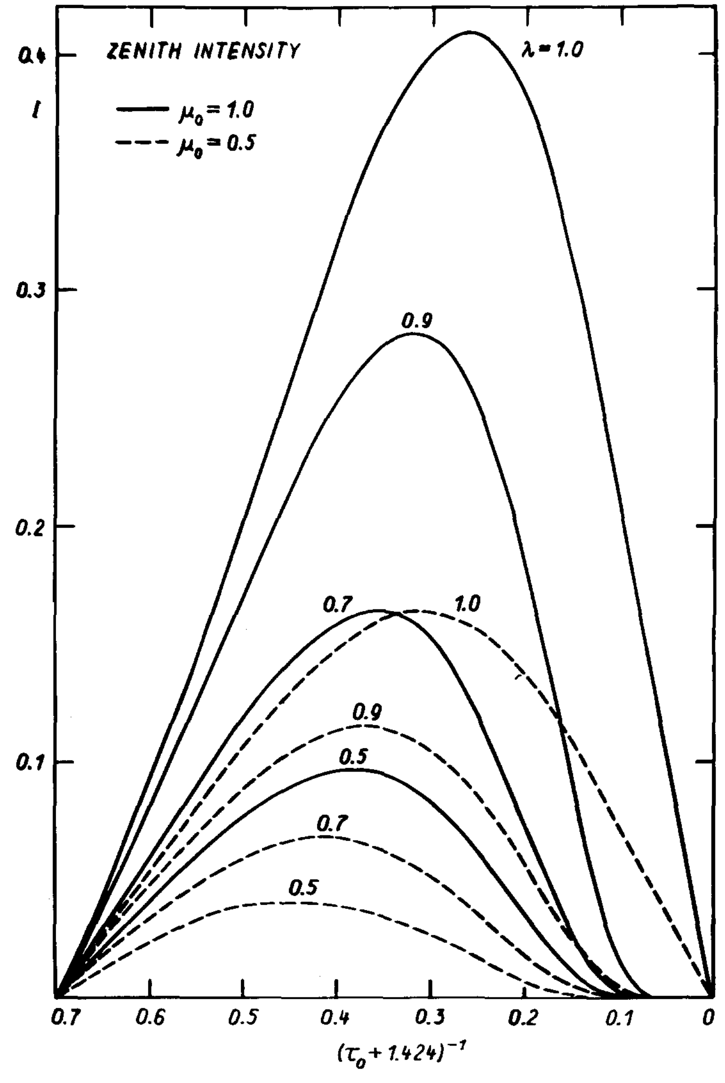

Fig. 1. The intensity of the solar radiation in the zenith as a function of the optical thickness of the Rayleigh scattering atmosphere for different angles of incidence and albedos of single scattering.

The Rubenson degree of polarization (RDP) in zenith

$$P = \frac{I_r(0, 1, \mu_0, \tau_0) - I_l(0, 1, \mu_0, \tau_0)}{I_r(0, 1, \mu_0, \tau_0) + I_l(0, 1, \mu_0, \tau_0)}$$

as a function of the optical thickness of the atmosphere is given for different altitudes of the sun, albedo of single scattering, factors of depolarization and states of polarization of the incident flux (Figures 3, 4 and 5).

It appears that for the natural incident flux ($F_l = F_r = 0.5$) the RDP is high for

