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Diffusive Thermal Conductivity of Superfluid ^{3}He-B

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The diffusive thermal conductivity $\kappa(T)$ of superfluid ^{3}He-B is calculated in the s-p-d-*wave approximation by solving the Boltzmann equation for the Bogoliubov–Valatin quasiparticles variationally. A new set of Landau parameters calculated from recent heat capacity data as well as old ones given in Wheatley's review are used to estimate the scattering amplitudes of the collision integral. Landau parameters F_2^s, F_1^a, and F_2^a are treated as free parameters under the constraint that $\kappa_{exact}(T_c) = \kappa_{exp}(T_c)$, where κ_{exact} and κ_{exp} are the exact theoretical value and the experimental value, respectively. We have varied F_2^s, F_1^a, and F_2^a over a wide range $\{-10 \le A_l^{s,a} \equiv F_l^{s,a}/[1+F_l^{s,a}/(2l+1)] \le 10\}$ and found the possible range of the reduced diffusive thermal conductivity $\tilde{\kappa}(T) = \kappa(T)T/\kappa(T_c)T_c$. The behavior of $\tilde{\kappa}(T)$ in the* s-p-d-*wave approximation does not much depend on the values of the Landau parameters, and $\tilde{\kappa}(T)$ decreases monotonically with decreasing temperature.*

1. INTRODUCTION

Since the discovery of the superfluid phases of ^{3}He in 1972, a large number of investigations have been carried out on transport phenomena in the superfluid phases.[1–3,*]

The diffusive thermal conductivity $\kappa(T)$ of superfluid ^{3}He-B has been studied by several authors.[4–9] Pethick *et al.*[6,7] have calculated the diffusive thermal conductivity in the low-temperature region ($T \ll T_c$). In the limit $T = 0$, they have found an exact solution of the Boltzmann equation for the Bogoliubov–Valatin (BV) quasiparticles, and have obtained the exact expression of the diffusive thermal conductivity: The reduced diffusive thermal conductivity $\tilde{\kappa}(T) = \kappa(T)T/\kappa(T_c)T_c$ in the *s-p*-wave approximation for the scattering amplitude is about 0.7 in this limit at 21 bar.

*For a theoretical review of the transport properties of superfluid ^{3}He, see Wölfle.[1] For a review of the experimental properties of superfluid ^{3}He, see Wheatley.[2,3]

Wölfle and Einzel[8,9] have studied the temperature dependence of the diffusive thermal conductivity $\kappa(T)$ of superfluid ^{3}He-B over the whole temperature range by using the Boltzmann equation for the BV quasiparticles. They have calculated the collision integral, approximating its kernel by a separable form. Instead of using the partial wave approximation for the scattering amplitude, they have estimated the relevant parameters for the diffusive thermal conductivity, $\tau_N(0)$ and λ_1^-, from other experimental data. The reduced diffusive thermal conductivity $\tilde{\kappa}(T)$ calculated by them first increases with decreasing temperature and then decreases toward the low-temperature limiting value $\tilde{\kappa}(0) \simeq 0.74$.[32]

In previous papers,[10–12] which will be referred to as I, II, and III, we have investigated the shear viscosity $\eta(T)$ over the whole temperature range by solving the Boltzmann equation for the BV quasiparticles variationally. In the calculation of the collision integral, the scattering amplitude was estimated in the s-wave approximation (I), and in the s-p-wave and s-p-d-wave approximations (II and III).

The results of the reduced shear viscosity $\tilde{\eta}(T) = \eta(T)/\eta(T_c)$ in the s-p-d-wave approximation are in good agreement with experiments at 20 bar,[13–16] and the behavior of $\tilde{\eta}(T)$ does not much depend on the values of the Landau parameters as long as we impose the condition that $\eta_{\text{exact}}(T_c) = \eta_{\text{exp}}(T_c)$, where η_{exact} and η_{exp} are the exact theoretical value[17–19] and the experimental value,[2] respectively.

The purpose of this paper is to investigate the diffusive thermal conductivity $\kappa(T)$ of superfluid ^{3}He-B by making use of the variational method to solve the Boltzmann equation, which was also employed in I, II, and III. For the variational solution of the Boltzmann equation, we adopt the following trial function:

$$\tau_v(E) = \tau_\kappa\{1 + C_1/[\pi^2 + (\beta E)^2] + C_2(\beta E)^2\} \tag{1}$$

where E is the excitation energy of the BV quasiparticles, τ_κ and the C_i are constant parameters, and $\beta = 1/k_B T$. In the limit $T = 0$, if we choose the appropriate variational parameters (i.e., $C_1 = C_2 = 0$), this trial function becomes the exact solution of the Boltzmann equation.[6,7] Furthermore, this trial function yields a very good value of the diffusive thermal conductivity at the transition temperature T_c (see Section 3). Therefore our trial function is expected to be a good solution of the Boltzmann equation over the whole temperature range $0 \le t = T/T_c \le 1$.

