

Balanced-budget redistribution as the outcome of political competition

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Abstract

This paper models balanced-budget redistribution between socio-economic groups as the outcome of electoral competition between two political parties. Equilibrium is unique in the present model, and a sufficient condition for existence is given, requiring that there be enough 'stochastic heterogeneity' with respect to party preferences in the electorate. The validity of Hotelling's 'principle of minimum differentiation', and of 'Director's Law', are examined under alternative hypotheses concerning administrative costs of redistributions, and voter's possibilities both of abstaining from voting and of becoming campaign activists for one of the parties. The policy strategy of expected-plurality maximization is contrasted with the strategy of maximizing the probability of gaining a plurality. Incomes are fixed and known, so lump-sum taxation is feasible. However, constraints on tax/transfer differentiation between individuals are permitted in the analysis.

1. Introduction

The driving forces behind government-induced redistributions of income and wealth are still not well understood. Theoretical models trying to explain such redistributions have emphasized changes in the size-distribution of income, i.e. 'vertical' redistributions, with the Median Voter Theorem perhaps being the most important achievement. This paper instead focuses on redistributions between (socio-economic) groups of individuals in general, permitting both 'vertical' and 'horizontal' redistributions. The Median Voter Theorem does not apply because of the multi-dimensionality of such redistributions: with m groups and one government budget constraint, every redistribution scheme has dimension $m-1$.

More specifically, we consider the competition for votes between two political parties in a representative democracy by means of political programs,

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part of which are schemes for (balanced budget) fiscal redistribution of income (or wealth) across the electorate. Since we want to focus on the parties' selection of such schemes, other parts of their programs are assumed to be fixed. However, as we shall see, also differences between the parties concerning other aspects than their redistribution policies are important for the outcome of party competition. In this study, we take gross incomes (or initial endowments of wealth) to be fixed and known. Hence, first-best (individual) lump-sum redistributions are in principle feasible. However, for realism we permit the possibility that the electorate is subdivided into groups, with the requirement that taxes and transfers should be uniform within groups. This classification into groups could be based on any criteria, e.g. income, family size, residential location etc.

A basic assumption of the analysis is that voters derive utility both from consumption (in the mandate period) and from policies that are not related to consumption (in the same period). Thus, one component of every voter's welfare depends on fiscal policy through its effects on his consumption. This component is known by both parties. The other component of his welfare, derived from other policies in the parties' political programs, or from personal attributes of the candidates, is only imperfectly observed by the parties. Therefore, they assign probability distributions to individuals' party preferences.¹ Both parties are assumed to make the same probability assignments, an assumption which is natural if they have access to the same information concerning the party preference distribution in the electorate, e.g. via opinion polls (perhaps disaggregated in terms of socio-economic and geographic characteristics). The feature of the real world which we want to catch by assuming complete information concerning consumption preferences but incomplete information as to political preferences is that consumption is a relatively simple and visible phenomenon, whereas many aspects of politics, frequently related to ideological considerations and politicians personalities, are much more difficult to define and observe.

Every individual votes for that party which best promotes his own welfare, and each of the two political parties selects its redistribution policy so as to maximize its expected plurality. Hence, the voters make use of the parties to obtain a government that promotes their welfare and the parties make use of the voters to get power – the political system thus being formed by the interaction between two categories of self-interested, maximizing agents.

In other words, we investigate the explanatory power of the 'economic man' paradigm in the context of political competition: '... the social meaning or function of parliamentary activity is no doubt to turn out legislation and, in part, administrative measures. But in order to understand how democratic policies serve this end, we must start from the competitive struggle for power and office and realize that the social function is fulfilled,

as it were, incidentally – in the same sense as production is incidental to the making of profits' (Schumpeter, 1950: 282 – quoted in Downs, 1957: 29).

This is the basic setup of our model, developed in Sections 2–4 below. Section 5 considers some extensions covering administrative costs associated with the implementation of redistribution schemes, as well as the possibility of abstention from voting and the need for active party supporters in order for the policies to be known by the electorate. Section 6 analyzes the equilibria that arise if the parties instead strive to maximize the probability of gaining a plurality, i.e. of 'winning' the election. The conclusions are summarized in Section 7, where some directions for further research are also discussed. Mathematical proofs are collected in an appendix at the end of the paper.

To the best of our knowledge, balanced-budget redistributions across socio-economic groups have rarely been analysed in terms of political equilibrium, perhaps partly due to the easily encountered non-existence of equilibrium when the policy space is multi-dimensional (cf. the example in Section 3 below). An exception is Kramer (1983), who actually analyses electoral competition between an incumbent party and an opposition by means of non-distortionary balanced-budget redistributions, such as in the present model. However, his model differs from ours in two important respects: he assumes complete information about voters' preferences, and the incumbent party must commit itself to a policy before the opposition does.

A two-party probabilistic voting model, in which both parties strive to maximize their expected votes, is developed in Enelow and Hinich (1982). Though their model belongs to the Downsian tradition of 'spatial' political modelling, it is similar to ours in the sense that voters' utility is additively separable in two components, one deterministic, relating to the policy question under consideration, the other stochastic and exogenous. In contrast to our approach, Enelow and Hinich treat a one-dimensional policy space in which the voters evaluate a policy by its Euclidean distance from an 'ideal position'. Moreover, they restrict the analysis to the case of two homogeneous groups of voters, and to the case of normally distributed random utility components.

The existence of (pure strategy) expected-plurality equilibria in probabilistic voting models have earlier been studied by Hinich, Ledyard and Ordeshook (1972), Denzau and Kats (1977), Coughlin and Nitzan (1981), and Wittman (1983).² However, these studies do not explicitly examine redistribution policies but mainly focus on the question of existence of political equilibrium in general, while our primary goal is to identify specific properties of redistribution equilibria. While their models are more general than ours in some respects, they are, as will be seen below, less general in others.

2. The basic model

There are n voters, indexed by i . The exogenous and fixed *gross incomes* (or initial endowments) are given by the vector $\omega = (\omega_1, \dots, \omega_n) > 0$. Let $I = \{1, \dots, n\}$, and suppose I is partitioned into m disjoint subsets, where the k :th subset I_k contains n_k individuals ($\sum n_k = n$). We assume $2 \leq m \leq n$ and each n_k positive. The subset of individuals I_k is referred to as *group* k (a special case being $m = n$, i.e. one voter in each group). A *balanced-budget redistribution* is a vector $z = (z_1, \dots, z_m) \in \mathbb{R}^m$, where z_k ($k = 1, \dots, m$) is the transfer to each member of group k , and z satisfies the budget equation $\sum n_k z_k = 0$ and the requirement that every individual's consumption (or net income) be positive. Let $X = \{z \in \mathbb{R}^m; \omega_i + z_k > 0 \text{ for all } k \text{ and } i \in I_k\}$ and $X_0 = \{z \in X; \sum n_k z_k = 0\}$, i.e. X_0 is the set of (feasible) balanced-budget redistributions. Finally, let $k(i)$ signify the group identity of individual i .

Before the election, the two parties, A and B, promise balanced-budget redistributions x and y , respectively. Each voter i derives utility $v_i(c_i)$ from consumption, where $c_i = \omega_i + x_{k(i)}$ if party A wins and $c_i = \omega_i + y_{k(i)}$ if B does. We assume $v_i'(c_i) > 0$ and $v_i''(c_i) < 0$ for i and $c_i > 0$, i.e. increasing utility and decreasing marginal utility of consumption. Moreover, to avoid corner solutions, we suppose that the marginal utility falls from infinity to zero as consumption rises from zero to infinity.

