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Transport and Relaxation Properties of Superfluid ^{3}He. II. Viscosity, Thermal Conductivity, and Relaxation of the Normal Fluid Density

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The kinetic equation for Bogoliubov quasiparticles is solved in a well-controlled approximation and the kinetic coefficients of shear viscosity, second viscosity, and thermal conductivity are calculated for all temperatures. The results are in good agreement with data on the shear viscosity. Finally, the rate of relaxation of the normal fluid density from a state of disequilibrium with the superfluid component is considered.

1. INTRODUCTION

Using the scalar kinetic equation and the results for the Bogoliubov quasiparticle relaxation time derived in the preceding paper (hereafter referred to as I), we shall calculate in this paper the transport coefficients associated with the orbital part of the linearized hydrodynamics of ^{3}He-B. We shall demonstrate that, due to the isotropy of the B-phase energy gap, a quantitative calculation of the transport coefficients is possible for all temperatures, at least in the weak coupling limit and as far as the quasiparticle scattering amplitude is known or may be inferred from other experiments.

Calculations of the transport coefficients of an s-wave pairing Fermi gas have been reported by Shumeiko.[2] The shear viscosity of superfluid ^{3}He has been considered by Seiden,[3] Soda and Fujiki,[4] Shazamanian,[5] and Geilikman and Chechetkin.[6] All of these calculations had a more or less exploratory character. On the other hand, there are some exact results available on the shear viscosity and thermal conductivity at low temperatures[7] and on the shear viscosity and the second viscosity near the transition.[8,11] For the case of isotropic quasiparticle scattering there is also a

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recent calculation of the B-phase shear viscosity for the whole temperature range by Ono *et al.*[12]

The paper is organized as follows. In Section 2 we deal with the coefficient of shear viscosity and include a detailed comparison with experiment. In Section 3 the coefficients of second viscosity are calculated. Section 4 is devoted to the calculation of the diffusive thermal conductivity. In Section 5 we consider the relaxation of the normal fluid density from a state of disequilibrium with the superfluid component.

2. SHEAR VISCOSITY

The coefficient of shear viscosity η describes the response of the momentum current density $\delta\pi_{ij}$ to a transverse velocity field v_i^n imposed on the normal component

$$\delta\Pi_{ij} = -\eta\left(\frac{\partial v_i^n}{\partial x_j}+\frac{\partial v_j^n}{\partial x_i}-\frac{2}{3}\delta_{ij}\frac{\partial v_l^n}{\partial x_l}\right) \tag{1}$$

$\delta\pi_{ij}$ is expressed in terms of the Bogoliubov quasiparticle distribution function $\delta\nu_{\mathbf{p}}$ by

$$\delta\Pi_{ij} = \sum_{\mathbf{p}} p_i(\mathbf{v_p})_j\,\delta\nu'_{\mathbf{p}} \tag{2}$$

where $\delta\nu'_p$ characterizes the deviation from local equilibrium and

$$\mathbf{v_p} = \boldsymbol{\nabla}_p E_{\mathbf{p}} = (\xi_p/E_{\mathbf{p}})\mathbf{p}/m^* \tag{3}$$

is the quasiparticle velocity.

In the case of a stationary transport situation it is sufficient to approximate $\delta\nu_{\mathbf{p}}$ on the lhs of the kinetic equation (I.33) by the local equilibrium distribution in the rest frame of the moving fluid

$$\delta\nu_{\mathbf{p}} = f'_p(\delta E_p - \delta E_p^{\text{ext}}) + O(\boldsymbol{\nabla}\cdot\mathbf{v}^n) \tag{4}$$

where $\delta E_p^{\text{ext}} = \mathbf{p}\cdot\mathbf{v}^n$ denotes the energy change of a quasiparticle in the velocity field $\mathbf{v}^n$. Assuming the velocity field along the x axis and its gradient along $\hat{\mathbf{y}}$, the kinetic equation has the form

$$\frac{\xi_p}{E_p}\frac{p_y q_y}{m^*}p_x v_x^n f'_p = -\frac{i}{\tau_p}\delta\nu'_p + I_p^{\text{in}} \tag{5}$$

$\delta\nu'_p$ is seen to be an odd function of ξ_p, and I_p^{in} is consequently given by Eq. (I.59). I_p^{in} is closely connected with τ_p^{-1}, as demonstrated by the exact properties, Eqs. (I.75) and (I.76). Since the eigenfunctions of I_p^{in} are not known, an exact solution of the integral equation (5) cannot be given.

However, the dependence of the collision operator on energy and temperature is smooth and it can be hoped that good approximate solutions can be easily found. Approximate solutions may be generated in two ways. The more common method is the variational solution of the kinetic equation (5). It may be shown by standard methods that the coefficient of shear viscosity is bounded from below by

$$\eta \geq \frac{1}{5} p_F v_F \eta \frac{\left[\int_{-\infty}^{\infty} d\xi_1 f'_{p_1} \frac{\xi_1}{E_1} \Psi(\xi_1)\right]^2}{\left\{\int_{-\infty}^{\infty} d\xi_1 f'_{p_1} \frac{\xi_1}{E_1} \left[\frac{\Psi(\xi_1)}{\tau(E_1)} - \int_{-\infty}^{\infty} d\xi_2 f'_{p_2} C(\xi_1, \xi_2) \Psi(\xi_2)\right]\right\}^{-1}} \tag{6}$$

where

$$\begin{aligned} C(\xi_1, \xi_2) = & \frac{1}{\tau_N(0)} \frac{\xi_1}{E_1} \frac{\xi_2}{E_2} \prod_{i=1}^{2} \cosh \frac{E_i}{2T} \\ & \times \sum_{\mu_1 \cdots \mu_4} \int_0^{\infty} d\xi_3 \, d\xi_4 \, \delta(\mu_1 E_1 + \mu_2 E_2 + \mu_3 E_3 + \mu_4 E_4) \\ & \times \prod_{j=3}^{4} \operatorname{sech} \frac{E_j}{2T} \left[\lambda_2 + \gamma_2 \mu_3 \mu_4 \frac{\Delta^2}{E_3 E_4}\right] \end{aligned}$$

Here $\Psi(\xi)$ is a trial function. The simplest choice would be $\Psi(\xi) = \xi/E$. For this case η has been evaluated numerically by Ono *et al.*[12] with the additional approximation of isotropic scattering. In general, Ψ is expected to be given by a power series

$$\Psi(\xi) = (\xi/E) \sum_{n=-\infty}^{\infty} a_n E^n$$

The coefficients a_n can in principle be determined by seeking the minimum of the functional (6) in the space of the a_n.