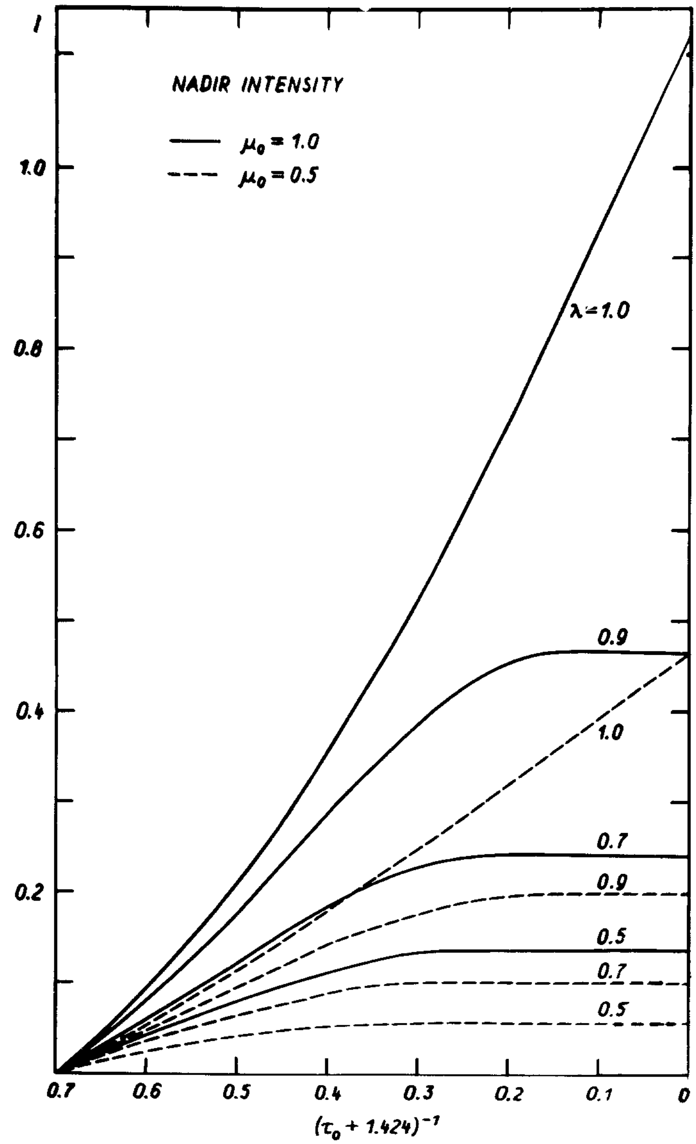

Fig. 2. The intensity of the solar radiation of the nadir as a function of the optical thickness of the Rayleigh scattering atmosphere for different angles of incidence and albedos of single scattering.

thinner atmospheres where the pure Rayleigh scattering dominates when the sun is low. The RDP decreases towards optically thicker atmospheres reaching zero for semi-infinite atmospheres. This decrease is more steep for higher values of λ (Figure 3).

If the incident flux is linearly polarized in the l direction (Figure 4) or in the r direction (Figure 5) the RDP shows roughly the same behaviour – it decreases (in absolute value) towards optically thicker atmospheres.

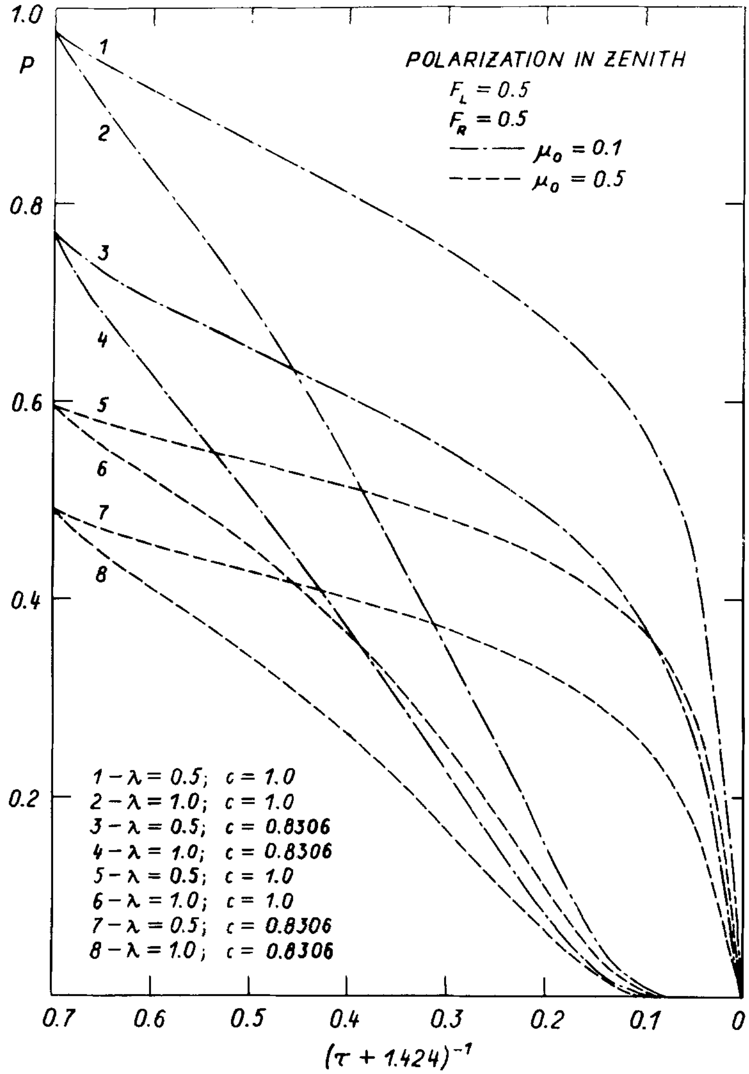

Fig. 3. The Rubenson degree of polarization at zenith for the solar radiation as a function of the optical thickness of the atmosphere, the altitude of the sun, albedo of single scattering and the factor of depolarization.