The individual's total welfare is assumed to be additively separable as follows:

$$\begin{aligned}
 u_i(x, a) &= v_i(\omega_i + x_{k(i)}) + a_i \quad \text{if A wins} \\
 u_i(y, b) &= v_i(\omega_i + y_{k(i)}) + b_i \quad \text{if B wins}
 \end{aligned}
 \tag{1}$$

where a_i is the utility that individual i derives from other policies in party A's political program and likewise with b_i .³ Individual i is assumed to vote for party A if $u_i(x, a) > u_i(y, b)$, for B if $u_i(x, a) < u_i(y, b)$, and otherwise abstain. Hence, his choice is deterministic, and it is a discontinuous function of the utility differential between the two party programs. However, the parties, which cannot observe the terms a_i and b_i exactly, treat them as random variables when selecting their redistribution policies. Each random utility differential $b_i - a_i$ is assigned by both parties a twice continuously differentiable probability distribution function F_i , with everywhere positive density $f_i = F_i'$. Then the parties' probability assignment for an individual i to vote for party A is a continuous function of the consumption-utility differential between the policies:

$$p_i = \Pr((u_i(y, b) < u_i(x, a)) = F_i[v_i(\omega_i + x_{k(i)}) - v_i(\omega_i + y_{k(i)})], \tag{2}$$

the probability that he will vote for B being $q_i = 1 - p_i$.⁴

Let n_A denote the associated random number of votes for party A and n_B the votes for B. Then the plurality expected (by both parties) for party A is $E(n_A - n_B) = \Sigma(p_i - q_i)$.⁵ Viewing the expected outcome as a function of both promised redistributions, we call a pair $(x^*, y^*) \in X_0^2$ a (pure strategy) *Nash equilibrium (NE) in the expected-plurality game* if $E(n_A - n_B \mid x, y^*) \leq E(n_A - n_B \mid x^*, y^*) \leq E(n_A - n_B \mid x^*, y)$ for all x and y in X_0 .

When interpreting subsequent results, it is sometimes useful to refer to the special case of unimodal, symmetric and translate symmetric density functions. We call a density function f_i *unimodal* if it has a unique maximum, to the left (right) of which it is strictly increasing (decreasing). It is said to be *symmetric* if $f_i(-t) = f_i(t)$ for all $t \in \mathbb{R}$. If there is a (unimodal and) symmetric density function f and scalars α_i such that $f_i(t) = f(t + \alpha_i)$ for all i and t , then we say that the f_i 's are *translates* of a common (unimodal and) symmetric density. In this case, each α_i is both the mean value and the median of the parties' probability assignment for the party bias $a_i - b_i$ of individual i . Hence, α_i may be referred to as the *expected party bias* of individual i in favour of party A.

Two particular types of such densities are especially useful: the logit and probit models. In both models, a_i and b_i are random variables of the form $\alpha_i + \epsilon_{i1}$ and ϵ_{i2} , respectively, where α_i is a scalar. In the logit model, ϵ_{i1} and ϵ_{i2} are independent and identically distributed (i.i.d.) according to the doubly exponential distribution function $\Phi(t) = \exp(-\exp(-t))$. It can be shown that then $F_i(t - \alpha_i) = e^t / (e^t + 1)$ (cf. e.g. McFadden (1973)), so in this case $f_i(t - \alpha_i) = (e^{t/2} + e^{-t/2})^{-2}$. In the probit model, all differences $\epsilon_{i1} - \epsilon_{i2}$ are i.i.d according to the normal distribution, so then $f_i(t - \alpha_i) = (2\pi)^{1/2} \exp(-t^2/2)$. In each case, the f_i 's are translates of a common unimodal and symmetric density function. More generally, it is readily verified that if a_i and b_i are i.i.d., then f_i is symmetric, and if moreover their common density function is unimodal, then so is f_i (cf. Lemma A in the appendix).

3. Characteristics of equilibria

Suppose (x, y) is a NE. Since there is no abstention from voting, $z = x$ then maximizes $E(n_A \mid z, y)$ subject to $z \in X_0$, and $z = y$ minimizes $E(n_A \mid x, z)$ subject to $z \in X_0$, where $E(n_A) = \Sigma p_i$ and $X_0 = \{z \in X; \Sigma n_k z_k = 0\}$. Each goal function being continuously differentiable on the open set X , there are Lagrangians $\lambda, \mu > 0$ such that for all k

$$\sum_{i \in I_k} v_i'(\omega_i + x_k) f_i(t_i) = \lambda n_k \quad (3)$$

$$\sum_{i \in I_k} v_i'(\omega_i + y_k) f_i(t_i) = \mu n_k \quad (4)$$

where the probability densities are evaluated at the corresponding consumption-utility differential: $t_i = v_i(\omega_i + x_k) - v_i(\omega_i + y_k)$. These equations state that, from the viewpoint of parties A and B, respectively, the per capita marginal gain in expected votes, with respect to marginal shifts in transfers, should be equal for all groups; otherwise the expected number of votes on a party could be improved without violation of the public budget constraint.

It follows from these first-order conditions that the ratios

$$\rho_k = \frac{\sum_{i \in I_k} v_i'(\omega_i + x_k) f_i(t_i)}{\sum_{i \in I_k} v_i'(\omega_i + y_k) f_i(t_i)} \quad (k = 1, \dots, m)$$

should be equal for all k . Now suppose $x \neq y$. Then the budget requirement $\sum n_k x_k = \sum n_k y_k = 0$ implies that there are groups k and h such that $x_k < y_k$ and $x_h > y_h$. However, since the marginal utilities v_i' are assumed to be decreasing functions, this would imply $\rho_k > 1 > \rho_h$, contradicting the requirement that the ratio be equal for all groups. Hence, $x = y$ is a necessary condition for equilibrium. Inserting this equality into eq. (3), we have proved the following multidimensional analogue of Hotelling's 'principle of minimum differentiation':⁶

Theorem 1: If (x, y) is a NE in the expected-plurality game, then $x = y$, and there is a $\lambda > 0$ such that for all k

$$\sum_{i \in I_k} v_i'(c_i) f_i(0) = n_k \lambda. \quad (5)$$

Let us consider two special cases of preference variations across the electorate. First, if all individuals have been assigned the same party preference distribution, then all factors $f_i(0)$ in eq. (5) are identical, so in this case the average marginal utility of consumption is equal in all groups. The resulting political equilibrium is thus identical with the utilitarian optimum achieved when maximizing the social welfare function $\sum v_i(c_i)$ subject to $x \in X_0$, where individual consumer preferences are represented by the cardinal utility functions employed in the probability assignments (cf. note 5). In other words, *in this special case democratic electoral competition for the votes of selfish individuals produces the same income distribution as would an omnipotent Benthamite government.*⁷

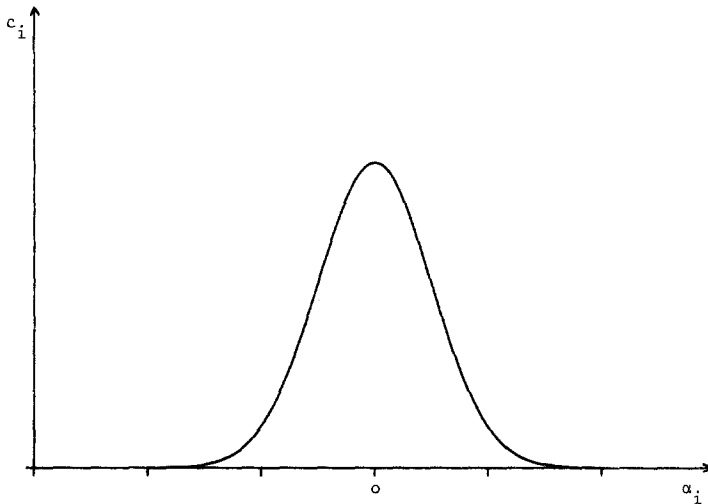


Figure 1. Equilibrium in the expected-plurality game. Consumption as a function of the expected party bias.

Secondly, suppose all individuals have the same consumption preferences (i.e. $v_i = v_j$ for all i and j), while party preference distributions differ between groups but are identical within groups (i.e. $f_i = f_j$ iff $k(i) = k(j)$). Further, assume that all f_i 's are translates of a unimodal and symmetric density function f . Then eq. (5) implies that the levels of consumption in a group are functions of the expected party bias of the group members: $\Sigma v'(c_i)/n_k = \lambda/f(\alpha_i)$ (where $i \in I_k$). Since the marginal utility of consumption by assumption is decreasing, the per capita transfer to a group is a decreasing function of the absolute value of the expected party bias in the group, in this special case (cf. Figure 1 above). In other words, in equilibrium both parties will favour those groups in the electorate whose expected partisan biases are weak, i.e. 'marginal' voters (or 'swing' voters).