Since the numerical evaluation of the integrals in Eq. (6) is rather laborious and the result for the simple trial function ξ/E is expected to be off by up to 30%, as may be inferred from a discrepancy of this size of the variational solution and the exact solution in the normal state, we follow a different line of approach. Instead of finding an approximate solution of the exact collision integral, we construct an approximation of the collision integral that allows an exact solution. The approximation should preserve as many features of the exact collision integral as possible, while at the same time allowing a closed solution. An approximation of this type is

$$I_p^{\text{in}} = i\lambda_2 \frac{\xi_p}{E_p} \frac{f'_p}{\tau_0(\xi_p)} \frac{\sum_{\mathbf{p}'} 5P_2(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')(\xi_{p'}/E_{p'})[\delta\nu'_{p'}/\tau_0(\xi_{p'})]}{\sum_{\mathbf{p}'} [f'_{p'}/\tau_0(\xi_{p'})]}$$
$$+ i\gamma_2 \frac{\xi_p}{E_p} \frac{f'_p}{\tau_I(\xi_p)} \frac{\sum_{\mathbf{p}'} 5P_2(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')(\xi_{p'}/E_{p'})[\delta\nu'_{p'}/\tau_I(\xi_{p'})]}{\sum_{\mathbf{p}'} [f'_{p'}/\tau_I(\xi_{p'})]} \tag{7}$$

where

$$\tau_0^{-1}(\xi) = \tau_N^{-1}(0, T) I_0(E), \qquad \tau_I^{-1}(\xi) = \tau_N^{-1}(0, T)\gamma_0 \frac{\Delta^2(T)}{\Delta^2(0)} I_2(E) \tag{8}$$

Equation (7) satisfies the relations for the exact collision integral given by Eqs. (I.75) and (I.76), indicating that the energy and temperature dependence of the integral operator in (7) is simulated quite closely.

In the vicinity of T_c one has $\tau_I^{-1} \ll \tau^{-1}$ and $\tau_0^{-1} \approx \tau^{-1}$. In the opposite limit of $T \to 0$ the backscattering integral I_p^{in} is small, of order T/Δ_0, due to the factors ξ/E. In order to save effort in the solution of the kinetic equation (5), we have therefore neglected τ_I^{-1} and approximated τ_0^{-1} by τ^{-1}.

Introducing a dimensionless function $\phi(\xi)$ by

$$\delta\nu'_p = -i \frac{q_y v_x^n}{m^*} p_x p_y f'_p \frac{\xi_p}{E_p} \phi(\xi_p) \tag{9}$$

and performing the angular integration in the backscattering integral I_p^{in}, one then obtains the separable integral equation

$$\tau(\xi) = \phi(\xi) - \lambda_2 \frac{\langle(\xi^2/E^2)\phi(\xi)/\tau\rangle_{f'}}{\langle 1/\tau\rangle_{f'}} \tag{10}$$

where we have introduced the abbreviation

$$\langle A(\xi)\rangle_{f'} = \int_{-\infty}^{\infty} d\xi\, (-f'(E)) A(\xi)$$

Equation (10) is readily solved for $\phi(\xi)$. Substituting the result into Eq. (5), one finds for the shear viscosity coefficient

$$\eta = \frac{1}{5} n p_F v_F \left[\left\langle \frac{\xi^2}{E^2} \tau \right\rangle_{f'} + \lambda_2 \frac{\langle \xi^2/E^2 \rangle_{f'}^2}{\langle (1 - \lambda_2 \xi^2/E^2)(1/\tau) \rangle_{f'}} \right] \tag{11}$$

Here $\tau(E)$ is the exact quasiparticle lifetime given by Eq. (I.65).

In the normal state we have $\xi^2/E^2 = 1$. Noting the simple structure of $\tau(\xi)$ given by Eq. (I.73), the energy integrals may be evaluated analytically, with the result

$$\eta(T)=\frac{1}{5}np_{\mathrm{F}}v_{\mathrm{F}}\tau_{\mathrm{N}}(0,T)f_\eta(\lambda_2); \qquad T\geqslant T_c \tag{12}$$

where

$$f_\eta(\lambda_2)=\tfrac{1}{12}\pi^2+\tfrac{3}{4}\lambda_2/(1-\lambda_2) \tag{13}$$

Due to the T^2 dependence of the relaxation rate $\tau_N(0,T)$, η is proportional to T^{-2}. Comparing the result (12) with the exact result,[9,10] it is found that the corrections are less than 1% for all values of $\lambda_2(0<\lambda_2<1)$. Thus for all practical purposes the result (12) is exact.

The first term in an expansion of η in powers of Δ/T_c below T_c may be easily calculated. Observing that

$$\lim_{\Delta\to 0}\frac{\xi^2}{E^2}=1-\Delta\lim_{\Delta\to 0}\frac{\Delta}{\xi^2+\Delta^2}=1-\pi\Delta\,\delta(\xi) \tag{14}$$

we can do the energy integrals in Eq. (11) with the aid of the delta function. The result is

$$\eta(T)/\eta(T_c)=1-a(\lambda_2)[1-(T/T_c)]^{1/2} \tag{15}$$

where

$$a(\lambda_2)=\frac{\pi^2}{4}\left(\frac{2}{3}\frac{\Delta C}{C_N}\right)^{1/2}\frac{[1+\frac{3}{4}\lambda_2/(1-\lambda_2^2)]^2}{f_\eta(\lambda_2)} \tag{16}$$

Here we have inserted $\Delta(T)$ as given by (I.83). $a(\lambda_2)$ is a strong function of λ_2 for λ_2 values of interest ($0.5\leq\lambda_2\leq 0.8$). The function $a(\lambda_2)$ may also be calculated by employing the exact eigenfunctions of the normal-state collision operator.[8] Equation (16) deviates from the exact result by higher order terms in the eigenstate expansion, which are typically of the order of 1%.