For further illustration, Figures 6, 7, 8 and 9 show the RDP of light diffusely reflected from a semi-infinite atmosphere in the principal plane as a function of λ, c and the state of polarization of the incident flux. Figure 6 for $\lambda = 1$ coincides with the Figure 16.2 of van de Hulst (1980). Since every act of scattering reduces the RDP, Figures 6 and 7 show the larger scale of RDP values for absorbing atmospheres. If the incident flux is linearly polarized in the l direction, i.e., in the principal plane, there arises a small area of negative RDP for the zenith source

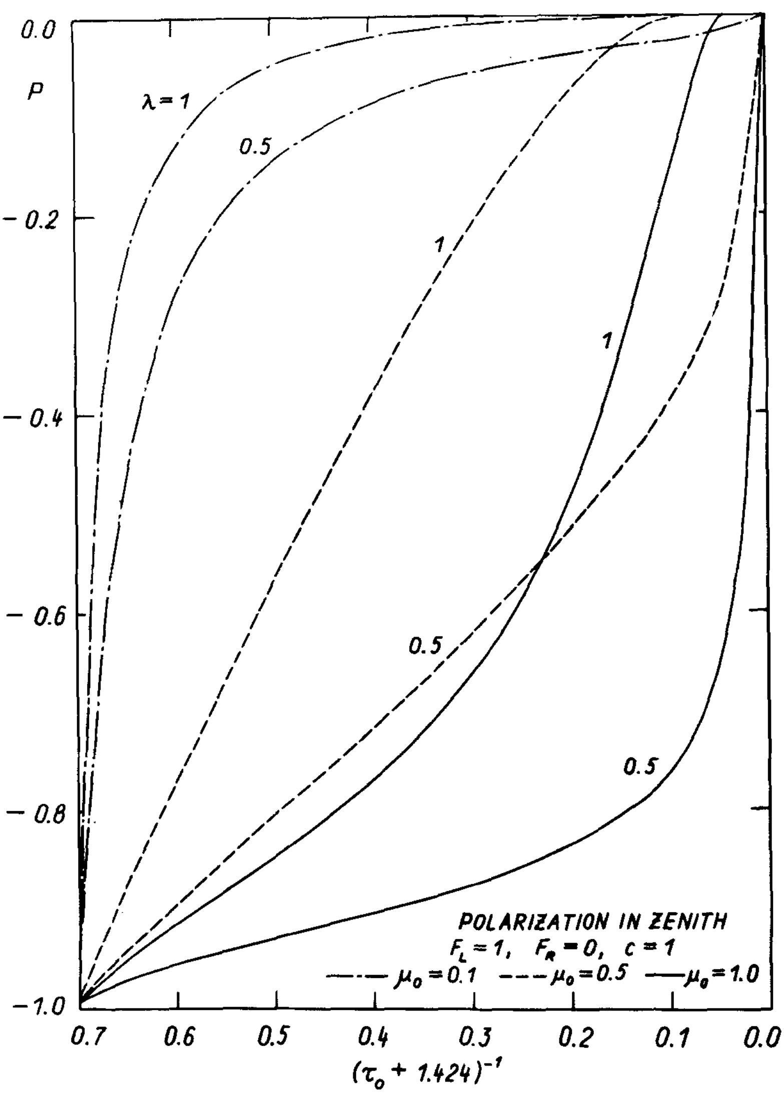

Fig. 4. The Rubenson degree of polarization at zenith for the linearly polarized incident flux ($F_L = 1$, $F_R = 0$) as a function of the optical thickness of the Rayleigh scattering atmosphere, the altitude of the source and albedo of single scattering.

and grazing view angles ($\varphi = 0$) and for the same situation in the antisolar direction (Figure 8).

If the incident flux is linearly polarized in the r direction the isolines of the RDP are symmetric with respect to the line $\mu = 1$ (Figure 9). For grazing incidence and view angles the RDP reaches nearly 100%, since the radiation at these angles cannot penetrate deep in the atmosphere and the single-scattering approximation dominates. In Figure 10 we display the isolines RDP = 0 as functions of τ_0, μ and

