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Continuous crack growth or quantized growth steps?

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ABSTRACT
The assertion that a non-vanishing Griffith energy release rate requires an r^{-1} type singularity at the tip of a crack for the energy intensity, i.e. the product of stress and strain, is examined. When the existence of such a singularity is denied on physical grounds continuum mechanics energy balance considerations suggest that initial unstable crack extension is by a discrete growth step of characteristic size Δa.

Notation

a = half crack length
b = normal displacement from the crack plane, at the centre of the crack
C = constant in equation (1)
c = subscript denoting critical values
d = total displacement conjugate to force F
d_1 = total crack tip displacement
E = modulus of elasticity
$f(\xi)$ = displacement function
F = crack separation force on Δa per unit thickness
G = Griffith energy release rate
G^Δ = crack separation energy rate
K_I = Irwin's mode I stress intensity factor
r = distance from initial crack tip measured in x-direction
s = distance from extended crack tip or tip of cohesive zone towards the centre of the crack $(=\Delta a - r)$
v = vertical displacement from crack plane
v_1 = crack tip displacement from crack plane
x, y = coordinates
$\alpha = d_1/\Delta a$
$\eta = d/\Delta a$
Δa = crack extension
ΔW = energy absorbed at Δa during separation of the surfaces
$\xi = s/\Delta a$
$\bar{\xi}$ = value of ξ at which the crack opening displacement is d
σ = normal stress on potential crack surfaces
σ_∞ = remotely applied normal stress
$\bar{\sigma}$ = average normal stress over Δa
σ_y = yield stress in tension
ν = Poisson's ratio
Λ = absorption energy density $(=\frac{1}{2}\bar{\sigma}\eta)$

1. Introduction

It has been recognised for some time tacitly or explicitly [1–9] that a non-vanishing Griffith energy release rate at the tip of a crack necessitates an r^{-1} type singularity for the products of stress and strain in the crack tip region. The following is intended as a

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heuristic explanation of concepts underlying this phenomenon which is sometimes referred to as an energy sink and leads to the suggestion of a possible mode of unstable crack extension by discrete growth step, not dependent on the existence of crack tip singularities.

2. Discussion

Consider a sharp crack of length $2a$ in an infinite plane in plane strain, loaded by a remotely applied stress σ_∞ normal to the crack surfaces, i.e. the Inglis configuration; see Fig. 1. Now imagine that the crack extends under constant load by an amount Δa which is small compared to a and that this occurs by the proportional quasi-static release of the cohesive forces holding the surfaces together at Δa. Let s be the distance measured towards the centre from the extended crack tip and let ξ equal to $s/\Delta a$ be the normalized distance, as shown in Fig. 2. The normal stress in Δa before extension is $\sigma(\xi)$ and the average stress over Δa is $\bar{\sigma}$ equal to $\int_0^1 \sigma(\xi)\,d\xi$. The normal force per unit thickness acting on Δa is F equal to $\bar{\sigma}\Delta a$. Let the vertical displacement of the top separating surface from the plane of the crack be $v(\xi)$ equal to $(K_I/K_{IC})f(\xi)\Delta a$, where $f(\xi)$ is a function of ξ and possibly of the applied load factor (K_I/K_{IC}). Here K_I equal to $\sigma_\infty(\pi a)^{1/2}$ and K_{IC} are Irwin's mode I stress intensity factor and its critical value, respectively. Let the total displacement at the tip be d_1 equal to $2v_1$ or to $\alpha\Delta a$ where α is $2(K_I/K_{IC}f(1)$. However, we are more concerned here with the displacement d equal to $\eta\Delta a$, conjugate to the force F where η is a number. The

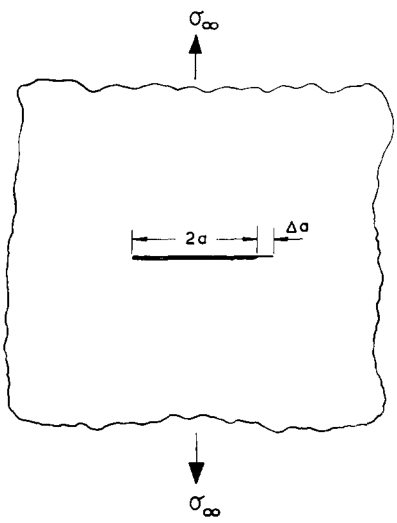

Figure 1. Crack in an infinite plane under uniaxial tension.