In the limit of low temperatures, $T/\Delta\ll 1$, the corresponding limit of the qp relaxation rate, Eq. (I.79), may be substituted in Eq. (11). It is seen that the contribution from the backscattering term I_p^{in} is of order T/Δ_0 small and can be neglected. Thus the dependence of the viscosity on the parameter λ_2 drops out. As noted by Pethick *et al.*,[7] the viscosity is given by the usual gas kinetic expression

$$\eta(T=0)=\tfrac{1}{5}\rho_n\overline{v_p^2}\tau \tag{17}$$

where

$$\rho_n=\sum_{\mathbf{p}}p_x^2(-f_p')=m^*(p_{\mathrm{F}}^3/3\pi^2)Y(T) \tag{18}$$

is the mass density of the normal component [Y is defined by Eq. (I.78)] and

$$\overline{v_p^2}=\sum_{\mathbf{p}}\left(\frac{\mathbf{p}}{m^*}\right)^2\left(\frac{\xi_p}{E_p}\right)^2(-f_p')\Big/\sum_{\mathbf{p}}(-f_p')=\frac{T}{\Delta_0}v_F^2 \tag{19}$$

is the averaged thermal quasiparticle velocity squared. It is remarkable that the temperature dependences of ρ_n, $\overline{v_p^2}$, and τ compensate such that η attains a temperature-independent limiting value for $T\to 0$, which is of the order of the viscosity at T_c on dimensional grounds. Inserting the expression (I.79) for τ, one finds

$$\frac{\eta(T=0)}{\eta(T_c)}=\frac{2}{3}\pi\left(\frac{T_c}{\Delta_0}\right)^2\frac{1}{f_\eta(\lambda_2)w_0} \tag{20}$$

with $f_\eta(\lambda_2)$ and w_0 defined by Eqs. (11) and (I.79), respectively. The exact result by Pethick *et al.*[7] is recovered from Eq. (20).

We have evaluated Eq. (11) numerically with $\tau(E)$ and $\Delta(T)$ given by Eqs. (I.65) and (I.83) for various values of λ_2. For purposes of comparison with other theoretical results, we first consider the case of isotropic quasiparticle scattering, i.e., $A_s(\theta,\phi)=S_0$, $A_t(\theta,\phi)=0$. In this case one has $\lambda_2=\frac{1}{5}$. In Fig. 1 the normalized viscosity coefficient is plotted vs. $(1-T/T_c)^{1/2}$. The solid curve is the result of the present calculation, the dashed curve represents the result of a variational calculation by Ono *et al.*,[12] and the limiting behavior as $T\to T_c$ and as $T\to 0$ according to Pethick *et al.*[7,8] is shown as dash-dotted straight lines. The agreement is good in general. The deviation of the dashed line from the exact low-temperature limit is caused by the variational approximation for $\eta(T_c)$. It is seen that the straight line at T_c approaches $\eta(T)$ only very close to T_c.

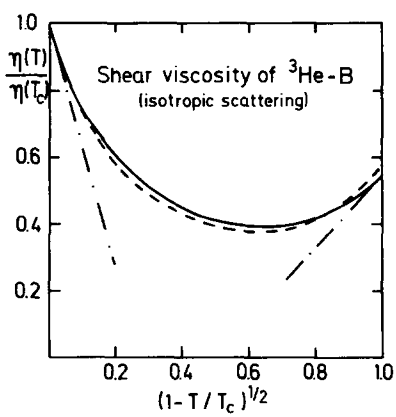

Fig. 1. Normalized shear viscosity of the BW state for isotropic scattering vs. $(1-T/T_c)^{1/2}$: our theory (solid curve), exact asymptotic results[7,8] (dash-dotted straight lines), variational calculation[12] (dashed line).

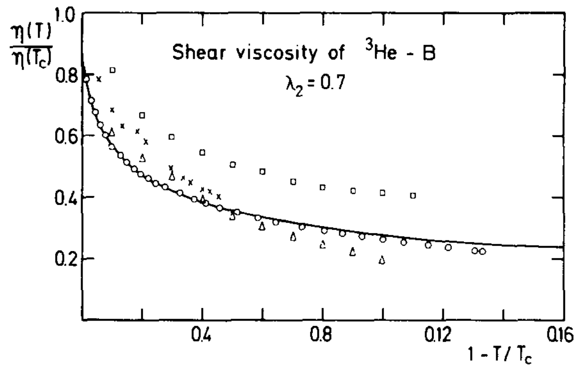

Fig. 2. Shear viscosity of ^{3}He-B normalized to the value at T_c vs. reduced temperature near 20 bar pressure: our theory (solid curve), heat flow data after Johnson *et al.*[13] (squares), torsion pendulum data after Main *et al.* (triangles), Guernsey *et al.*[15] (crosses), and Parpia *et al.*[16] (circles).

The scattering of quasiparticles in ^{3}He-B is far from isotropic. Attempts to calculate η in the normal state by employing the $s-p$ approximation for the scattering amplitude have been successful at vapor pressure, but yield results that are too small by a factor of two in the range from 5 bar to the melting pressure. We therefore adopt the point of view that $\tau_N(0)$ and λ_2 are parameters to be determined from other experiments. Given values of $\tau_N(0)$ extracted from orbital and spin relaxation measurements as well as from the sound absorption closely below T_c, λ_2 may be calculated from the shear viscosity or the zero-sound attenuation in the normal state. In this way values for λ_2 ranging from 0.65 at melting pressure to about 0.7 at 20 bar are found.