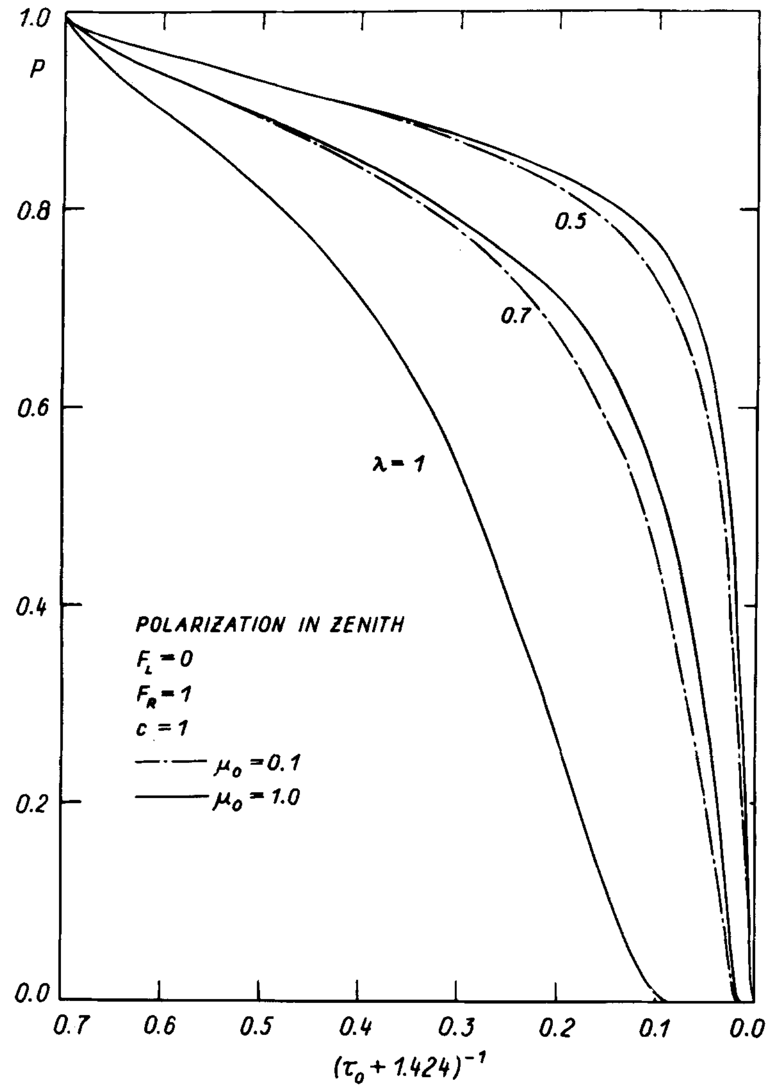

Fig. 5. The Rubenson degree of polarization at zenith for the linearly polarized incident flux (F_L 0, $F_R = 1$) as a function of the optical thickness of the Rayleigh scattering atmosphere, the altitude of the source and albedo of the single scattering.

λ for the natural incident flux at the angle ~81°.4 both for the reflected and transmitted light. In the (μ_0, τ_0) plane the increase in absorption reduces the areas with negative RDP. As indicated by Dave and Furukawa (1966) for $\lambda = 1$ we observe four neutral points for the transmitted light in region $\tau_0 \approx 1.25 - 1.65$. For $\lambda = 0.8$ this region is only $\tau_0 \approx 0.8 - 1.05$. We studied the neutral curves for the transmitted light in more detail for two values of $\lambda - 1.0$ (Figure 11) and 0.5 (Figure 12). Both figures show the neutral curves in (μ, τ_0) plane. In the conserv-

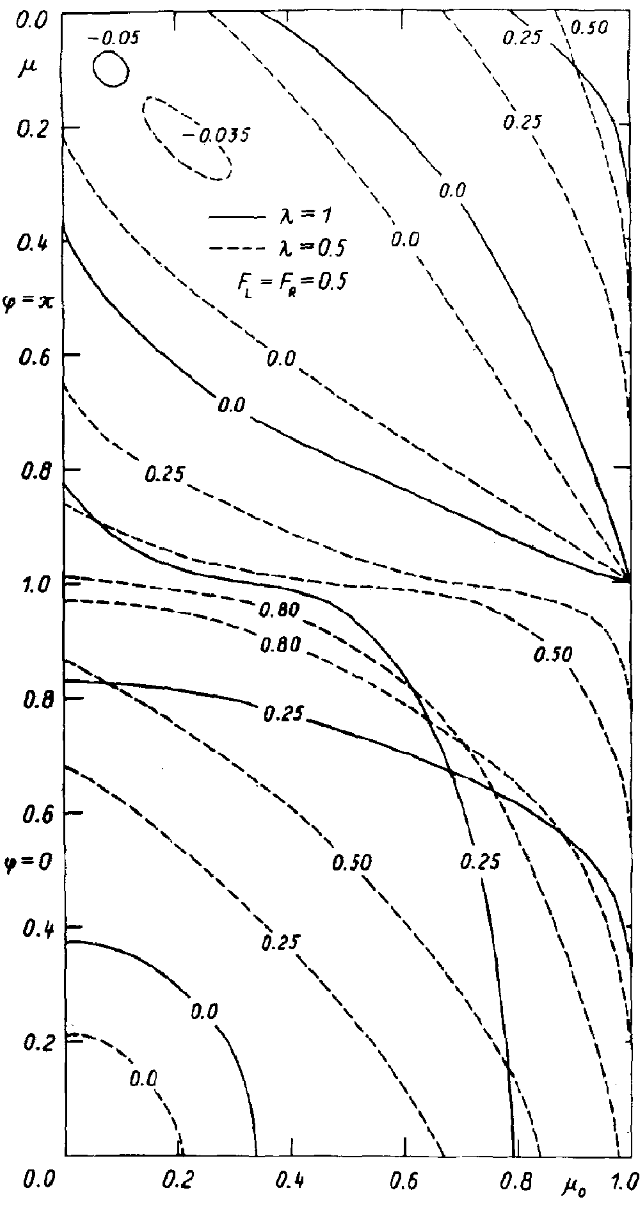

Fig. 6. The Rubenson degree of polarization for the solar radiation diffusely reflected by a semi-infinite Rayleigh scattering atmosphere in the principal plane as a function of the albedo of single scattering.