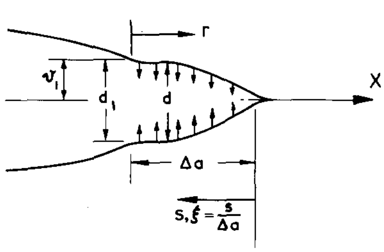

Figure 2. Separation of surfaces during stress relaxation in region Δ_a.

work absorbed during separation as the crack extends can be written

$$\Delta W = CF\ d = \tfrac{1}{2}\bar{\sigma}\eta\Delta a^2 \tag{1}$$

where C is a number having a value between 0 and 1. When the stress-displacement law at the separating surfaces is linear C equals 0.5 and for convenience the value of C equal to 0.5 is assumed in (1). Note that any assumed value of C can be absorbed in the definition of d.

The displacement d conjugate to F can be the actual displacement at some point ξ equal to $\bar{\xi}$. When the form of the stress distribution over Δa before extension and that of the crack profile after extension do not depend on Δa, $\bar{\xi}$ also does not depend on Δa. This is true when the material is linear elastic responding linearly at the crack surfaces and for a number of Barenblatt type models [10, 11, 12] if Δa is identified with the cohesive zone size. In the former case $\bar{\xi}$ equals $\pi^2/16$; see Appendix I. In the Barenblatt type models the buld of the material is assumed to be linear elastic but the response in the cohesive zone is not linear. One of the limitations of Barenblatt type models is that the maximum stress under the assumed stress-displacement law is far in excess of the stress which the material can withstand without yielding, assuming a quasi-static loading situation. In real elastic-plastic materials the stress distribution over Δa and $\bar{\xi}$ vary with the crack tip plastic zone size $/\Delta a$ ratio but this fact in itself is not likely to invalidate the main conclusions in what follows.

The non-dimensional quantities α equal to $d_1/\Delta a$ and η equal to $d/\Delta a$ can be looked upon as angles or strains to which the crack tip region is subjected during the crack extension process. If we define an energy absorption density Λ by

$$\Lambda = \tfrac{1}{2}\bar{\sigma}\eta \tag{2}$$

(1) becomes

$$\Delta W = \Lambda\Delta a^2 \tag{3}$$

The crack separation energy rate [14, 15] is the energy available for surface separation over the distance Δa, divided by Δa and it is given by

$$G^\Delta = \frac{\Delta W}{\Delta a} = \Lambda\Delta a \tag{4}$$

It is assumed that the cohesive strength G_c is a material property and G_c is equal to the value of G^Δ at fracture. Using the subscript c to denote critical values,

$$G_c^\Delta = \Lambda_c\Delta a_c = G_c \tag{5}$$

Note that the Griffith energy release rate for an elastic material, G, can be obtained by making Δa infinitesimal in (4), i.e.

$$G = \lim_{\Delta a \to 0} G^\Delta \tag{6}$$

The inescapable conclusion drawn from (4) is that G given by (6) can be different from zero or finite only if Λ varies as $(\Delta a)^{-1}$ as Δa tends to zero. In a linear elastic material the stress is proportional to the strain and the absorption energy density Λ is therefore proportional to $\bar{\sigma}^2$ and also to η^2. It follows that $\bar{\sigma}$ and η must vary as $(\Delta a)^{-1/2}$. This is of course the case since

$$\eta = \frac{K_I(1-\nu^2)}{E}\left(\frac{2\pi}{\Delta a}\right)^{1/2} \tag{7}$$

see Appendix I, and the normal stresses ahead of the crack tip are given by the familiar expression

$$\sigma(r) = \frac{K_I}{\sqrt{2\pi r}} \text{ where } r = \Delta a - s \tag{8}$$