As a further illustration of this special case, suppose gross incomes are equal within groups but differ between groups. If furthermore low-income groups have an expected bias in favour of party A ($\alpha_i > 0$) and high income groups an expected bias towards B ($\alpha_i < 0$), then *in equilibrium both parties will favour middle-income earners at the expense of both low- and high-income earners, and to such an extent that the middle income earners will have the highest net income of all*. This corresponds, in an extreme form, to what George Stigler has termed 'Director's law' (Stigler 1970).⁸

A general consequence of Theorem 1 is that in equilibrium $E(n_A) = \Sigma p_i^0 = \Sigma F_i(0)$, where each term p_i^0 is the 'prior' probability that individual i votes for party A (cf. eq. (2)). In particular, if there are no party biases (more exactly if all $F_i(0) = 1/2$), then the expected number of votes is $n/2$ for both parties.⁹ In fact, there is a direct and intuitive argument that

$E(n_A \mid x, y) = \Sigma p_i^0$ is a necessary condition for any pair (x, y) of balanced-budget redistribution policies to be an equilibrium. For if this condition were not met, then one of the parties could gain votes by adopting the other party's redistribution policy.

4. Existence and uniqueness

It is straightforward to show *uniqueness* of equilibrium in the expected-plurality game analyzed above. For suppose that both (x, λ) and (x', λ') satisfy the necessary first-order condition (5) in Theorem 1. If $\lambda = \lambda'$, then $x = x'$ by (strict) concavity of the utility functions v_i . On the other hand, if $\lambda < \lambda'$ would be the case, then $x > x'$ by the same property. However, this would exclude the possibility that both x and x' are balanced-budget redistributions. Hence there can be at most one solution $(x, \lambda) \in X_0 \times (0, +\infty)$ of eq. (5).

The presence of uncertainty is crucial for *existence* of equilibrium in our model. For if both parties knew in advance and with certainty every individual's party preferences a_i and b_i , then the 'expected' number of votes would be a discontinuous function of the party policies (each F_i would be a step function, cf. eq. (2)). Suppose, for example, that there were no party biases in the electorate (i.e. $a = b$), and that each group consisted of only one individual (i.e. $m = n$). Then, for any balanced-budget policy y that party B might suggest, party A could obtain $n - 1$ votes by choosing e.g. the balanced-budget policy $x = (y_1 - \epsilon, y_2 + \epsilon/(n - 1), \dots, y_n + \epsilon/(n - 1))$, for some ϵ in $(0, \omega_1 + y_1)$. This would give party B an incentive to change its policy, since it could switch from 1 to $n - 1$ votes in the same way. Hence, for $n > 2$ no equilibrium exists in this deterministic example.

In the present model with uncertainty, however, infinitesimal shifts in policies give rise, not to finite, but to infinitesimal shifts in votes. Of the usual four sufficient conditions for existence of (pure strategy) equilibrium in zero-sum games, viz. (i) compactness and (ii) convexity of the individual strategy sets, (iii) continuity and (iv) concavity/convexity of the pay-off function (cf. e.g. Rosen (1965, Th.1) or Owen (1982, Th.IV.6.2)), only two are fulfilled in the present model, viz. (ii) and (iii). However, for certain preferences and probability distributions, (iv) is also met, i.e. $E(n_A - n_B \mid x, y)$ is concave in x and convex in y ('decreasing marginal returns' in votes to redistribution promises, cf. Lemma B in the appendix). Even though (i) is not fulfilled, existence can then be shown directly by means of the first-order conditions (3), (4), and (5).

Theorem 2: If condition C1 below holds, then there exists a (unique) NE of the expected-plurality game.

C1: $\beta_i \leq \gamma_i$ for all i ,

Where $\beta_i = \sup |f_i'(t)|/f_i(t)$ and $\gamma_i = \inf |v_i''(s)|/(v_i'(s))^2$. (A weaker condition, but still sufficient for existence, is given in the appendix, cf. proof of Lemma B.)

For given probability density functions, condition C1 requires every utility function v_i to have a concavity index exceeding β_i , in the terminology of Debreu and Koopmans (1980).¹⁰ For given consumption preferences, on the other hand, the condition is more easily satisfied, the larger is the degree of uncertainty about (or 'stochastic heterogeneity' in) individual party preferences.¹¹ For if one lets all random utility terms a_i and b_i be increased by a factor $\sigma > 1$, then each index β_i becomes a factor $1/\sigma$ smaller. In this sense, condition C1 defines a critical degree of uncertainty (for some utility functions and probability distributions infinitely high) above which equilibrium exists.¹² For example, the condition is fulfilled for logarithmic utility functions in the logit model, since then $\gamma_i = \beta = 1$ for every i , while no utility functions satisfy C1 in the probit model, since then $\beta_i = +\infty$ for every i (cf. the discussion of the logit and probit models in Section 2 above).¹³

What if, at given consumption preferences, the actual degree of uncertainty falls short of the critical level in condition C1? Actually, it is not difficult to establish a necessary second-order condition for existence of equilibrium, a condition that essentially requires this degree to exceed another (lower) critical level. Suppose (x, y) is a NE. Then $x = y$ by Theorem 1, and with c_i (as usual) denoting the consumption of individual i in equilibrium:

$$\begin{aligned} \text{C2: (i)} \quad & \sum_k (z_k - x_k)^2 \sum_{i \in I_k} [v_i''(c_i) f_i(0) + (v_i'(c_i))^2 f_i'(0)] \leq 0 \text{ for all } z \in X_0, \\ \text{(ii)} \quad & \sum_k (z_k - x_k)^2 \sum_{i \in I_k} [v_i''(c_i) f_i(0) - (v_i'(c_i))^2 f_i'(0)] \leq 0 \text{ for all } z \in X_0 \end{aligned}$$

(cf. Lemma C in the appendix). This necessary condition is met trivially in the special case of symmetric party-preference distributions, since then $f_i'(0) = 0$ for all i .¹⁴ More generally, it holds if $|f_i'(0)|/f_i(0) \leq |v_i''(c_i)|/(v_i'(c_i))^2$ for all i . Consequently, condition C1 implies C2, and a multiplication of all random utility terms by some scalar σ shows, just as in the case of C1, that C2 is more easily satisfied the larger is the degree of uncertainty.

Similar conclusions as to the role of uncertainty, or 'stochastic heterogeneity', in establishing the existence of equilibrium have been reached in another context by de Palma, Ginsburgh, Papageorgiou and Thisse (1985), who analyzed the validity of Hotelling's 'principle of minimum differentia

tion' under multi-firm competition in one-dimensional location space. Their study was inspired by the discovery by d'Aspremont, Gabszewicz and Thisse (1979) of an important error in Hotelling's analysis, implying that his principle in fact is invalid in the deterministic (price and location) model he used. De Palma et al. assume that firms cannot observe individual preferences exactly, and that they therefore endow consumers with a probabilistic choice rule according to the logit model. In this setting, Hotelling's principle is restored provided consumers' preferences are sufficiently heterogenous, as expressed by the scale parameter σ discussed above.

5. Extensions

The basic model outlined above is very simple and it abstracts away many real-life complications. Here we investigate, in three examples, the robustness of the results with respect to some extensions in the direction of increased realism.

5.1 Administration costs

Suppose that there is an aggregate cost of the additive form $\sum g_i(z_{k(i)})$ associated with the implementation of any redistribution vector z in X , restricting the analysis to the case in which each component $g_i(z_k)$ depends smoothly on the amount transferred in such a way that neither the cost nor the marginal cost diminishes as the transferred amount increases. More precisely we assume each $g_i: \mathbb{R} \rightarrow \mathbb{R}_+$ to be twice continuously differentiable with $g_i(t) \geq 0$ for $t > 0$, $g_i(t) \leq 0$ for $t < 0$, and $g_i''(t) \geq 0$ for all $t \in \mathbb{R}$. The administrative costs are assumed to be drawn from the public budget, so the previous budget equation $\sum n_k z_k = 0$ is generalized to

$$\sum_k (n_k z_k + \sum_{i \in I_k} g_i(z_k)) = 0, \quad (6)$$

and accordingly X_0 is now the subset of redistributions in X satisfying eq. (6) (the basic model being the special case $g_i = 0$ for every i).

