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Vertex Operator Algebras and Representations of Affine Lie Algebras

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Abstract. We announce the construction of an explicit basis for all integrable highest weight modules over the Lie algebra $A_1^{(1)}$. The construction uses representations of vertex operator algebras and leads to combinatorial identities of Rogers–Ramanujan-type.

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1. Introduction

In a sequence of papers culminating in [9, 10], J. Lepowsky and R. L. Wilson have given a Lie theoretical interpretation and proof of the Rogers–Ramanujan identities. The product sides of the identities were obtained from a 'principal' specialization of the Weyl–Kac character formula for the level 3 integrable highest weight modules of the Lie algebra $A_1^{(1)}$. The sum sides were obtained by constructing an explicit basis of the modules parametrized by partitions.

In this paper, we summarize recent work [11] leading to an explicit basis for each integrable highest weight module over the Lie algebra $A_1^{(1)}$. In contrast to [9, 10], this basis is associated to the so-called 'homogeneous' gradation of the modules. Combining our results with the Weyl–Kac character formula, we obtain a series of apparently new combinatorial identities.

After introducing some notation we formulate in Section 2 the end result (Theorem 2) by giving the conditions required on the 'colored' partitions that parametrize the basis. Vertex operator algebras (VOA) enter into the proof of Theorem 2 and in Section 3 we give a definition of VOA and describe some of their properties. We also announce Theorem 4 giving a way to construct VOA's by a kind of generators and relations. Section 4 then outlines part of the proof of Theorem 2. The details will appear elsewhere.

2. Bases of Integrable Highest Weight Modules

Let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra. In the associated affine Lie algebra

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

we introduce the notation $x(n) = x \otimes t^n$ for $x \in \mathfrak{g}$, $n \in \mathbb{Z}$, and identify $\mathfrak{g}$ as the subalgebra $\mathfrak{g} \otimes t^0$ of $\tilde{\mathfrak{g}}$. Set $\mathfrak{h} = \mathfrak{h}^0 \oplus \mathbb{C}c \oplus \mathbb{C}d$, with $\mathfrak{h}^0$ a Cartan subalgebra of $\mathfrak{g}$. Let $\Lambda_0 \in \mathfrak{h}^*$ denote the linear form satisfying

$$\Lambda_0(\mathfrak{h}^0) = 0, \qquad \Lambda_0(d) = 0, \qquad \Lambda_0(c) = 1.$$

Then Λ_0 is a fundamental weight. For arbitrary $\Lambda \in \mathfrak{h}^*$, let $M(\Lambda)$ be the Verma module with highest weight Λ, and let $L(\Lambda)$ be the unique irreducible quotient of $M(\Lambda)$. We shall also need the Heisenberg algebra

$$\hat{\mathfrak{h}} = \bigoplus_{n \neq 0} \mathfrak{h}^0 \otimes t^n \oplus \mathbb{C}c,$$

and its Abelian subalgebra

$$\hat{\mathfrak{h}}^- = \bigoplus_{n<0} \mathfrak{h}^0 \otimes t^n.$$

There is then an irreducible (Fock) representation of $\hat{\mathfrak{h}}$ on $S(\hat{\mathfrak{h}}^-)$ with c acting as the identity operator. Let Q be the root lattice of $\mathfrak{g}$ with respect to $\mathfrak{h}^0$.

THEOREM 1 ([5]). *If $\mathfrak{g}$ is of type ADE then the fundamental $\tilde{\mathfrak{g}}$-module $L(\Lambda_0)$ can be realized as*

$$L(\Lambda_0) = S(\hat{\mathfrak{h}}^-) \otimes \mathbb{C}[Q].$$

The operators $x_\alpha(n)$ (x_α root vectors of $(\mathfrak{g}, \mathfrak{h}^0)$) are then determined from the generating function $X(\alpha, z) = \sum_{n\in\mathbb{Z}} x_\alpha(n) z^{-n-1}$ through the vertex operator formula

$$X(\alpha, z) = E^-(\alpha, z) E^+(\alpha, z) \otimes e_\alpha z^{\alpha(0)},$$

$$E^\pm(\alpha, z) = \exp\left(- \sum_{\substack{n\in\mathbb{Z} \\ \pm n>0}} \frac{\alpha(n) z^{-n}}{n} \right),$$

where e_α up to a ± 1-valued cocycle is multiplication by a group element in the group algebra $\mathbb{C}[Q]$.