ative atmosphere the neutral curves for small μ_0 both for solar and antisolar directions are approximately symmetric and equal in area. The larger the angle arc cos μ_0 the smaller area is covered by the neutral curve in antisolar direction and if $\mu_0 \approx 0.5$ this area vanishes completely (for the absorbing atmosphere this

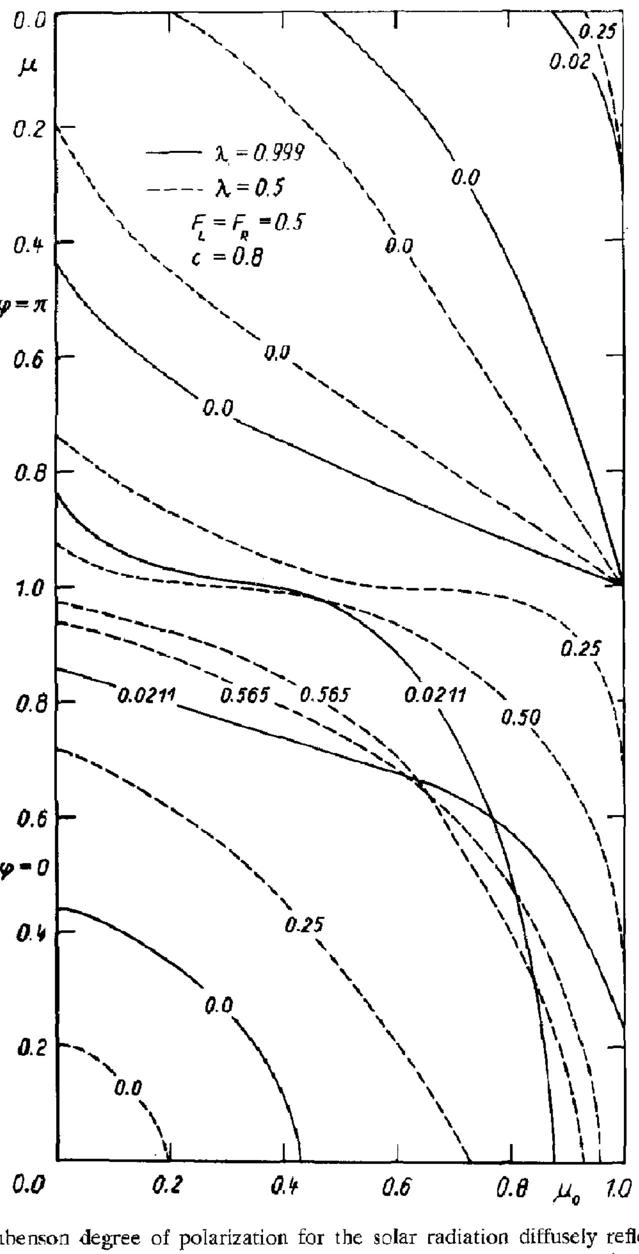

Fig. 7. The Rubenson degree of polarization for the solar radiation diffusely reflected by a semi-infinite Rayleigh–Cabannes scattering atmosphere in the principal plane as a function of the albedo of single scattering.

situation arrives at $\mu_0 \approx 0.3$). In solar directions the neutral bubble becomes larger with the growth of μ_0 and 'takes off' from the $\mu = 0$ axis. The bubble flattens and if $\mu_0 = 1$ it degenerates into the line $\mu = 1$. The evolute of the neutral curves is approximately a straight line