Since $n_k z_k + \sum g_i(z_k)$ is the cost of a per capita transfer z_k to group k (summing over $i \in I_k$), one would expect all marginal costs of public funds, $n_r + g'_i(x_k)$ and $n_k + \sum g'_i(y_k)$ ($k = 1, \dots, m$), to be positive in equilibrium, since otherwise one of the parties could gain votes by 'costlessly' increasing the transfer to a group with a negative marginal cost. In fact, a proof of this conjecture is easily given (see proof of Theorem 3 in the Appendix).

In view of the egalitarian character of the equilibrium obtained in the absence of administrative costs, one may wonder whether the competition for

votes now induces both parties to suggest excessively costly redistributions. However, it can be shown that Theorem 1 remains valid in this more general case under the natural qualification that the right-hand side in the first-order condition (5) should be multiplied by the corresponding marginal cost:

Theorem 3: If (x, y) is a NE in the expected-plurality game with administrative costs, then $x = y$, and there is a $\lambda > 0$ such that for all k

$$\sum_{i \in I_k} v_i(c_i) f_i(0) = [n_k + \sum_{i \in I_k} g_i'(x_k)]\lambda. \quad (7)$$

In other words, electoral competition in the absence of party preference variations ($f_i(0) = f_j(0)$ for all i and j) still leads to a *utilitarian welfare optimum*: eq. (7) then is the first-order condition associated with the maximization of the Benthamite social welfare function $\sum v_i(c_i)$ subject to $x \in X_0$ (recall the particular cardinalization of preferences, cf. note 4). More generally, eq. (7) shows that, for any probability assignments f_i , the equilibrium redistribution vector is (constrained) Pareto efficient.¹⁵ In this sense, *competition for votes does not induce the parties to promote excessively costly redistribution programs*. Note also that now it is not the average marginal utility *per unit of transfer* that should be equal for all groups, but the more general notion of average marginal utility *per marginal unit of societal cost*. Thus, when administrative costs are introduced, then the previous equilibrium redistribution is modified so that groups for which marginal costs are relatively high receive less.

5.2 'Exit' and 'voice'

Two important features of the electoral process which have been neglected so far are (i) the possibility of abstaining from voting, i.e. 'exit', and (ii) the possibility for an individual of becoming an active supporter, or promotor, of the party in the sense that he actively 'voices' his opinions to induce others to vote for the party (to use a general terminology by Hirschman (1970) in a somewhat special meaning). In both cases, an element is introduced that may counteract the 'Hotelling tendency' towards 'the middle ground'.

For the purpose of analysing the exit option, suppose that there for every individual i is a nonnegative welfare threshold ϵ_i , such that he votes for party A if $u_i(x, a) - u_i(y, b) > \epsilon_i$, party B if $u_i(x, a) - u_i(y, b) < -\epsilon_i$, and otherwise abstains. In other words, the utility differential between the two parties' policies must now exceed a certain level in order for the voter to find it worthwhile to vote at all, i.e. 'nonvoting due to indifference'. With the basic model as the special case $\epsilon_i = 0$ for all i , we have defined an *expected-*

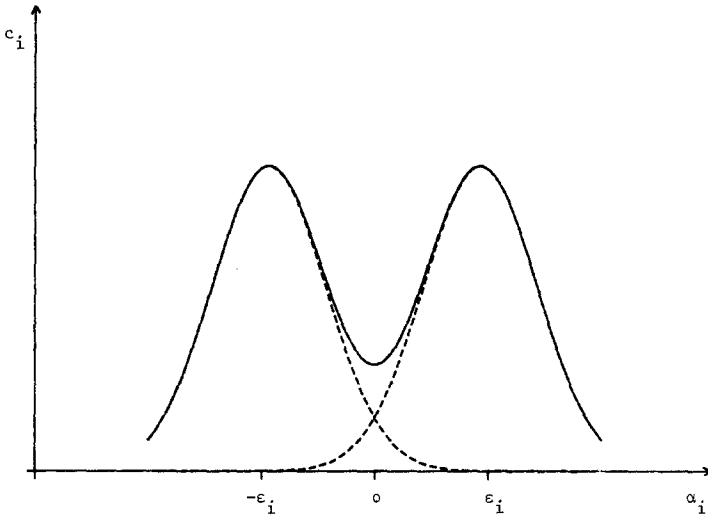


Figure 2. Equilibrium in the expected plurality game with ‘exit’. Consumption as a function of the expected party bias.

plurality game with ‘exit’. In the present model, the effect of this extension is not strong enough to prevent the parties from still selecting the same policy in equilibrium:

Theorem 4: If (x, y) is a NE in the expected-plurality game with ‘exit’, then $x = y$, and there is a $\lambda > 0$ such that for all i

$$\sum_{i \in I_k} v_i(c_i) [f_i(-\epsilon_i) + f_i(\epsilon_i)] = n_k \lambda. \tag{8}$$

To illustrate this result, suppose consumption preferences are identical and that all density functions f_i are translates of a unimodal and symmetric density f . It then follows that each factor $[f_i(-\epsilon_i) + f_i(\epsilon_i)] = [f(\alpha_i - \epsilon_i) + f(\alpha_i + \epsilon_i)]$ is a two-peaked symmetric function of the expected party bias α_i , for sufficiently large thresholds ϵ_i (in the sense of $F_i(\epsilon_i)$ being close to one), with the two maxima placed close to $-\epsilon_i$ and $+\epsilon_i$ (cf. Figure 2 above). Hence, if the party preference distributions are identical within groups, the most favoured groups (by both parties) in equilibrium are partitioned into two distinct classes, viz. those that are expected to be in favour of either party A or B with an absolute bias $|\alpha_i|$ close to ϵ_i – *contradicting Director’s law*.

Note however, that the favoured groups are still those consisting of ‘marginal’ voters, this time not in the sense of indifference between the parties (such voters tend not to vote when ϵ_i is large), but *in the sense of indifference between voting and not voting*.¹⁶

In the real world, political parties rely heavily on active support of a group of loyal party workers and sympathizers. This leads us to the opposite reaction to ‘exit’, viz. ‘voice’. For the purpose of analyzing this aspect of the electoral process, we assume that individual i becomes an activist for party A if $u_i(x, a) - u_i(y, b) > \delta_i$ and of party B if $u_i(x, a) - u_i(y, b) < -\delta_i$, where $\delta_i > 0$ is another welfare threshold. As a simple way of introducing ‘campaign activists’ we assume that, in order for its policy to be known by the electorate, each party must select its policy so that the expected number of activists is at least νn ($\nu \geq 0$). Let s_A and s_B be the random number of activists of party A and B, respectively. Then party A faces the additional constraint $E(s_A | x, y) \geq \nu n$, and party B $E(s_B | x, y) \geq \nu n$.¹⁷ In this *expected-plurality game with ‘voice’* (with the basic model as the special case $\nu = 0$), Hotelling’s principle of minimum differentiation ceases to be generally valid, since if the inequality

$$\min\{\Sigma F_i(-\delta_i), n - \Sigma F_i(\delta_i)\} < \nu n \quad (9)$$

holds, then at least one of the parties has an insufficient expected number of promoters in any potential equilibrium in which both parties’ policies coincide. Hence, if (9) holds and (x, y) is a NE in the expected-plurality game with ‘voice’, then necessarily $x \neq y$.

6. Alternative policy strategy: Maximizing the probability of winning

So far, it has been assumed that both parties strive to obtain as large expected plurality as possible. In the analytical literature on electoral competition this is the most common assumptions as to party objectives, a natural alternative for which is the maximization of the probability of gaining a plurality. Therefore, a comparison of the equilibria of these two games seems highly relevant.¹⁸ In particular, one may wonder whether an equilibrium policy which maximizes the probability of obtaining a majority favours a smaller fraction of the electorate than does an equilibrium policy that maximizes the expected number of votes.