Let $\mathfrak{k} = \mathfrak{sl}(2, \mathbb{C})$ and consider the case $\mathfrak{g} = \mathfrak{k}$ of Theorem 1. The special case $e^A e^B = e^B e^A e^{[A,B]}$ of the Baker–Campbell–Hausdorff formula in a Heisenberg group gives

$$X(\alpha, z_1)X(\alpha, z_2)$$
$$= (z_1 - z_2)^2 E^-(\alpha, z_1)E^-(\alpha, z_2)E^+(\alpha, z_1)E^+(\alpha, z_2) \otimes e_\alpha^2 z_1^{\alpha(0)} z_2^{\alpha(0)}$$

on $L(\Lambda_0)$. This shows that it is well-defined to take $z_1 = z_2$ in $X(\alpha, z_1)X(\alpha, z_2)$ and implies that $X(\alpha, z)^2 = 0$ on $L(\Lambda_0)$. A similar argument shows that $X(\alpha, z)^2 = 0$ on $L(\Lambda_1)$, Λ_1 the other fundamental weight of $A_1^{(1)}$. Let $\Lambda = k_0\Lambda_0 + k_1\Lambda_1$, $k_0, k_1 \in \mathbb{N}$ be a dominant integral weight of $\tilde{\mathfrak{k}}$, and set $k = k_0 + k_1$. One can then realize

$$L(\Lambda) \subset L(\Lambda_0)^{\otimes k_0} \otimes L(\Lambda_1)^{\otimes k_1}$$

as the submodule generated by the tensor product of highest weight vectors. Then

$$X(\alpha, z)^{k+1} = 0 \quad \text{on } L(\Lambda) \tag{1}$$

follows from

$$X(\alpha, z)^{k+1} = \sum_{j_1+\cdots+j_k=k+1} X(\alpha, z)^{j_1} \otimes \cdots \otimes X(\alpha, z)^{j_k},$$

since by the Dirichlet box principle some $j_r \geqslant 2$ in each term.

Let $\{x, h, y\}$ be the usual basis of $\mathfrak{k} = \mathfrak{sl}(2, \mathbb{C})$, set

$$\overline{B} = \{x(n), h(n), y(n) \mid n \in \mathbb{Z}\},$$

$$\overline{B}_- = \{x(n), h(n), y(n) \mid n < 0\} \cup \{y(0)\},$$

$$\mathcal{P}(\overline{B}_-) = \{\pi\colon \overline{B}_- \to \mathbb{N} \mid \pi \text{ has finite support}\}.$$

Order $\overline{B}_-$ so that $x(n-1) < y(n) < h(n) < x(n)$ and let for $\pi \in \mathcal{P}(\overline{B}_-)$, $u(\pi) = \prod_{b \in \overline{B}_-} b^{\pi(b)}$ (ordered product) be an element in the enveloping algebra $U(\tilde{\mathfrak{k}})$. Our main result then states:

THEOREM 2. *For dominant integral $\Lambda \in \mathfrak{h}^*$, the $\tilde{\mathfrak{k}}$-module $L(\Lambda)$ has a basis consisting of the vectors $u(\pi)v_\Lambda$ (v_Λ a highest weight vector), where $\pi \in \mathcal{P}(\overline{B}_-)$ satisfies*

$$\pi(y(j-1)) + \pi(h(j-1)) + \pi(y(j)) \leqslant k,$$

$$\pi(h(j-1)) + \pi(x(j-1)) + \pi(y(j)) \leqslant k,$$

$$\pi(x(j-1)) + \pi(y(j)) + \pi(h(j)) \leqslant k,$$

$$\pi(x(j-1)) + \pi(h(j)) + \pi(x(j)) \leqslant k,$$

$$\pi(x(-1)) \leqslant k_0, \quad \pi(y(0)) \leqslant k_1.$$

Note that since $x(j-1)^a x(j)^{k+1-a}$ is a coefficient in $X(\alpha, z)^{k+1}$, (1) implies part of the 4th constraint on π.