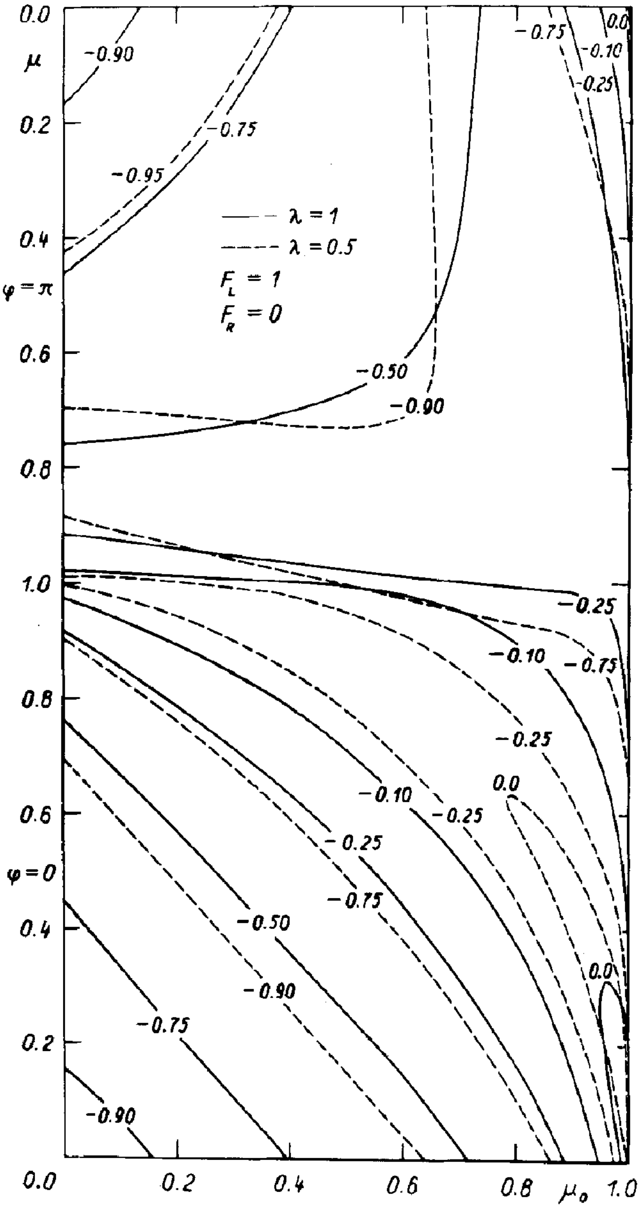

Fig. 8. The Rubenson degree of polarization for the linearly polarized incident flux ($F_L = 1$, $F_R = 0$) diffusely reflected by a semi-infinite Rayleigh scattering atmosphere in the principal plane as a function of the albedo of single scattering.

$$\mu = 0.215\tau_0 - 0.312 \qquad (\mu_0 = 0.1, \ldots, 0.9)\,.$$

The maximum optical thickness of an atmosphere for which there exists a region with negative polarization in the principal plane (for given μ_0) may be approximated by a simple formula

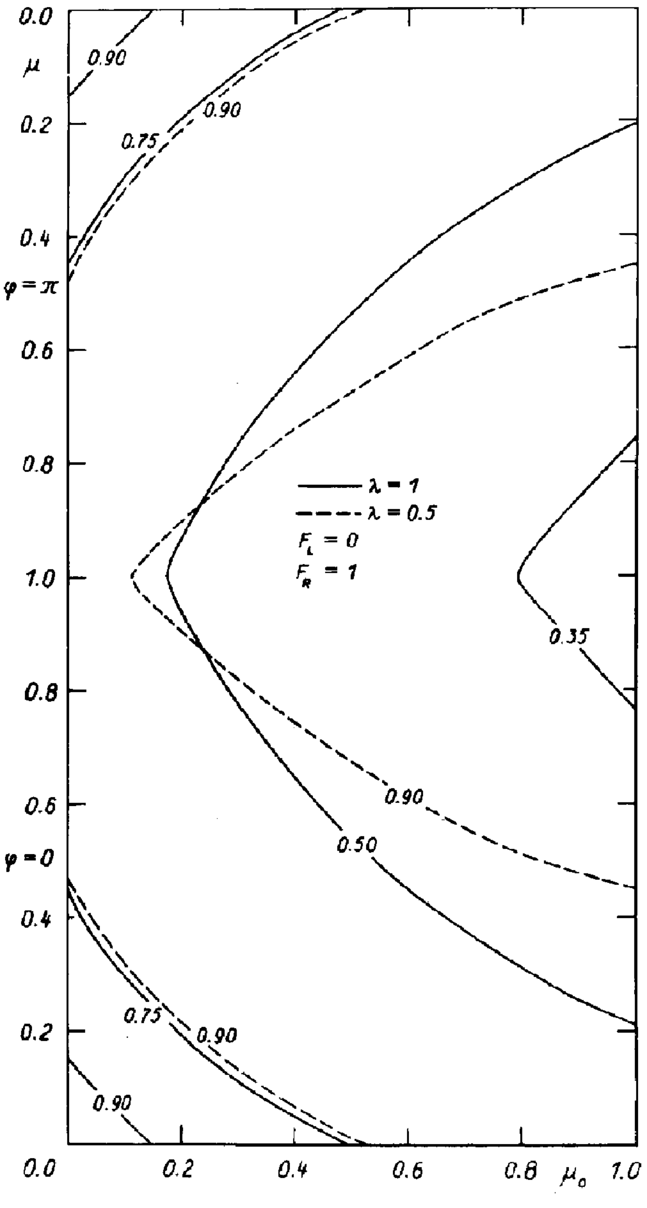

Fig. 9. The Rubenson degree of polarization for the linearly polarized incident flux ($F_L = 0$, $F_R = 1$) diffusely reflected by a semi-infinite Rayleigh scattering atmosphere in the principal plane as a function of the albedo of the single scattering.

$$\tau_0 = 0.950\mu_0^2 + 3.525\mu_0 + 1.598\ .$$

For the case $\lambda = 0.5$ (Figure 12) the largest area with negative RDP appears to be for $\mu_0 \approx 0.55$. With the further increase in the sun's altitude the area with negative RDP decreases rapidly.