For technical reasons, we from now on assume that all random utility differentials $b_i - a_i$ are statistically independent (which is the case e.g. in the logit and probit models). Secondly, and more importantly, we assume $m = n$, i.e. that all groups consist of only one individual. Albeit unrealistic, the latter assumption does not seem to hamper the possibility that the equilibria of the two games differ. (In this sense, the assumption does not seem to reduce the explanatory power of Theorem 5 below.)

6.1 *Exact specification*

In a first, exact approach to this question, we also assume that the number n of individuals in the electorate is odd. For every voter i , we introduce a random variable that indicates whether that particular individual votes for A or not: let $e_i = 1$ or 0 according to whether his vote is cast for party A or B (cf. note 5). Then the probability that party A will gain a plurality can be written $\pi_A = \Pr(\sum e_i > n/2)$, with $\pi_B = 1 - \pi_A$, defining a constant-sum game.¹⁹ Viewing π_A as a function of the promised redistributions x and y , we call a pair $(x^*, y^*) \in X_o^2$ a (pure strategy) equilibrium of *the probability-of-winning game* if $\pi_A(x, y^*) \leq \pi_A(x^*, y^*) \leq \pi_A(x^*, y)$ for all x and y in X_o .

It is not difficult to show that in equilibrium both parties will again select the same policy in this game. For, by the assumed independence of the individual random utility differentials $b_i - a_i$, any two vote variables e_i and e_j are statistically independent. Focusing on a particular voter i , one may hence write the probability that party A gains a plurality as the sum of two terms, one being the probability that all other votes together give a plurality for A, the other being the probability that they result in a tie, multiplied by the probability that i votes for A:²⁰

$$\pi_A(x, y) = \Pr[\sum_{j \neq i} e_j > n/2] + \Pr[\sum_{j \neq i} e_j = (n-1)/2]p_i. \tag{10}$$

Here only p_i depends on x_i and y_i since by assumption every group has only one member. Hence, if a pair (x, y) of balanced-budget redistributions is a NE, then there are Lagrangians $\lambda, \mu > 0$ such that for all i

$$\Pr[\sum_{j \neq i} e_j = (n-1)/2]v_i'(\omega_i + x_i) f_i(t_i) = \lambda \tag{11}$$

$$\Pr[\sum_{j \neq i} e_j = (n-1)/2]v_i'(\omega_i + y_i) f_i(t_i) = \mu \tag{12}$$

(cf. eqs. (3) and (4)). By the same argument as in the proof of Theorem 1, this implies $x = y$. Moreover, when $x = y$ and the ‘prior’ voting probabilities $p_i^0 = F_i(0)$ are identical, then all random variables e_1, \dots, e_n are not only independent but also identically distributed ($\Pr(e_i = 1) = F_i(0)$ when $x = y$). Consequently, all probability factors (for $i = 1, \dots, n$) in eqs. (11) and (12) are then identical. In other words, *if ‘prior’ voting probabilities are identical, then the necessary first-order condition is the same as in the expected-plurality game.* In sum, we have proved

Theorem 5: Suppose n is odd, $m = n$ and $\{b_i - a_i\}$ independent. If (x, y) is a NE in the probability-of-winning game, then $x = y$. If moreover $p_i^0 = p_j^0$ for all i and j , then x satisfies eq. (5) for some $\lambda > 0$.

Hence, under the hypothesis of this theorem an equilibrium policy which is aimed at obtaining a majority *does not* favour a smaller (or larger) fraction of the electorate than does a policy aimed at obtaining as many votes as possible.

6.2 Approximate specification

What if there are differences in ‘prior’ voting probabilities in the electorate? Unfortunately, the first-order conditions for the equilibria of the probability-of-winning game seem analytically intractable in the presence of such variations. However, if the electorate is large, then the number of votes on each party is approximately normally distributed by the Central Limit Theorem, so then tractability can be obtained by way of approximation of the probability distribution for the number of votes for each party. More precisely, by Liapounoff’s version of this theorem, the *divergence condition*.

$$C3: \sum_1^n p_i q_i \rightarrow \infty \text{ as } n \rightarrow \infty$$

is sufficient (and in the present case also necessary) for Σe_i to be asymptotically normal with mean Σp_i and standard deviation $(\Sigma p_i q_i)^{1/2}$ (cf. e.g. Cramer (1946, Section 17.4)). In the present model, there is no reason to assume that the individual choice variances $p_i q_i$ decrease as the number of individuals in the electorate increases, so condition C3 seems justified. Hence $\pi_A(x, y)$ can be approximated by

$$\hat{\pi}_A(x, y) = (2\pi)^{-1/2} \int_{\psi(p)}^{+\infty} \exp(-t^2/2) dt, \quad (13)$$

where $p = (p_1, \dots, p_n)$ and

$$\psi(p) = (n/2 - \Sigma p_j)(\Sigma p_i(1 - p_i))^{-1/2}. \quad (14)$$

Substituting $\hat{\pi}_A(x, y)$ for $\pi_A(x, y)$ in the above definition of equilibrium, we have constructed an *approximate-probability-of-winning game*.²¹ As before, let $p_i^0 = F_i(0)$ and $q_i^0 = 1 - p_i^0$.

Theorem 6: If (x, y) is a NE in the approximate-probability-of-winning game, then $x = y$ and there is a $\lambda > 0$ such that for every i

$$v_i'(c_i) f_i(0) [\Sigma p_j^0 q_j^0 + (p_i^0 - q_i^0) \Sigma (p_j^0 - q_j^0) / 4] = \lambda. \quad (15)$$

In other words, *the Hotelling convergence result is not affected by the approximation of probabilities, but the necessary first-order condition (15) generally differs from the corresponding condition (5) in the expected-plurality game* by a factor depending on ‘prior’ voting probabilities.²² If all individuals are assigned equal probabilities, then eq. (15) collapses to condition (5) – in agreement with the exact result in Theorem 6. The present theorem shows that this equivalence between the two games can be extended to all situations in which the parties are equally popular: *if $\Sigma p_i^o = \Sigma q_i^o$ then the first-order conditions of the expected-plurality and approximate-probability games are identical.*

On the other hand, if one of the parties is more popular than the other, then the first-order conditions of the two games generally differ. For example, suppose that party A is more popular than B, i.e. $\Sigma p_j^o > \Sigma q_j^o$, and that all f_i ’s are translates of a symmetric density function f . Suppose also that there are two voters i and j who have identical consumer preferences, and equally strong (expected) partisan biases but in opposite directions: $v_i = v_j$ and $\alpha_i = -\alpha_j > 0$. It then follows from Theorem 1 and symmetry of f that they receive the same net income in the equilibrium of the expected-plurality game ($f_i(0) = f(\alpha_i) = f(\alpha_j) = f_j(0)$, so $c_i = c_j$ by eq. (5)), while in the approximate-probability-of-winning game voter i is favoured at the expense of voter j : we still have $f_i(0) = f_j(0)$, but now $p_i^o - q_i^o > p_j^o - q_j^o$, so by eq. (15) $v_i'(c_i) < v_j'(c_j)$, i.e. $c_i > c_j$ since i and j have the same consumption preferences.

If $f_i(0) = f_j(0)$ for all i and j (e.g. $\alpha_i = \pm \alpha$ for all i), then this would mean that the (symmetric) utilitarian equilibrium of the expected-plurality game would be ‘tilted’ in favour of the sympathizers of the more popular party if the parties instead were to maximize the probability of obtaining a plurality. In this sense, the latter type of equilibrium favours a more narrow fraction of the electorate than does the expected-plurality equilibrium.

In sum: *parties that strive to maximize the probability of gaining a plurality tend to favour, in equilibrium, those voters that are partisans of the more popular party, as compared with the corresponding equilibrium of the expected-plurality game.* In particular, Director’s law does not apply generally when parties strive to maximize the probability of gaining a majority, but only if ‘prior’ voting probabilities are sufficiently balanced.