3. Vertex Operator Algebras

The proof of Theorem 2 uses calculations in an associated vertex operator algebra. Vertex operator algebras (VOA), also known in the physics literature under names such as chiral algebras and meromorphic conformal field theories, were introduced in mathematics in [1, 4, 6] in connection with the 'moonshine' module for the monster sporadic group. A *vertex operator algebra* is a quadruple $(V, Y, \mathbf{1}, \omega)$ where

$$V = \bigoplus_{n \geqslant -N} V_n, \quad \dim V_n < \infty,$$

is a graded complex vector space, $\mathbf{1}, \omega \in V$ are distinguished vectors and

$$\begin{aligned} Y\colon V &\longrightarrow (\operatorname{End}_{\mathbb{C}} V)[[z, z^{-1}]] \\ v &\longmapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \end{aligned}$$

is a linear map. The formal series $Y(v, z)$ are called vertex operators. Let $V' = \bigoplus_{n \geqslant -N} V_n^*$ be the graded dual to V. The main axiom of a VOA, then states that to any $u, v, w \in V$, $w' \in V'$ there is a rational function

$$f(z_1, z_2) = \frac{g(z_1, z_2)}{z_1^l z_2^m (z_1 - z_2)^n}, \quad g(z_1, z_2) \in \mathbb{C}[z_1, z_2],$$

such that as analytic functions of (z_1, z_2),

$$\langle w', Y(u, z_1) Y(v, z_2) w \rangle = f(z_1, z_2) \quad \text{when } |z_1| > |z_2|,$$

$$\langle w', Y(v, z_2) Y(u, z_1) w \rangle = f(z_1, z_2) \quad \text{when } |z_2| > |z_1|,$$

$$\langle w', Y(Y(u, z_1 - z_2) v, z_2) w \rangle = f(z_1, z_2) \quad \text{when } |z_1 - z_2| < |z_2|. \tag{2}$$

Other axioms demand

$$v_{-1}\mathbf{1} = v, \qquad v_n \mathbf{1} = 0 \quad \text{for } n \geqslant 0,\ v \in V,$$

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},$$

where the operators L_n generate a Virasoro algebra

$$[L_m, L_n] = (m - n) L_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} cI, \quad c = \operatorname{rank}(V) \in \mathbb{C}.$$

Furthermore

$$[L_{-1}, Y(v,z)] = \frac{\mathrm{d}}{\mathrm{d}z}Y(v,z) = Y(L_{-1}v, z), \tag{3}$$

$$L_0|_{V_n} = n\,\mathrm{id}_{V_n}.$$

Some consequences of the axioms are the formulas

$$[u_m, v_n] = \sum_{i=0}^{\infty} \binom{m}{i} (u_i v)_{m+n-i}, \tag{4}$$

$$v_n(V_p) \subset V_{p+m-n-1} \quad \text{for } v \in V_m,$$

$$\begin{aligned} Y(u_{-n}v, z) \;=\; & \left(\frac{(\mathrm{d}/\mathrm{d}z)^{n-1}}{(n-1)!} Y(u,z)^-\right) Y(v,z) + \\ & + Y(v,z)\left(\frac{(\mathrm{d}/\mathrm{d}z)^{n-1}}{(n-1)!} Y(u,z)^+\right), \quad n \geqslant 1, \end{aligned} \tag{5}$$

for $u, v \in V$, where we set

$$Y(u,z)^+ = \sum_{n\geqslant 0} u_n z^{-n-1}, \qquad Y(u,z)^- = Y(u,z) - Y(u,z)^+.$$

In the affine Lie algebra $\tilde{\mathfrak{g}}$, let

$$\tilde{\mathfrak{g}}_{\geqslant 0} = \mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

be a parabolic subalgebra. For $k \in \mathbb{C}$ consider the generalized Verma module

$$N(k\Lambda_0) = U(\tilde{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{g}}_{\geqslant 0})} \mathbb{C}v_{k\Lambda_0}.$$

Let $h^\vee$ be the dual Coxeter number of $\mathfrak{g}$.

THEOREM 3 ([7]). *For $k \neq -h^\vee$ there is a unique structure of VOA on $N(k\Lambda_0)$ such that $\mathbf{1} = v_{k\Lambda_0}$ and*

$$Y(x(-1)\mathbf{1}, z) = \sum_{n\in\mathbb{Z}} x(n) z^{-n-1}$$

for $x \in \mathfrak{g}$.

The VOA of Theorem 3 is related to the Wess–Novikov–Zumino–Witten model in the physics literature. In [11], Theorem 3 is derived as a corollary of a 'generators and relations' type construction of vertex operator algebras that we proceed to describe.

Let V be a complex vector space, and assume given operators L_0, L_{-1} on V such that $[L_0, L_{-1}] = L_{-1}$. Set

$$V_n = \{v \in V \mid L_0 v = nv\},$$

and assume that

$$V = \bigoplus_{n \geqslant -N} V_n.$$

Let $\mathbf{1} \in V_0$. For the following theorem cf. [3, 8, 12].