In Fig. 2 the result of a numerical evaluation of Eq. (11) for $\lambda_2 = 0.7$ and values of γ_0 and δ_0 quoted in Section 4 of I is compared with heat flow data of the La Jolla group,[13] and torsion pendulum data of the Manchester group,[14] and of the Columbia group.[15] All of these experiments were of an exploratory nature. In the calculation $\Delta(T)$ was approximated by Eq. (I.83), with $\Delta C/C_N = 1.55$.

Also shown in Fig. 2 are the recent, very precise torsion pendulum data of the Cornell group.[16]

In Fig. 3 our theoretical result for $\lambda_2 = 0.65$, $\gamma_0 = 0.16$, and $\delta_0 = 0.29$, as well as a specific heat discontinuity of $\Delta C/C_N = 2.0$, appropriate at melting pressure, is compared with the vibrating wire data of the Helsinki group.[17] The agreement of theory with experiment would be even better if we had allowed for strong coupling enhancement of the zero-temperature gap in Eq. (I.83). The deviation at low temperatures is presumably due to finite mean free path effects.

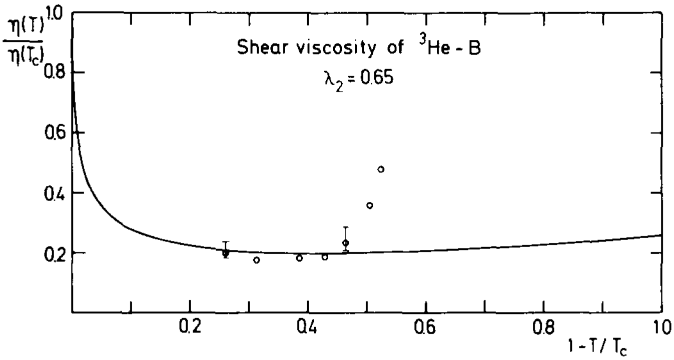

Fig. 3. Shear viscosity of ^{3}He-B normalized to the value at T_c vs. reduced temperature at melting pressure: our theory (solid curve), vibrating wire data after Alvesalo *et al.*[17] (circles).

In Fig. 4 a comparison of theory is made with the high-precision data of the Cornell group[16] by plotting the square of the relative change of the viscosity near T_c against T/T_c. The agreement is seen to be quite satisfactory, apart from a systematic deviation of these viscosity data toward lower values at lower temperatures. It has been suggested[18] that this is

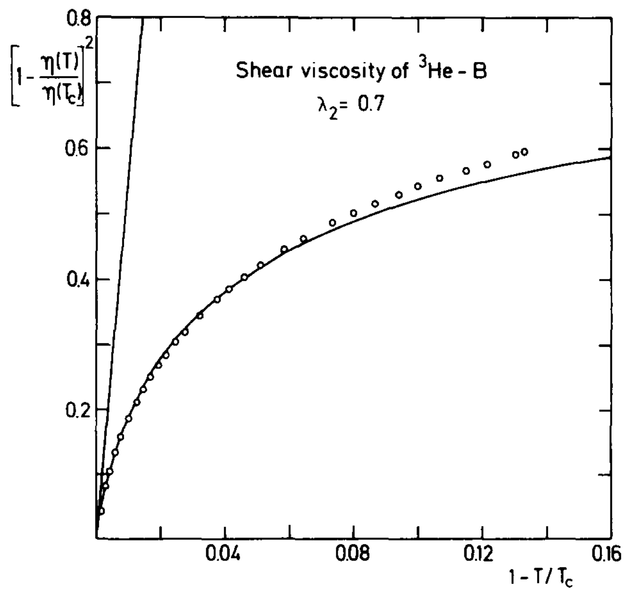

Fig. 4. Normalized change of the shear viscosity squared of ^{3}He-B vs. reduced temperature: our theory (solid curve), torsion pendulum data after Reppy *et al.*[16] (circles). Asymptotic behavior near T_c is shown by the straight line.[8]

probably associated with finite mean free path effects in the narrow slab geometry (50 μm) used in this experiment.

In a temperature region of width $10^{-3}T_c$ about the transition a behavior linear in $1-T/T_c$ of $\eta-\eta(T_c)$ was observed in the Cornell experiment. Such behavior has been predicted by one of the authors for the so-called gapless regime $\tau\Delta \ll 1$ [Eq. (83) of Ref. 19]. It turns out, however, that the coefficient of $(1-T/T_c)$ is larger than the experimentally observed one by a factor of 40. The observed behavior is probably due to fluctuations above T_c.

3. SECOND VISCOSITY

The hydrodynamic equations of an isotropic superfluid contain three coefficients of second viscosity appearing in the momentum conservation law and the acceleration equations of the superfluid.

The corresponding terms in the stress tensor are

$$\Pi_{ij} = -\delta_{ij}\{\zeta_1 \operatorname{div}[\rho_s(\mathbf{v}^s-\mathbf{v}^n)]+\zeta_2 \operatorname{div}\mathbf{v}^n\} \tag{21}$$

In the limit $T\to T_c$ the ζ_1 term vanishes, while ζ_2 is identified with the usual coefficient of bulk viscosity. ζ_1 and ζ_2 describe the response of the momentum current to longitudinal velocity fields $\mathbf{v}_n$ and $\mathbf{v}_s$. The change in the distribution function induced by such a disturbance is isotropic. The diagonal part of the momentum current is given by

$$\begin{aligned}\tfrac{1}{3}\operatorname{tr}[\Pi_{ij}] &= \tfrac{1}{3}\sum_{\mathbf{p}} \mathbf{p}\nabla_p E_p\,\delta\nu'_p \\ &= \tfrac{1}{3}\sum_{\mathbf{p}} p v_p(\xi_p/E_p)\,\delta\nu'_p \\ &= \tfrac{1}{3}p_F v_F \sum_{\mathbf{p}} (\xi_p/E_p)\,\delta\nu'_p + O(T^2/\varepsilon_F^2)\end{aligned} \tag{22}$$

with

$$\begin{aligned}\delta\nu'_p &= \delta\nu_p - f'_p\frac{\xi_p}{E_p}(\delta\varepsilon_p-\delta\mu) \\ &= \delta\nu_p - f'_p\frac{\xi_p}{E_p}\left(f_0-\frac{\partial\mu}{\partial n}\right)\delta n + O\left(\frac{T^2}{\varepsilon_F^2}\right)\end{aligned}$$