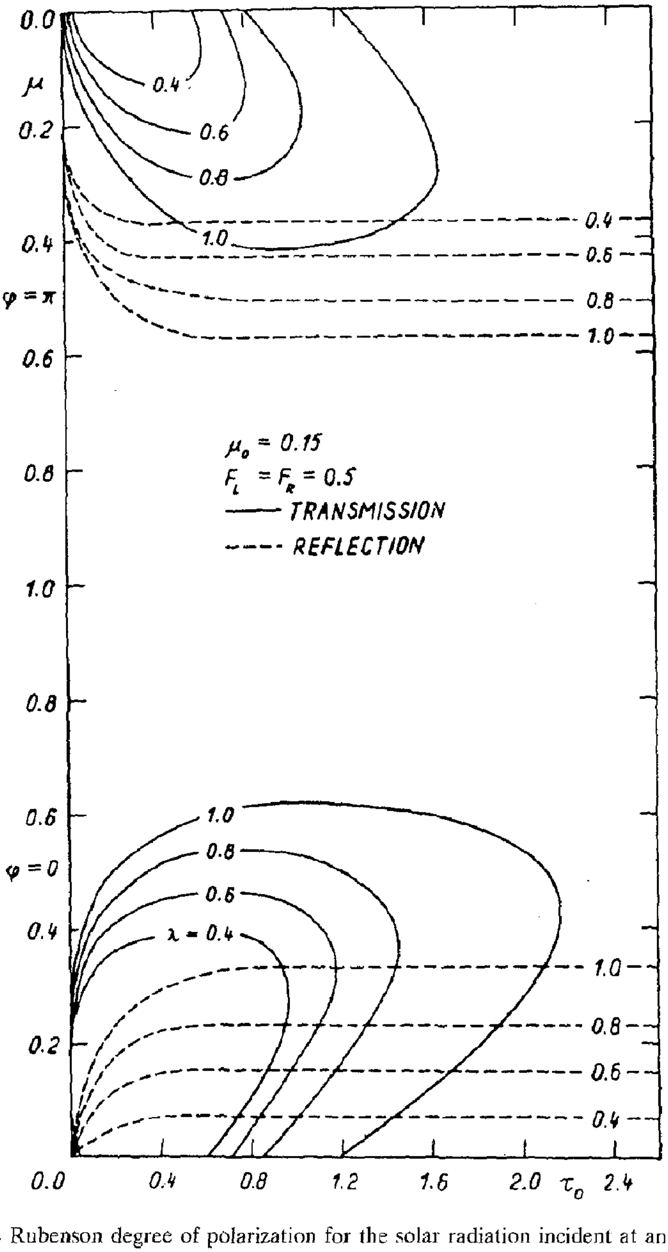

Fig. 10. The Rubenson degree of polarization for the solar radiation incident at an angle 81.4° as a function of the optical thickness of the Rayleigh scattering atmosphere and the albedo of single scattering for both the reflected and transmitted diffuse radiation.

It is of importance to estimate the accuracy of the method proposed. Firstly, we compared our results with those by Schnatz and Siewert (1971) and Siewert and Maiorino (1980). They coincided to the last given figure. Secondly, we gradually increased the number of quadrature points and compared the respective results for