THEOREM 4. *With $V, L_0, L_{-1}, \mathbf{1}$ as above, let U be a subspace of V and assume there is a linear map*

$$Y\colon U \longrightarrow (\operatorname{End}_{\mathbb{C}} V)[[z, z^{-1}]]$$
$$u \longmapsto Y(u, z) = \sum_{n \in \mathbb{Z}} u_n z^{-n-1}$$

such that

$$[L_{-1}, Y(u, z)] = \frac{\mathrm{d}}{\mathrm{d}z} Y(u, z), \tag{6}$$

$$u_{-1}\mathbf{1} = u, \qquad u_n \mathbf{1} = 0 \quad \text{for } u \in U,\ n \geqslant 0, \tag{7}$$

for all $u_1, u_2 \in U$ there is $N = N(u_1, u_2) \in \mathbb{N}$ such that

$$(z_1 - z_2)^N [Y(u_1, z_1), Y(u_2, z_2)] = 0, \tag{8}$$

$$V = \mathbb{C}\text{-span}\{u^{(1)}_{n_1} \cdots u^{(k)}_{n_k} \mathbf{1} \mid k \in \mathbb{N}, u^{(i)} \in U, n_i \in \mathbb{Z}\}. \tag{9}$$

Then Y has a unique extension to V making V into a VOA (except that we only get the operators L_0, L_{-1} out of the Virasoro algebra).

Theorem 3 follows by taking $V = N(k\Lambda_0)$, $U = \mathfrak{g}(-1)\mathbf{1} \simeq \mathfrak{g}$.

There is a natural notion of module over a VOA V. A V-module W is a graded complex vector space

$$W = \bigoplus_{n \in q + \mathbb{N}} W_n, \quad q \in \mathbb{C}, \quad \dim W_n < \infty,$$

together with a linear map

$$Y_W\colon V \longrightarrow (\operatorname{End}_{\mathbb{C}} W)[[z, z^{-1}]]$$
$$v \longmapsto Y_W(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}$$

satisfying the same properties as the vertex operators on V except that we do not require the existence of a distinguished 'vacuum vector' $\mathbf{1}$ in W. The operators Y_W 'represent' the structure V through the appearance of Y_V in the analogue of (2)

$$\langle w', Y_W(Y_V(u, z_1 - z_2)v, z_2)w\rangle = f(z_1, z_2), \quad \text{for } |z_1 - z_2| < |z_2|.$$

THEOREM 5 (cf. [7]). *Let W be a highest weight $\tilde{\mathfrak{g}}$-module of highest weight Λ and level $k = \Lambda(c)$. If $k \neq -h^\vee$ then W has a structure of VOA-module over $N(k\Lambda_0)$ such that*

$$Y_W(x(-1)\mathbf{1}, z) = \sum_{n\in\mathbb{Z}} x(n)z^{-n-1}$$

for $x \in \mathfrak{g}$.

The proof of Theorem 5 is based on a construction of 'intertwining' operators $Y\colon\ W \to \mathrm{Hom}(V, W)[[z, z^{-1}]]$ that are 'local' with respect to the operators $Y(x(-1)\mathbf{1}, z)$, $x \in \mathfrak{g}$.

4. Relations on Integrable Modules

We shall now describe how the VOA-structure enters into the proof of Theorem 2. Let from now on $\mathfrak{g} = \mathfrak{k} = \mathfrak{sl}(2, \mathbb{C})$ with the standard basis $\{x, h, y\}$. Let $N^1(k\Lambda_0)$ be the kernel of the quotient map $N(k\Lambda_0) \to L(k\Lambda_0)$ so that we have an exact sequence

$$0 \longrightarrow N^1(k\Lambda_0) \longrightarrow N(k\Lambda_0) \longrightarrow L(k\Lambda_0) \longrightarrow 0.$$

Take $k \in \mathbb{Z}$, $k \geqslant 1$. As is well known, $N^1(k\Lambda_0)$ is then generated by the vector

$$r_{k+1} = x(-1)^{k+1}\mathbf{1} = f_0^{k+1} v_{k\Lambda_0} \in N^1(k\Lambda_0).$$