A heuristic explanation for this, somewhat surprising, result can be given in terms of eq. (10). By differentiation,

$$\begin{aligned} \partial \pi_A(x, y) / \partial x_i &= \Pr[\sum_{j \neq i} e_j = (n-1)/2] \partial p_i / \partial x_i = \\ &\Pr[\sum_{j \neq i} e_j = (n-1)/2] \partial E(n_A | x, y) / \partial x_i \end{aligned} \tag{16}$$

(recall $m = n$). In other words: the ‘marginal return’ to the probability of

gaining a plurality, from a marginal increase in the transfer to any individual i , is proportional to the corresponding return to the expected number of votes, the proportionality factor being the probability that all other votes together result in a tie, i.e. that individual i becomes a 'pivot' voter. Now suppose both parties have chosen the same redistribution policy ($x = y$). Then individual i is more likely to be a pivot voter the stronger is his bias in favour of the more popular party, since the exclusion of such an individual from the electorate leaves the remaining electorate as little biased as possible, and hence also as likely as possible to produce a tie.

7. Conclusions and directions for further research

Assuming that each of two competing political parties tries to maximize its number of votes from selfish voters, an equilibrium solution for a redistribution policy has been shown to exist, provided a sufficient degree of uncertainty prevails concerning party preferences in the electorate. It was also found that in equilibrium both parties will select the same redistribution policy, a conclusion which may be regarded as a multi-dimensional version of Hotelling's well-known 'principle of minimum differentiation'.

In the special case when no systematic variations in party preferences are observed, while consumption preferences are known to vary, redistributions will be pursued until the average marginal utility of consumption is equal in all (socio-economic) groups, implying that democratic electoral competition results in the same distribution of income as would be imposed by an omnipotent Benthamite government maximizing a utilitarian social welfare function. In the opposite special case of identical consumption preferences, but with observed differences in party preferences between groups, both parties will instead in equilibrium favour groups with weak party preferences, i.e. 'marginal voters'. If moreover gross incomes do not vary within groups, and low-income groups are biased in favour of one party while high-income groups are biased in favour of the other, then both parties will favour middle-income earners, and indeed to such an extent that these voters will get the highest net income of all. This conclusion corresponds, in an extreme form, to what George Stigler has called 'Director's Law' of redistribution policy.

It turns out that the conclusions are basically the same if redistributions entail administration costs, financed from the government budget (provided marginal redistribution costs do not fall with the volume of redistribution). One modification of the previous conclusions, though, is that groups for which marginal costs of redistribution are high will receive relatively less than low-cost groups.

It was also found that the conclusions did not change much if it is as-

sumed that people abstain from voting (hence ‘exiting’ from the electoral process) when the utility differential between the programs of the political parties is small. Both parties will still propose identical programs, though now the tendency to favor voters with weak partisan preferences disappears; thus while Hotelling’s principle still holds, Director’s Law is not generally valid. Instead, both parties tend to favour voters who are ‘marginal’ in the special sense of being indifferent vis-à-vis voting at all.

Alternatively, if it is assumed that parties need active supporters (‘party activists’) for a successful election campaign, and that voters require a certain utility difference between the parties to be willing to support a party *actively* i.e. by ‘voicing’ their opinions to other members of the electorate) Hotelling’s principle of minimum differentiation ceases also to be generally valid.

It was finally established that Hotelling’s principle also holds in the basic set-up (without administrative costs, ‘exit’ or ‘voice’) when the assumption of maximization of expected plurality is replaced by the assumption that the parties maximize the probability of gaining a plurality. Moreover, in the absence of party preference variations in the electorate, it was found that the first-order condition for equilibrium is identical to that of the expected-plurality game. As a consequence, there is no tendency in this case for parties to favor a smaller fraction of the electorate than they would do if they instead aimed at attracting as many voters as possible. In the case of varying party preferences in the electorate, some further conclusions were derived for large electorates, in which the sum of votes for a party is approximately normally distributed. It then turns out that the first-order conditions for equilibrium coincide with the corresponding conditions of ‘expected plurality maximization’, assuming both parties are equally popular (in the sense of expecting equally many votes if their redistribution policies were identical). If instead one of the parties is more popular than the other, then, while Hotelling’s principle still holds, in equilibrium both parties will favour those voters who are biased in favor of the more popular party – as compared with the situation when the parties’ objective is to maximize the number of votes. In this special sense, an equilibrium policy which maximizes the probability of obtaining a plurality favours a narrower fraction of the electorate than does an equilibrium policy that maximizes the expected plurality.

The analysis in this paper, which may also be applied to certain internal decision processes within organizations (such as trade unions), rests on a number of restrictive assumptions. For instance, redistribution has here been visualized as a simultaneous ‘one-shot’ process, while in the real world only *some* redistributions are considered at a time (for instance during a single election campaign). Thus a model of redistribution as a *recursive* process may be preferable (cf. e.g. Kramer (1977)). In that case it would also be natural to consider the consequences of ‘irreversibilities’ in the sense that

the *removal* of a grant is valued differently by the voter than never getting the grant to begin with. Moreover, it is often asserted that voters have non-symmetric reactions to taxes and transfers, in the sense that a reduction in taxes buys fewer (or more) votes than does an equally large increase in transfers. Also the existence of 'reaction thresholds' has been asserted, implying that for instance more votes are gained by large transfers to a minority than are lost by small tax increases, of the same total value, for a majority – the reason being that voters do not notice or find it worthwhile to react to small changes in taxes and transfers (for a discussion of these and related issues, see Lindbeck, 1985).

It would certainly be of interest to extend the analysis to multi-party systems. However the difficulties seem substantial in view of the possibility of party coalitions, and hence also the difficulty of specifying what a vote for a (small) party actually supports (cf. Downs, 1957: 9.II.A). Entry of new parties, or just the threat of entry, may invalidate the Hotelling principle (but not necessarily so, cf. de Palma et al., 1985). It is, of course, also of great importance to extend the analysis to political parties with a broader interest than just getting votes and related political power, as well as to consider the costs of redistributions which arise due to distortions of the allocation of resources in addition to administration costs.

These various extensions of the present model certainly seem worth studying, as some of them may change the possibility and meaning of equilibrium, as well as efficiency properties of the political process.

NOTES

1. This stochastic approach is in line with the random-utility approach to discrete choice theory, cf. the seminal contribution in McFadden (1973) and the survey in Small and Rosen (1981).
2. Kramer (1978) provides sufficient conditions for the existence of mixed strategy equilibria in deterministic voting games with a continuum of voters.
3. The net party bias $a_i - b_i$ plays a similar role as the 'incumbency premium' p_i in the deterministic model in Kramer (1983), a variable representing the extra credibility attached to a promise by a party which already is in power during the election campaign.
4. Ties can be ignored since $u_i(y, b) - u_i(x, a)$ is a random variable with density. Note also that, for any given distribution functions F_i , the preference representations v_i are cardinal, invariant only under addition of a constant. What really matters, however, is that the resulting probability assignments p_i , as functions of x and y , are the same.
5. Let e_i be a random variable indicating the vote of individual i as follows: $e_i = 1$ if his/her vote goes to A and $= 0$ if it goes to B. Then $E(n_A) = E(\sum e_i) = \sum E(e_i) = \sum p_i$ and $E(n_B) = \sum q_i$.
6. In Hotelling's original model the two players compete in two dimensions, location and price, while here they compete in $m - 1$ dimensions (since each of them selects a vector in R^m satisfying the budget equation).
7. A similar conclusion, obtained under other assumptions and in a different model, is Result