Taking into account $\partial\mu/\partial n = f_0 - N_F^{-1}$ and the expression for the density change

$$\delta n = \sum_{\mathbf{p}}\left[\frac{\xi_p}{E_p}\delta\nu_p - \frac{\Delta^2}{2E_p^3}\left(\tanh\frac{E_p}{2T}\right)(\delta\varepsilon_p-\delta\mu)\right] \tag{23}$$

it is readily verified that

$$\sum_{\mathbf{p}} \frac{\xi_p}{E_p} \delta\nu'_p = 0 \tag{24}$$

and therefore

$$\tfrac{1}{3}\,\mathrm{tr}\,[\Pi_{ij}] = O(T^2/\varepsilon_F^2) \tag{25}$$

In the case of ζ_2 there is an additional factor of T_c^2/ε_F^2 from the streaming part of the kinetic equation such that ζ_2 is of order T^4/ε_F^4 small compared to the coefficient of first viscosity. ζ_1 is of order T^2/ε_F^2 small. Both coefficients are therefore negligible in ^{3}He-B.

The remaining coefficient ζ_3 governs the response of the chemical potential to normal–superfluid counterflow, which introduces dissipation into the equation of motion of $\mathbf{v}_s$,

$$m\dot{\mathbf{v}}^s = -\nabla\{\mu^l - m\zeta_3\,\mathrm{div}\,[\rho_s(\mathbf{v}^s - \mathbf{v}^n)] - m\zeta_1\,\mathrm{div}\,\mathbf{v}^n\} \tag{26}$$

Here μ^l is the local equilibrium value of the chemical potential. $\delta\mu^l$ is determined by the requirement that the local equilibrium distribution

$$\delta\nu_p^l = f'_p \frac{\xi_p}{E_p}(\delta\varepsilon_p - \delta\mu^l) \tag{27}$$

gives the correct density change,

$$\delta n^l = \sum_p \left[\frac{\xi_p}{E_p}\delta\nu_p^l - \frac{\Delta^2}{2E^3}\left(\tanh\frac{E_p}{2T}\right)(\delta\varepsilon_p^l - \delta\mu^l)\right] \equiv \delta_n \tag{28}$$

The isotropic part of $\delta\varepsilon_p$, given by

$$\delta\varepsilon_p = f_0\,\delta n \tag{29}$$

is equal to its local equilibrium value.

Taking the difference of δn and δn^l, one obtains

$$\delta n - \delta n^l = \sum_p \left[\frac{\xi_p}{E_p}(\delta\nu_p - \delta\nu_p^l) + \frac{\Delta^2}{2E_p^3}\left(\tanh\frac{E_p}{2T}\right)(\delta\mu - \delta\mu^l)\right] = 0 \tag{30}$$

Introducing $\delta\mu' = \delta\mu - \delta\mu^l$, the deviation from local equilibrium, and recalling that $\delta\nu'_p$ in the kinetic equation is defined by

$$\delta\nu'_p = \delta\nu_p - f'_p(\xi_p/E_p)(\delta\varepsilon_p - \delta\mu) = \delta\nu_p - \delta\nu_p^l + f'_p(\xi_p/E_p)(\delta\mu - \delta\mu^l)$$

we derive the following relation between $\delta\mu'$ and $\delta\nu'_p$ from Eq. (30):

$$\sum_{\mathbf{p}} \left(-\frac{\xi_p^2}{E_p^2} f'_p + \frac{\Delta^2}{2E_p^3}\tanh\frac{E_p}{2T}\right)\delta\mu' = N_F\,\delta\mu' = -\sum_{\mathbf{p}} \frac{\xi_p}{E_p}\delta\nu'_p \tag{31}$$

Here N_F may be interpreted as the thermodynamic derivative of the density with respect to the chemical potential at fixed quasiparticle distribution.

The left-hand side of the kinetic equation in this case is approximated by substituting the local equilibrium distribution function

$$\delta\nu_p^{\text{loc}} = f'_p(\delta E_p^l - \mathbf{p}\cdot\mathbf{v}^n) \tag{32}$$

with the isotropic part of δE_p^l given by

$$\delta E_p^l = \frac{\xi_p}{E_p}(f_0\ \delta n - \delta\mu^l) = \frac{\xi_p}{E_p}\frac{1}{N_F}\delta n \tag{33}$$

Making use of the continuity equation to eliminate $\omega\ \delta n$ in favor of $\mathbf{q}\cdot\mathbf{j}$, we obtain for the isotropic part of the kinetic equation

$$f'_p\frac{\xi_p}{E_p}\frac{1}{N_F}\mathbf{q}(\mathbf{j}-\rho\mathbf{v}^n) = -\frac{i}{\tau_p}\delta\nu'_p + I_p^{\text{in}} \tag{34}$$

The relevant part of the inscattering term, isotropic and odd in ξ_p, is approximated by

$$I_p^{\text{in}} = \frac{i}{\tau_p}\frac{\xi_p}{E_p}f'_p\frac{\sum_{\mathbf{p}'}(\xi_{p'}/E_{p'})(\delta\nu'_{p'}/\tau_{p'})}{\sum_{\mathbf{p}'}f'_{p'}/\tau_{p'}} \tag{35}$$

The expression (35) again satisfies approximately the relations (I.75) and (I.76) of the exact inscattering integral. A more precise approximation involving two relaxation times could again be given in a form similar to Eq. (7).