By (5) the vertex operator associated to r_{k+1} is

$$Y(r_{k+1}, z) = : X(\alpha, z) \cdots X(\alpha, z) := X(\alpha, z)^{k+1}$$

since all operators $x(n)$, $n \in \mathbb{Z}$ commute. As $X(\alpha, z)^{k+1}$ vanishes on $L(\Lambda)$ for $\Lambda = k_0\Lambda_0 + k_1\Lambda_1$, $k_0, k_1 \in \mathbb{N}$ (1), it follows from (5) that $Y(r, z)$ vanishes on $L(\Lambda)$ for each $r \in N^1(k\Lambda_0)$, in particular for each $r \in R$ where $R = U(\mathfrak{k})r_{k+1}$ is a $2k+3$-dimensional irreducible $\mathfrak{k}$-submodule. As R is the subspace of highest degree in $N^1(k\Lambda_0)$, (4) implies that

$$[a(m), r_n] = (a \cdot r)_{m+n}$$

for $a \in \mathfrak{k}$, $r \in R$, $m, n \in \mathbb{Z}$ so that

$$\overline{R} = \{r_n \mid r \in R,\ n \in \mathbb{Z}\} \simeq R \otimes \mathbb{C}[t, t^{-1}]$$

is an irreducible loop module for $\tilde{\mathfrak{k}}$. The fact that $\overline{R} \cdot L(\Lambda) = 0$ can be alternatively seen as a consequence of a theorem of Chari and Pressley [2]: if $\overline{R}$ is an irreducible loop module and $L(\Lambda)$ an integrable highest weight module, then $\overline{R} \otimes L(\Lambda)$ is (except for a small number of exceptions) irreducible. Since clearly $\overline{R} \otimes L(\Lambda) \not\simeq L(\Lambda)$ this forces $\overline{R} \cdot L(\Lambda) = 0$.

Let $M^1(\Lambda)$ be the maximal submodule of the Verma module $M(\Lambda)$. It is easy to show that the set $\{u(\pi)r_n v_\Lambda \mid u(\pi) \in U(\tilde{\mathfrak{k}}),\ r \in R,\ n \in \mathbb{Z}\} \subset M(\Lambda)$ spans $M^1(\Lambda)$. To prove Theorem 2 we must reduce this spanning set to a basis. This requires nontrivial relations among the operators r_n, $r \in R$, $n \in \mathbb{Z}$, acting on $M(\Lambda)$. These are produced by expressing suitable vectors in $N(k\Lambda_0)$ in two different ways and then identifying the associated vertex operators.

First, from

$$L_{-1}x(-1)^{k+2}\mathbf{1} = (k+2)x(-2)x(-1)^{k+1}\mathbf{1}$$

it follows from (5) and (3) that, with the notation $x(z) = \sum_{n\in\mathbb{Z}} x(n)z^{-n-1}$, $h(z) = \sum_{n\in\mathbb{Z}} h(n)z^{-n-1}$, $y(z) = \sum_{n\in\mathbb{Z}} y(n)z^{-n-1}$,

$$\frac{\mathrm{d}}{\mathrm{d}z}\Big(x(z)r_{k+1}(z)\Big) = (k+2) : \left(\frac{\mathrm{d}}{\mathrm{d}z}x(z)\right)r_{k+1}(z) : . \tag{10}$$

In addition, the Sugawara construction of the Virasoro algebra

$$\begin{aligned} L_n &= \frac{1}{2(k+2)}\sum_{m\in\mathbb{Z}} 2 : x(m)y(n-m) : + \\ &\quad + : h(m)h(n-m) : +2 : y(m)x(n-m) : \end{aligned}$$

implies that

$$\begin{aligned} L_{-1}r_i &= \frac{1}{k+2}\Big\{(k+2-i)x(-1)r_{i-1} + ih(-1)r_i + \\ &\qquad +(k+2+i)y(-1)r_{i+1}\Big\} \end{aligned}$$

for a suitable basis $\{r_i \mid -k-1 \leqslant i \leqslant k+1\}$ of R. Formulas (5) and (3) give

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}z}Y(r_i,z) \\ &\quad = \frac{1}{k+2}\Big\{(k+2-i) : x(z)Y(r_{i-1},z) : + \\ &\qquad\qquad +i : h(z)Y(r_i,z) : +(k+2+i) : y(z)Y(r_{i+1},z) :\Big\}. \end{aligned} \tag{11}$$

Extracting the coefficients of z^n in (10) and (11) gives enough relations among the operators r_n, $r \in R$, $n \in \mathbb{Z}$ to prove Theorem 2. The proof of Theorem 2 can now be completed by combinatorial arguments involving a total order on a Poincare–Birkhoff–Witt type basis of $M(\Lambda)$.

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