- 3 in Hinich, Ledyard and Ordeshook (1972). For a general class of probabilistic voting models, Coughlin (1982) shows that whenever there is an electoral equilibrium, this is Pareto efficient.
8. According to Stigler (1970), Aaron Director proposed that 'public expenditures are made for the primary benefit of the middle classes, and financed with taxes that are borne in considerable part by the poor and rich' (op. cit., p. 1).
 9. Note however, that the actual outcome in most cases gives one of the parties a majority – always when n is odd and for almost all combination of vectors a and b if n is even and large. For in equilibrium the actual number of votes for party A is $\#\{i \in I; a_i > b_i\}$ and for $\#\{i \in I; a_i < b_i\}$, cf. eq. (1).
 10. They define a convexity index which has the property that for a twice continuously differentiable function v on R , with $v' > 0$, it takes the value $\sup v'' / (v')^2$. Cf. also the notion of 'r-convexity' in Lindberg (1980).
 11. The term 'stochastic heterogeneity' might be preferable to 'uncertainty'. For suppose that the electorate were represented as a continuum, partitioned into discrete groups within which taxes and transfers were not allowed to vary. Then an analogue condition to C1 would be sufficient for existence of equilibrium even in the absence of uncertainty. In this situation, C1 would be a requirement on heterogeneity within groups.
 12. After we had finished our study, cf. Lindbeck and Weibull (1985), a manuscript by Melvin J. Hinich came to our attention, in which the model of Enelow and Hinich (1982) is generalized in certain respects. In this new paper (Hinich, 1984), preferences are still Euclidean, but the policy space is multidimensional and the random components of voters' preferences are identically distributed, according to a general probability density function. In this context, sufficient conditions are given for the existence and uniqueness of Nash equilibrium; some of these conditions together play a similar role as our condition C1.
 13. The most general result on the existence of expected-plurality voting equilibria that we have found is due to Denzau and Katz (1977). The present model satisfies their conditions (A.1), (A.2) and (A.3), but (A.4) only in the special case when every density f_i is symmetric. Our condition C1 implies their condition (A.2'). Moreover, their result, as well as those of Hinich, Ledyard and Ordeshook (1972) and Wittman (1982), require both parties' policy sets to be compact, which they are not in the present model since X_0 is an open set. As a consequence, their equilibria may be corner solutions while ours are always interior.
 14. As a by-product, this shows that the counter-example against existence of equilibrium, in the deterministic version of the model discussed above, does not apply as soon as there is any amount of (symmetric) uncertainty in the parties' assessment of individual party preferences.
 15. Clearly the solution x of eq. (7) maximizes the (Paretian and concave) social-welfare function $\sum f_i(0)v_i(\omega_i + x_{k(i)})$ s.t. $x \in X_0$ (recall $f_i(0) > 0$ by assumption). Hence $[v_i(\omega_i + z_{k(i)}) \geq v_i(\omega_i + x_{k(i)})$ for all i] implies $[z = x]$.
 16. The probability of voting in equilibrium is $p_i + q_i = F(\alpha_i - \epsilon_i) + 1 - F(\alpha_i + \epsilon_i)$. Hence, if $\alpha_i = 0$ and f is symmetric, then $p_i + q_i = 2(1 - F(\epsilon_i))$, which is close to zero if ϵ_i is relatively large. Likewise, if $|\alpha_i| = \epsilon_i$ and f is symmetric, then $p_i + q_i = 1/2 + 1 - F(2\epsilon_i)$, which is close to 1/2 if ϵ_i is relatively large.
 17. Formally, a pair $(x^*, y^*) \in X_A(y) \times X_B(x)$ is a NE of the expected-plurality game with 'voice' if $E(n_A - n_B | x, y^*) \leq E(n_A - n_B | x^*, y^*) \leq E(n_A - n_B | x^*, y)$ for all $(x, y) \in X_A(y) \times X_B(x)$, where $X_A(y) = \{x \in X_0; E(s_A | x, y) \geq \nu n\}$ and $X_B(x) = \{y \in X_0; E(s_B | x, y) \geq \nu n\}$. Ideally, we would like to make the number of votes for a party a monotone function of the size and 'enthusiasm' of the group of 'promoters'. However, for analytical reasons we have chosen a more primitive formalization.
 18. In his seminal contribution, Downs (1957) does not make a clear distinction between the two objectives: '... each party seeks to receive more votes than any other. Thus our

- reasoning has led us . . . to the vote maximizing government . . .' (op. cit. p. 31). However, this question has later been analyzed by Aranson, Hinich and Ordeshook (1974).
19. For n even, $\pi_B = \Pr(\Sigma e_i < n/2) = 1 - \pi_A - \Pr(\Sigma e_i = n/2)$.
 20. In models such as this one, a voter has an incentive (within the model framework) to participate only if his vote affects the outcome of the election. In deterministic models of voting participation in majority elections, this is the case only if all other votes together result in a tie. In contrast, the present stochastic formulation always gives every voter an incentive to vote, since his vote will in all situations influence the probability that his preferred party wins. However, the influence is miniscule and we would not argue that this is the main reason why people vote; see Riker and Ordeshook (1968) for a discussion of various motives for voting.
 21. With n not necessarily odd, but preferably large to make the approximation reasonable.
 22. A necessary and sufficient condition for eq. (15) to have a solution is that the expression η_i in square brackets be positive for every i . For clearly no solution exists if these factors have different signs, and moreover $\Sigma \eta_i > 0$, so all must be positive. Sufficiency follows from the argument given in the proof of Theorem 2. A sufficient condition for all η_i to be positive is clearly $|\Sigma(p_i^0 - q_i^0)| < 4\Sigma p_i^0 q_i^0$, i.e. that the over-all party bias should not be 'too strong' in either direction.

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APPENDIX

First note that $\Pr(n_A + n_B = n) = 1$ in all but one cases studied – the exception being Theorem 4. Hence, in all other cases $E(n_A - n_B) = 2E(n_A) - n$, so maximization (minimization) of $E(n_A - n_B)$ is equivalent to maximization (minimization) of $E(n_A)$, and $E(n_A - n_B)$ is continuously differentiable, concave etc. whenever $E(n_A)$ is.

Lemma A: Let a and b be i.i.d. random variables with continuously differentiable density function ϕ , and let f be the density function of $a - b$. Then f is symmetric. Moreover, if ϕ is symmetric and $\phi'(t) > 0$ for $t < 0$, then f is also unimodal.

Proof: For symmetry, note that $f(t) = \int \phi(s+t) \phi(s) ds$, so $f(-t) = \int \phi(s-t) \phi(s) ds = \int \phi(u) \phi(u+t) du = f(t)$. For unimodality, note that for any $t \in \mathbb{R}$:

$$\begin{aligned} f'(t) &= \int_{-\infty}^{+\infty} \phi'(s+t) \phi(s) ds = \int_{-\infty}^0 \phi'(r) \phi(r-t) dr + \int_0^{+\infty} \phi'(r) \phi(r-t) dr \\ &= \int_{-\infty}^0 \phi'(s) \phi(s-t) ds + \int_{-\infty}^0 \phi'(-u) \phi(-u-t) du \\ &= \int_{-\infty}^0 \phi'(s) \phi(s-t) ds - \int_{-\infty}^0 \phi'(u) \phi(u+t) du \\ &= \int_{-\infty}^0 \phi'(s) [\phi(s-t) - \phi(s+t)] ds. \end{aligned} \quad (A1)$$

For any $s < 0$: $\phi(s-t) < \phi(s+t)$ for all $t > 0$, by unimodality and symmetry of ϕ , so $f'(t) < 0$ for all $t > 0$ and > 0 for all $t < 0$.

Lemma B: If condition C1 holds, then $E(n_A | x, y)$ is concave in x (on X), for any fixed $y \in X$, and convex in y (on X), for any fixed $x \in X$.

Proof: For every k and $i \in I_k$:

$$\partial^2 p_i / (\partial x_k)^2 = v_i''(\omega_i + x_k) f_i(t_i) + (v_i'(\omega_i + x_k))^2 f_i'(t_i) \quad (A2)$$

$$\partial^2 p_i / (\partial y_k)^2 = -v_i''(\omega_i + y_k) f_i(t_i) + (v_i'(\omega_i + y_k))^2 f_i'(t_i) \quad (A3)$$

where $t_i = v_i(\omega_i + x_k) - v_i(\omega_i + y_k)$. Moreover, $\partial p_i / \partial x_k \equiv 0$ for all i not in I_k and $\partial^2 p_i / (\partial x_k \partial x_h) \equiv 0$ for all i and $h \neq k$. Hence $E(n_A | x, y)$, defined on X^2 , is concave in its first argument and convex in its second if, for all $x, y \in X$ and k :

$$\sum_{i \in I_k} [v_i''(\omega_i + x_k) f_i(t_i) + (v_i'(\omega_i + x_k))^2 f_i'(t_i)] \leq 0 \quad (A4)$$

$$\sum_{i \in I_k} [v_i''(\omega_i + y_k) f_i(t_i) - (v_i'(\omega_i + y_k))^2 f_i'(t_i)] \leq 0. \quad (A5)$$

Condition C1 implies that every term in (A4) and (A5) is nonpositive.