Introducing a dimensionless function $\phi(\xi)$ by

$$\delta\nu'_p = -i\frac{\xi_p}{E_p}f'_p\frac{1}{N_F}\mathbf{q}(\mathbf{j}-\rho\mathbf{v}^n)\phi(\xi_p) \tag{36}$$

we can write the kinetic equation

$$\tau(\xi) = \phi(\xi) - \frac{\langle(\xi^2/E^2)\phi(\xi)/\tau\rangle_{f'}}{\langle 1/\tau\rangle_{f'}} \tag{37}$$

This separable integral equation is easily solved. Substituting the result into Eq. (34), we find the second viscosity coefficient as

$$\zeta_3 = \frac{1}{N_F}\left[\left\langle\frac{\xi^2}{E^2}\tau\right\rangle_{f'} + \frac{\langle\xi^2/E^2\rangle_{f'}^2}{\langle(\Delta^2/E^2\tau)\rangle_{f'}}\right] \tag{38}$$

Near the transition, the second term in the square brackets diverges as $1/\Delta$,

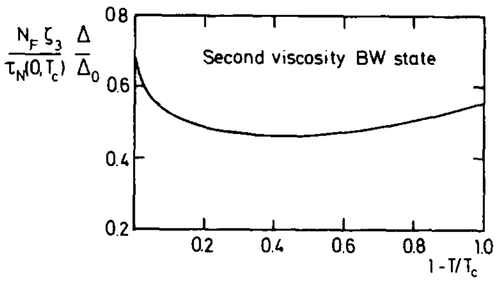

Fig. 5. Normalized second viscosity coefficient ζ_3 for the BW state vs. reduced temperature.

and one has

$$\zeta_3 = \frac{4}{m^* p_F} \tau_N(0) \left(\frac{2}{3} \frac{\Delta C}{C_N}\right)^{-1/2} \left(1 - \frac{T}{T_c}\right)^{-1/2} \tag{39}$$

This result has been derived previously by one of the authors by exactly solving the kinetic equation. Note, however, that the factor $\partial\mu/\partial n$ in Eq. (14) of Ref. 11 has to be taken at fixed qp distribution, i.e., $\partial\mu/\partial n|_\nu = N_F^{-1}$. The numerical estimates of ζ_3 in Ref. 11 therefore are too large by a factor of F_0^s.

In the limit of low temperatures ζ_3 becomes temperature independent

$$\lim_{T\to 0} \zeta_3 = \frac{2\pi}{3} \frac{1}{N_F} \left(\frac{T_c}{\Delta_0}\right)^2 \frac{1}{w_0} \tau_N(0, T_c) \tag{40}$$

This result is again exact.

The result of a numerical evaluation of Eq. (38) is plotted in Fig. 5 as $\zeta_3[\Delta(T)/\Delta(0)]/N_F\tau_N(0, T_c)$ vs. $1 - T/T_c$ using the values $\gamma_0 = 0.12$ and $\delta_0 = 0.29$ appropriate for a pressure of 20 bar.

4. THERMAL CONDUCTIVITY

The coefficient of thermal conductivity characterizes the response of the heat current $\mathbf{j}_Q$ to a temperature gradient

$$\mathbf{j}_Q = -\kappa \nabla T \tag{41}$$

Equation (41) describes heat transport by a random diffusion process of the thermal excitations. In addition, in a superfluid there is a convective contribution to the heat current, $\mathbf{j}_Q^c = S\mathbf{v}_n$ even in the absence of mass flow, due to the possibility of normal–superfluid counterflow. In the vicinity of T_c the convective transport process is much more effective than the diffusive one and it is difficult to measure κ. At lower temperatures the diffusive processes take over, since $\mathbf{v}_n$ is bounded by friction and by the finiteness of the qp velocity.

The diffusive heat current may be expressed in terms of the quasiparticle distribution function $\delta\nu_p$ as

$$\mathbf{j}_Q = \sum_{\mathbf{p}} E_p \nabla_p E_p \, \delta\nu'_p \tag{42}$$

In the presence of a stationary diffusive heat current the distribution function is close to a local equilibrium distribution function characterized by the local temperature deviation δT

$$\delta\nu_p = f'_p[\delta E_p - (E_p/T)\,\delta T] + O(\nabla T) \tag{43}$$

Substituting this into the left-hand side of the kinetic equation (I.33), one finds

$$\frac{\mathbf{p}\cdot\mathbf{q}}{m^*}\frac{\xi_p}{E_p} f'_p \frac{E_p}{T}\,\delta T = -\frac{i}{\tau_p}\,\delta\nu'_p + I_p^{\text{in}} \tag{44}$$

From the symmetry of the lhs of Eq. (44), it follows that the solution $\delta\nu'_p$ is odd in ξ_p and has $l=1$ angular symmetry. The energy dependence of the driving term has changed, as compared to the case of the viscosity, by the additional factor of E_p. Taking this modification into account, the inscattering term of the collision integral is approximated by

$$\begin{aligned} I_p^{\text{in}} &= i\lambda_1^- \frac{f'_p}{\tau_0(\xi_p)}\xi_p \frac{\sum_{\mathbf{p}'} \xi_{p'} P_1(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')[\delta\nu'_{p'}/\tau_0(\xi_{p'})]}{\sum_{\mathbf{p}'} f'_{p'}[E_{p'}^2/\tau_0(\xi_{p'})]} \\ &\quad + i\gamma_1 \frac{f'_p}{\tau_I(\xi_p)}\xi_p \frac{\sum_{\mathbf{p}'} \xi_{p'} P_1(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')[\delta\nu'_{p'}/\tau_I(\xi_{p'})]}{\sum_{\mathbf{p}'} f'_{p'}[E_{p'}^2/\tau_I(\xi_{p'})]} \end{aligned} \tag{45}$$

with λ_1^- and γ_1 given by Eq. (I.71) and τ_0 and τ_I defined by Eq. (8).