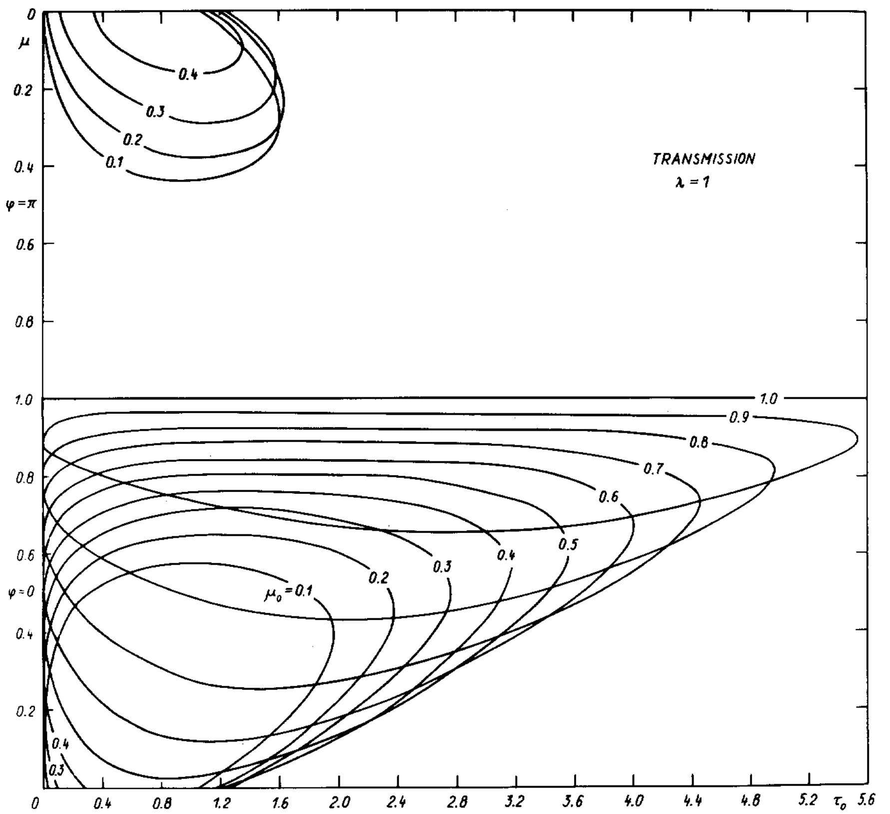

Fig. 11. The neutral curves in (μ, τ_0) plane for the conservative Rayleigh scattering atmosphere as the function of μ_0.

large number of model atmospheres. It appeared that there exists an approximate formula between the number of significant figures n and the number of quadrature points N $(N > 4)$ – namely,

$$n = \tfrac{1}{4}N + 4.$$

6. Conclusions

We discussed the solution of the vector equation of transfer in an atmosphere which absorbs radiation and scatters it according to Rayleigh–Cabannes' law. Despite the fact that the characteristic equation for the absorbing atmosphere cannot be factorized the method of discrete ordinates lends itself readily to finding the Stokes parameters both for internal and external radiation fields in a plane-parallel homogeneous atmosphere.

We suppose that this method is most suited for obtaining the benchmark results

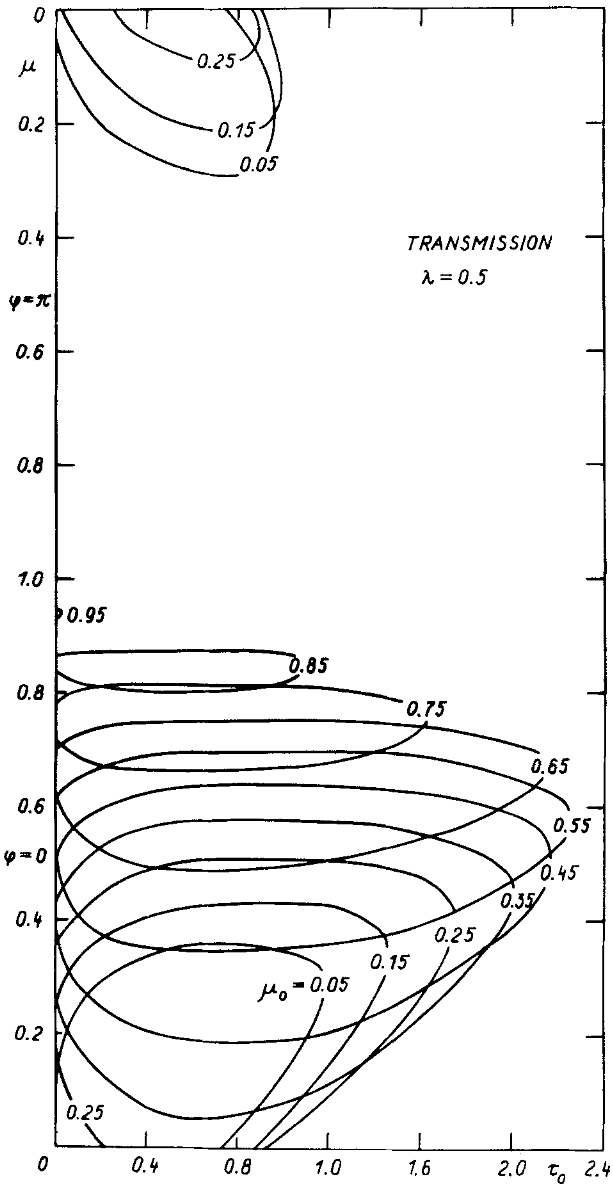

Fig. 12. The neutral curves in (μ, τ_0) plane for the non-conservative Rayleigh scattering atmosphere ($\lambda = 0.5$) as the functions of μ_0.

when elaborating powerful methods to solve the vector equation of transfer for atmospheres with realistic physical parameters.

Acknowledgements

I am grateful to Mrs V. Luud, Mrs E. Mark and Mrs M. Viik for kindly helping me at various stages of preparing the manuscript.

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