Proof of Theorem 2: First, we prove the existence and uniqueness of a solution $(x, \lambda) \in X_0 \times (0, +\infty)$ of eq. (5). By assumption every function v_i' is a strictly decreasing homeomorphism of $(0, +\infty)$ onto itself. Hence, for every $\lambda \in (0, +\infty)$ there exists a unique $x(\lambda) \in X$ satisfying eq. (5). Moreover, $\xi(\lambda) = \sum_{i \in I_k} v_i(x_k(\lambda))$ is continuous and strictly decreasing with $\xi(0) = +\infty$ and $\xi(+\infty) = -\sum_{i \in I_k} (n_k \min\{\omega_i; i \in I_k\}) < 0$. Thus, ξ has a unique zero. In sum: eq. (5) has exactly one solution (x^*, λ^*) in $X_0 \times (0, +\infty)$. As for the existence of a NE, we proceed to show that (x^*, λ^*) is a NE under condition C1. For this purpose, consider the program $\max E(n_A \mid x, x^*)$ s.t. $x \in X_0$. Since $E(n_A \mid x, x^*)$ is concave in x by Lemma B, the necessary first-order condition (3), with $y = x^*$ in the definition of t_i , is also sufficient for x to be a (global) maximum. However, as shown above, (x^*, λ^*) solves eq. (5), and hence also eq. (3) when $y = x^*$, so $z = x^*$ indeed maximizes $E(n_A \mid z, x^*)$ s.t. $z \in X_0$. The same argument applied to the convex program $\min E(n_A \mid x^*, y)$ s.t. $y \in X_0$ gives condition (4), with $x = x^*$, as necessary and sufficient for y to be a (global) minimum. Clearly (x^*, λ^*) also solves this equation, so $z = x^*$ indeed minimizes $E(n_A \mid x^*, z)$ s.t. $z \in X_0$.

Lemma C: If (x, y) is a NE in the expected-plurality game, then C2 holds.

Proof: Suppose (x, y) is a NE, and let $z \in X_0$ be arbitrary. Since X_0 is convex, $z(\epsilon) = z + \epsilon(z - x)$ belongs to X_0 for every ϵ in $(0, 1)$. Let $h(\epsilon, z) = E(n_A \mid z(\epsilon), y)$. By Theorem 1, $y = x$ and $h'_i(0, z) = 0$ for all $z \in X_0$. Moreover, $h''_i(0, z) \leq 0$ must hold for every $z \in X_0$, since otherwise x would not be an optimal policy for party A, contradicting that (x, y) is a NE. This second-order condition is equivalent to C2 (i), and C2 (ii) is the corresponding condition for optimality of y .

Proof of Theorem 3: The first-order conditions (3) and (4) are modified to

$$\sum v'_i(\omega_i + x_k) f_i(t_i) = \Sigma(1 + g'_i(x_k))\lambda \tag{A6}$$

$$\sum v'_i(\omega_i + y_k) f_i(t_i) = \Sigma(1 + g'_i(y_k))\mu \tag{A7}$$

where all summations are made over $i \in I_k$. Hence, if (x, y) is a NE, then both $\Sigma(1 + g'_i(x_k))\lambda$ and $\Sigma(1 + g'_i(y_k))\mu$ are positive for all k . Suppose $\Sigma(1 + g'_i(x_k)) < 0$ for some k . Since $h_k(t) = \Sigma(t + g_i(t))$ (still summing over I_k) by assumption is continuous on \mathbb{R} and unbounded on \mathbb{R}_+ ($g'_i \geq 0$ on \mathbb{R}_+), there exists some $x'_k > x_k$ such that $h_k(x'_k) = h_k(x_k)$. Substituting x'_k for x_k , we obtain a new vector $x' \in X_0$. Since each p_i , for $i \in I_k$, is strictly increasing in x_k , $E(n_A \mid x', y) > E(n_A \mid x, y)$, contradicting that (x, y) is a NE. Hence $\Sigma(1 + g'_i(x_k)) > 0$ for all k , and by symmetry also $\Sigma(1 + g'_i(y_k)) > 0$ for all k . Let

$$\rho_k = \frac{\sum v'_i(\omega_i + x_k) f_i(t_i)}{\sum v'_i(\omega_i + y_k) f_i(t_i)} \cdot \frac{\Sigma(1 + g'_i(y_k))}{\Sigma(1 + g'_i(x_k))} \tag{A8}$$

(still summing over $i \in I_k$). All factors being positive, and $g''_i \geq 0$ by assumption, ρ_k is strictly decreasing in x_k and strictly increasing in y_k . Suppose (x, y) is a NE and $x_k < y_k$ for some k . If it were the case that $x \leq y$, then $E(n_A \mid x, y) < E(n_A \mid y, y)$ by monotonicity with respect to x_k , and (x, y) would not be a NE. Hence $x_h > y_h$ for some h . But then $\rho_k > 1 > \rho_h$, contradicting (A6) and (A7). Thus $x = y$, and $t_i = 0$ for every i .

Proof of Theorem 4: In this case $p_i = F_i(t_i - \epsilon_i)$ and $q_i = 1 - F_i(t_i + \epsilon_i)$. Hence $E(n_A - n_B) = \Sigma[F_i(t_i - \epsilon_i) + F_i(t_i + \epsilon_i)] - n$, so the necessary first-order conditions (3) and (4) are modified to

$$\sum_{i \in I_k} v_i(\omega_i + x_k)[f_i(t_i - \epsilon_i) + f_i(t_i + \epsilon_i)] = n_k \lambda \tag{A9}$$

$$\sum_{i \in I_k} v_i(\omega_i + y_k) [f_i(t_i - \epsilon_i) + f_i(t_i + \epsilon_i)] = n_k \mu. \quad (\text{A10})$$

By the same argument as in the proof of Theorem 1, $x = y$ and $t_i = 0$ for all i .

Proof of Theorem 6: First note

$$\partial \bar{\pi}_A(x, y) / \partial x_i = -(2\pi)^{-1/2} (\partial \psi(p) / \partial p_i) \exp[-\psi^2(p)/2] \partial p_i / \partial x_i \quad (\text{A11})$$

$$\partial \bar{\pi}_A(x, y) / \partial y_i = -(2\pi)^{-1/2} (\partial \psi(p) / \partial p_i) \exp[-\psi^2(p)/2] \partial p_i / \partial y_i. \quad (\text{A12})$$

Hence, if (x, y) is a NE, then there are Lagrangians λ and μ such that for all i

$$-(\partial \psi / \partial p_i) \partial p_i / \partial x_i = \lambda \quad (\text{A13})$$

$$(\partial \psi / \partial p_i) \partial p_i / \partial y_i = \mu \quad (\text{A14})$$

(Note that $\exp[-\psi^2(p)/2]$ is positive and independent of i .) Differentiation of eq. (14) gives

$$\partial \psi / \partial p_i = -[\Sigma p_j q_j + (p_i - q_i) \Sigma (p_j - q_j) / 4] / (\Sigma p_j q_j)^{3/2} \quad (\text{A15})$$

Thus $\Sigma \partial \psi / \partial p_j < 0$, i.e. $\partial \psi / \partial p_i < 0$ for some i . But then $\partial \psi / \partial p_i < 0$ for all i at equilibrium, by (A13). Consequently $\lambda, \mu > 0$, and (A13) and (A14) together imply $v_i(\omega_i + x_i) / v_i(\omega_i + y_i) = \lambda / \mu$, which gives $x = y$ by the same argument as in the proof of Theorem 1. Now $x = y$ implies $p_i = p_i^0$ and $q_i = q_i^0$, so eq. (15) follows directly from eqs. (A13) and (A15).