The expression (45) is devised such that it shares the properties stated in Eqs. (I.75) and (I.76) with the exact collision integral, viz.,

$$\sum_{\mathbf{p}} Y_{1m}(\hat{p})\frac{E_p^2}{\xi_p} I_p^{\text{in}} = \sum_{\mathbf{p}} \frac{1}{3}\left(\frac{\lambda_1^-}{\tau_0(\xi_p)} + \frac{\gamma_1}{\tau_I(\xi_p)}\right) Y_{1m}(\hat{p})\,\delta\nu'_p \tag{46}$$

and

$$I_p^{\text{in}}\left\{ Y_{1m}(\hat{p})\frac{E_p^2}{\xi_p} f'_p \right\} = \frac{1}{3}\left(\frac{\lambda_1^-}{\tau_0(\xi_p)} + \frac{\gamma_1}{\tau_I(\xi_p)}\right) Y_{1m}(\hat{p})\,\delta\nu'_p \tag{47}$$

Adopting the argument following Eq. (8), we drop τ_I^{-1} and approximate

τ_0^{-1} by τ^{-1}. Introducing a dimensionless function $\phi(\xi)$ by

$$\delta\nu'_p = i\frac{\mathbf{p}\cdot\mathbf{q}}{m^*}\frac{\xi_p}{E_p}f'_p\frac{E_p}{T}\delta T\,\phi(\xi_p) \tag{48}$$

we find that the kinetic equation takes the form

$$\tau(\xi) = \phi(\xi) - \frac{\lambda_1^-}{3}\frac{\langle\xi^2\phi(\xi)(1/\tau)\rangle_{f'}^2}{\langle E^2/\tau\rangle_{f'}} \tag{49}$$

Solving for ϕ and substituting the result into Eq. (48), we find the coefficient of thermal conductivity as

$$\kappa = \frac{1}{3}\frac{n}{m^*T}\left[\langle\xi^2\tau\rangle_{f'} + \frac{\lambda_1^-}{3}\frac{\langle\xi^2\rangle_{f'}^2}{\langle(E^2-\frac{1}{3}\lambda_1^-\xi^2)(1/\tau)\rangle_{f'}}\right] \tag{50}$$

In the normal state ($T \geq T_c$) the energy integrals in Eq. (50) may be evaluated analytically with the result

$$\kappa = \frac{1}{3}C_N v_F^2 \tau_N(0)\left[3 - \frac{\pi^2}{4} + \frac{5}{12}\frac{\lambda_1^-}{3-\lambda_1^-}\right] \tag{51}$$

where $C_N = \frac{1}{3}\pi^2 N_F T$ is the specific heat of the normal Fermi liquid. $\varkappa$ is proportional to T^{-1}. Equation (51) deviates from the exact result[9,10] by typically 1%. In contrast to the viscosity, where the singular behavior of the quasiparticle velocity $\nabla_p E_p$ as a function of ξ_p for $T \to T_c$ caused the deviation of η from $\eta(T_c)$ to be nonanalytic, $\sim(1-T/T_c)^{1/2}$, the thermal conductivity varies smoothly as

$$\kappa(T) = \kappa(T_c) + a_\kappa(1 - T/T_c), \qquad T \simeq T_c \tag{52}$$

due to the smooth behavior of the energy current per quasiparticle, $E_p\nabla_p E_p$. The coefficient a_κ cannot be evaluated so easily from the exact kinetic equation, because it involves contributions from a variety of different sources.

In the limit $T \to 0$, the inscattering part of the collision integral vanishes due to the factors ξ/E, ξ'/E' scaling as T/Δ_0 and the kinetic equation can be solved exactly. Since this property is conserved by our approximate treatment, the result (50) reduces to the exact result in this limit. It turns out that κ diverges as $1/T$ as in the normal state, with a prefactor given by

$$\lim_{T\to 0}(\kappa T) = \frac{2}{3\pi}\frac{C_N}{T}v_F^2[T^2\tau_N(0)]\frac{1}{w_0} \tag{53}$$

The prefactor is roughly equal to the one in the normal state.

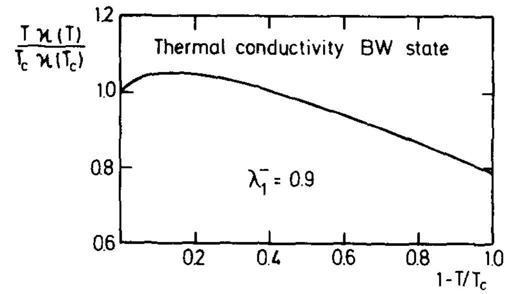

Fig. 6. Normalized thermal conductivity for the BW state at about 20 bar vs. reduced temperature.

We have evaluated Eq. (50) numerically using $\tau(\xi)^{-1}$ and $\Delta(T)$ as given by Eqs. (I.65) and (I.83), respectively, and taking values for γ_0, δ_0 as estimated in Section 4 of I. In Fig. 6 the result is plotted as κT vs. $1-T/T_c$, using the parameter value $\lambda_1^- = 0.9$ appropriate for ^{3}He at intermediate pressures. κT is seen to increase with decreasing temperature before tending to the low-temperature limiting value. There are no data on the diffusive thermal conductivity available at present.

5. RELAXATION OF THE NORMAL DENSITY COMPONENT

In this section we discuss situations in which by some means normal and superfluid components are brought out of equilibrium with respect to each other. For example, we might be able to increase the density of the normal component (instantaneously) by injection of hot quasiparticles into the system (a type of experiment which has been carried out on superconductors, where the device of tunnel junctions is available), thus increasing the normal density component ρ_n. We would expect ρ_n to relax subsequently to some equilibrium value. Conversely, if we had a chemical potential field that only acted on the Cooper pairs, we should find that the superfluid density component ρ_s adjusted in times of order $\Delta/\hbar$ to the instantaneous value of the field, forcing the superfluid component out of equilibrium with the normal component. The relaxation of the normal component to a mutual equilibrium state would take place via particle-number-nonconserving quasiparticle collisions in which quasiparticles are converted into Cooper pairs and vice versa. Indeed, the latter mechanism is realized for the spin density in ^{3}He, where the spin–orbit correlations induced by the nuclear dipole interactions provide a field acting only on the Cooper pair spins. This will be discussed in a subsequent paper dealing with the spin transport coefficients.