In an ideal perfectly plastic material $\bar{\sigma}$ is smaller or equal to $(2+\pi)\sigma_y/\sqrt{3}$ where σ_y is the yield stress in tension. Since $\bar{\sigma}$ is bounded η must vary as $(\Delta a)^{-1}$ and this is in agreement with Rice's solution [2] for this material. The Rice-Rosengren-Hutchinson solutions for power law hardening materials [7, 8, 9] also reveal an r^{-1} type singularity at the crack tip for the product of stress and strain. Generally, for all ideal materials exhibiting singularities satisfying this assumption one would expect the fracture criterion (5) to be equivalent to Griffith's fracture criterion, $G = G_c$.

In a real material crack extension is preceded by an initiation stage causing crack tip blunting and intense plasticity in the immediate vicinity of the crack tip. With increasing load some stable crack growth may follow if the material is sufficiently ductile, often ending in catastrophic crack propagation. This last stage can be subdivided into an initial transient stage followed possibly by near steady state crack propagation. The state with which we are concerned here is that of incipient instability or the very early stages of transient crack propagation, when the crack tip has moved sufficiently to be out of the close range effect of the concentrated residual plasticity incurred near the crack tip during the initiation stage. Nevertheless, the state considered is essentially quasi-static, on the general premise that even a running crack must walk before it can run. It is acknowledged that the energy balance for dynamic crack propagation must be different from that prevailing at incipient instability.

Since an infinite stress is not possible in a real material, the Griffith criterion equating G to G_c at fracture can be satisfied only if the required r^{-1} strain singularity is engendered at the tip. This would imply a blunt crack profile consistent with Rice's solution for a non-hardening material mentioned earlier. However if both stress and strain singularities are ruled out on physical grounds, then Λ is bounded and Griffith's criterion cannot be satisfied since G vanishes. A model often used to represent continuous crack extension satisfying the Griffith fracture criterion is a translating Barenblatt type cohesive zone of sufficient finite dimension to accommodate a tip displacement of the requisite magnitude. However it is difficult to reconcile the state of crack equilibrium in the cohesive zone with the difference which exists in the case of most materials between the maximum inter-atomic force in the Barenblatt model and the much smaller yield stress of the material. If on the other hand the stresses in the cohesive zone are taken to be of the order of the yield stress as in the Dugdale model, then the minimum tip displacement satisfying the energy rate requirements is probably much larger than the range of the forces acting on the fracture surfaces – leading to an inconsistency since the two values should be the same. Nevertheless the possibility of continuous translation of an equilibrium crack is not necessarily excluded as the actual surface separation processes may be of a much more complex nature than the simple one used in the model.

An alternative explanation has been proferred [14, 15] that initially the crack does not extend continuously in infinitesimal steps but that unstable propagation begins with a discrete growth step of size Δa. Note from (4) that Δa appears to have a "leverage" effect on the crack separation energy rate G^{Δ}. Hence for a finite value of Δa, the fracture criterion (5) can be satisfied even when Λ is bounded. The difference between the fracture criterion (5) and Griffith's criterion is now apparent. It would seem therefore that the value of Δa is partly determined by a quantization process dictated by (5). Other factors more difficult to assess are of a metallurgical nature and concern the micro-mechanisms connected with the dominant fracture mode. Although the elastic plastic continuum model may be used to derive certain general conclusions as we have done here, it is too simple to explain adequately the fracture process and a more realistic representation allowing for rate effects, microcracking and mechanical instability in the fracture zone is required [16]. These questions are not considered in the present analysis.

The need for considering a finite process zone ahead of the crack tip has been felt for many years by a number of workers [6, 10–15, 17–21]. B. Cotterell [22] refers to the formation of microcracks some distance below the root of a notch before the main fracture occurs; see also V.F. Zakay et al. [23]. For fracture nucleated by slip bands [24] Cotterell suggests that cleavage fracture occurs when the microcracks suddenly form a region of plastic instability. He adds that the presence of microcracks alone does not cause cleavage fracture; an essential condition for fracture to occur is that the size of the region of plastic instability reaches a critical minimum value. He also infers that the critical size is related to the critical crack opening displacement [25].