The change in density of the normal component may be defined by

$$\delta\rho_n = m \sum_{\mathbf{p}} (\xi_p/E_p)\,\delta\nu_p \tag{54}$$

and the change of the superfluid component by

$$\delta\rho_s = m \sum_{\mathbf{p}} \frac{\Delta^2}{2E^3}\left(\tanh\frac{E_p}{2T}\right)(\delta\mu - f_0^s\,\delta n) \tag{55}$$

where

$$\delta\rho_s + \delta\rho_n = m\,\delta n = \delta\rho \tag{56}$$

is the total change in mass density [cf. Eq. (23)].

Considering only the homogeneous case, the kinetic equation reads

$$\omega\,\delta\nu(\xi) = -[i/\tau(\xi)]\,\delta\nu'(\xi) + I^{\mathrm{in}}(\xi) \tag{57}$$

where $I^{\mathrm{in}}(\xi)$ is the component of the inscattering term isotropic in $\hat{p}$ and odd in ξ_p. The collision integral is identical to the one entering the calculation of the second viscosity coefficient ζ_3, and will be approximated by Eq. (35).

Let us assume that the chemical potential of the quasiparticle system changes suddenly by an amount $\delta\mu$. The quasiparticle collisions cause the distribution to relax toward a local equilibrium distribution

$$\delta\nu_p^l = -f_p'(\xi_p/E_p)\,\delta\mu \tag{58}$$

The change in the normal fluid density corresponding to this distribution is given by

$$\delta\rho_n^l = m \sum_{\mathbf{p}} \frac{\xi_p}{E_p}\,\delta\nu_p^l = \chi^{q0}\,\delta\mu \tag{59}$$

with the partial susceptibility at fixed superfluid component

$$\chi^{q0} = m \sum_{p} (\xi_p^2/E_p^2)(-f_p') \tag{60}$$

Multiplying Eq. (57) by ξ/E and summing on $\mathbf{p}$, one obtains

$$\omega\,\delta\rho_n = -[i/\tau_{\mathrm{eff}}(\omega)][\delta\rho_n - \delta\rho_n^l] \tag{61}$$

where τ_{eff} is calculated using the approximate collision integral and substituting $\delta\nu_p' = \delta\nu_p + f_p'(\xi/E)\,\delta\mu$ as

$$\tau_{\mathrm{eff}}(\omega) = \frac{\langle \xi^2/E^2\rangle_{f'}}{\langle(\Delta^2/E^2)(1/\tau)\rangle_{f'}} + \frac{\langle(\xi^2/E^2)[\tau/(\omega\tau+i)]\rangle_{f'}}{\langle(\xi^2/E^2)[1/(\omega\tau+i)]\rangle_{f'}} \tag{62}$$

There is a slight dependence on frequency due to the energy dependence of $\tau(\xi)$. In the limit of $T \to T_c$ the first term in Eq. (62) diverges like $1/\Delta$. This is a consequence of qp number conservation in the normal state. In the limit of low temperatures $\tau_{\text{eff}}(\omega)$ tends to the quasiparticle relaxation time

$$\lim_{T\to 0} \tau_{\text{eff}}(\omega) = \tau \tag{63}$$

In analogy to the case of intrinsic qp spin relaxation, one may consider relaxation at fixed total density variation $\delta\rho$ (in the spin case, total spin polarization), instead of fixed chemical potential $\delta\mu$. For given change in density the equilibrium change of the normal density component is given by

$$\delta\rho_n^0 = (\chi^{q0}/\chi^0)\,\delta\rho \tag{64}$$

where $\chi^0 = mN_F$ is the thermodynamic derivative $(\partial\rho/\partial\mu)_T$ in the absence of Fermi liquid effects.

One may eliminate $\delta\mu$ in favor of $\delta\rho$ from Eq. (59) by observing that the superfluid component of the density assumes its local equilibrium value

$$\delta\rho_s = \chi^{s0}\,\delta\mu \tag{65}$$

with $\chi^{s0} = \chi^0 - \chi^{q0}$, in times of the order of $\hbar/\Delta$, which is much more rapid than the quasiparticles.

With the aid of Eqs. (64) and (65), we may write the relaxation equation in the form

$$\omega\,\delta\rho_n = -(i/\tau_\rho)(\delta\rho_n - \delta\rho_n^0) \tag{66}$$

where the relaxation time is now given by

$$\tau_\rho = (1 - \chi^{q0}/\chi^0)\tau_{\text{eff}} \tag{67}$$

In the limit $T \to T_c$, τ_ρ tends to the normal-state qp relaxation time on the Fermi surface

$$\lim_{T\to T_c} \tau_\rho = \tau_N(0) \tag{68}$$

and for $T \to 0$ we have

$$\lim_{T\to 0} \tau_\rho = \tau \tag{69}$$

In Fig. 7 we have plotted the result of a numerical evaluation of Eq. (67) for $\omega\tau \ll 1$ as $\tau(E = \Delta, T)/\tau_\rho$ vs. $1 - T/T_c$.

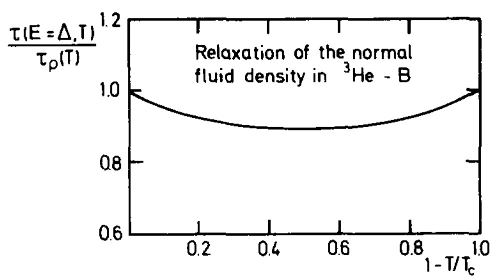

Fig. 7. Relaxation rate $1/\tau_\rho$ of the normal fluid density normalized to the quasiparticle relaxation rate at the Fermi surface for ^{3}He-B vs. reduced temperature.

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