3. Conclusions

Continuum mechanics energy balance considerations suggest that at incipient instability in a real material a crack may extend by a discrete growth step. A critical minimum size for the growth step must be attained in order that the rate of energy released be large enough to overcome the cohesive resistance of the material, equal to the energy per unit area required for surface separation. This necessary condition for fracture instability may possibly be satisfied by the formation of microcracks some distance ahead of the crack tip followed by the rapid spreading towards the crack tip of a zone of instability of the required minimum size, as suggested by Cotterell [22]. However the precise active micro-mechanism involved in the fracture process is left an open question.

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Appendix I

Values of $\bar{\xi}$, η and α for linear elastic materials

When a linear elastic material responds linearly also at the crack surfaces, the crack profile is elliptical and is given by $x^2/a^2 + y^2/b^2 = 1$, or

$$y = \pm b \frac{s}{a} \left(\frac{2a}{s} - 1 \right)^{1/2} \tag{A1}$$

(see Fig. A1) where a and b are the major and minor axes of the ellipse, respectively. When $s \ll a$, (A1) becomes approximately,

$$v = \pm b \left(\frac{2s}{a} \right)^{1/2} \tag{A2}$$

Now, $b = (2K_I(1 - \nu^2)/E)a/\pi^{1/2}$. Hence

$$\eta = d/\Delta a = \frac{4K_I(1 - \nu^2)}{E} \left(\frac{2\bar{\xi}}{\pi} \right)^{1/2} (\Delta a)^{-1/2} \tag{A3}$$

Using (8) and (A2) the work absorbed during separation is

$$\Delta W = \frac{2K_I^2(1 - \nu^2)}{E\pi} \int_0^{\Delta a} \left(\frac{\Delta a - r}{r} \right)^{1/2} dr$$

This is evaluated by making the substitution $(\Delta a/r - 1) = \tan^2\theta$ giving

$$\Delta W = \frac{K_I^2(1 - \nu^2)}{E} \Delta a \tag{A4*}$$

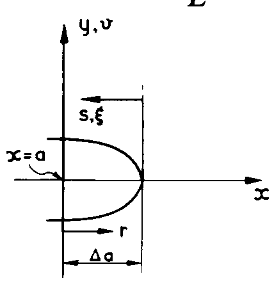

Fig. A I.

* The derivation of this equation has been carried out previously, e.g. G. R. Irwin [26].

or

$$G^{\Delta} = G \tag{A5}$$

The average stress is

$$\bar{\sigma} = \frac{1}{\Delta a} K_{\mathrm{I}}(2\pi)^{-1/2} \int_0^{\Delta a} r^{-1/2}\, dr = 2K_{\mathrm{I}}(\pi \Delta a)^{-1/2} \tag{A6}$$

Hence

$$\eta = \frac{2G^{\Delta}}{\bar{\sigma}\Delta a} = \frac{K_{\mathrm{I}}(1-\nu^2)}{E}(2\pi/\Delta a)^{1/2} \tag{A7}$$

Comparing (A3) with (A7) gives $\bar{\xi} = \pi^2/16 = 0.677$. From (A2)

$$\alpha = d_1/\Delta a = \frac{4K_{\mathrm{I}}(1-\nu^2)}{E}(2/\pi\Delta a)^{1/2} \tag{A8}$$

RÉSUMÉ

On examine la théorie suivant laquelle la vitesse de relaxation de l'énergie de Griffith non évanescente requiert une singularité du type r^{-1} au sommet d'une fissure pour exprimer l'intensité d'énergie, à savoir le produit de la contrainte et de la dilatation. Si l'existence de telle singularité est critiquée sur les bases physiques, des considérations d'équilibre d'énergie de mécanique des milieux continus suggèrent qu'une extension initiale d'une fissure instable s'effectue par un ressaut de croissance discrète caractérisé par une dimension Δa.