Transition probabilities of various scattering processes in the superfluid phase can be expressed in terms of the scattering amplitudes of the normal quasiparticles multiplied by various coherence factors. In calculating them, we treat the coherence factors correctly over the whole temperature range and estimate the scattering amplitudes in the s-p-d-wave approximation, using a new set of Landau parameters[20] as well as old ones given in Wheatley's review[2] (F_0^s, F_1^s, and F_0^a). Landau parameters F_2^s, F_1^a, and F_2^a, whose values are also needed to estimate the scattering amplitudes in the *s-p-d*-wave approximation, are treated as free parameters under the constraint given by

$$\kappa_{\text{exact}}(T_c) = \kappa_{\text{exp}}(T_c) \tag{2}$$

where κ_{exact} is the exact theoretical value of the diffusive thermal conductivity[17–19] and κ_{exp} is its experimental value.[2]

We have varied F_2^s, F_1^a, and F_2^a over a wide range ($-10 \leq A_2^{s,a}, A_1^a \leq 10$) and found the possible range of $\tilde{\kappa}(T)$. The values of the Landau parameters do not significantly affect the behavior of $\tilde{\kappa}(T)$ as long as we impose the constraint given by Eq. (2). The reduced diffusive thermal conductivity $\tilde{\kappa}(T)$ calculated in the *s-p-d*-wave approximation decreases monotonically with decreasing temperature,* in contrast to the result by Wölfle and Einzel.[32]

In the next section we microscopically derive the Boltzmann equation for the BV quasiparticles. The variational solution of the diffusive thermal conductivity is discussed in Section 3. In Section 4, the diffusive thermal conductivity is calculated in the *s-p-d*-wave approximation. A summary and discussion are given in Section 5.

2. BOLTZMANN EQUATION IN THE PRESENCE OF A TEMPERATURE GRADIENT

In this section, we derive the Boltzmann equation for the BV quasiparticles in the presence of a temperature gradient[22] (Ref. 22 will be referred to as IV).

As has been shown in IV, the deviation of the matrix Wigner distribution function from the local equilibrium distribution is given by the sum of the "particle" part δF_{+1} and the "hole" part δF_{-1}.

The matrix kinetic equation for δF_ν ($\nu = \pm 1$) in the presence of the temperature gradient is

$$-\frac{1}{2}\left(\nu\rho_3' + \frac{\xi}{E}\right)\frac{\partial f^<}{\partial E}\beta E\frac{\partial \xi}{\partial \mathbf{p}}\cdot\nabla\left(\frac{1}{\beta}\right) + \frac{i}{\hbar}[\varepsilon_0, \delta F_\nu] = I_\nu(\mathbf{p}) \tag{3}$$

where $[A, B] = AB - BA$ is the commutator, ξ is the kinetic energy of ^{3}He quasiparticles measured from the Fermi energy, $f^<(E) = 1/[\exp(\beta E) + 1]$ is the Fermi distribution function, and $I_\nu(\mathbf{p})$ is the collision term, which has the same form as Eq. (8) in II. The matrix $\varepsilon_0 = \xi\rho_3' + \Delta\rho_1'(\hat{\mathbf{p}})$ expresses the

*A similar result has recently been obtained by Dörfle *et al.*[21] within the relaxation time approximation.

equilibrium energy matrix and the ρ_i' are the Pauli matrices in the particle–hole space, given by

$$\rho_1'(\hat{\mathbf{p}}) = \frac{1}{\Delta}\begin{pmatrix} 0 & \Delta_{\hat{\mathbf{p}}} \\ \Delta_{\hat{\mathbf{p}}}^\dagger & 0 \end{pmatrix}, \qquad \rho_3' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

where $\Delta_{\hat{\mathbf{p}}}$ is the 2×2 matrix of the equilibrium energy gap of the BW state,[23] Δ is its magnitude, and $\hat{\mathbf{p}}$ is the unit vector parallel to $\mathbf{p}$.

Assuming that the solution of Eq. (3) can be expanded into a power series in terms of the parameter $\hbar/\Delta\tau$ (τ is a typical collision time) and also that the density of states has the particle–hole symmetry, we obtain the lowest order solution of Eq. (3) of the form[22]

$$\delta F_\nu = \tau(\xi)\frac{\partial f^<}{\partial E}\beta E\frac{\partial \xi}{\partial \mathbf{p}}\cdot\nabla\left(\frac{1}{\beta}\right)\frac{\xi}{E}\frac{1}{2}\left(1+\nu\frac{\varepsilon_0}{E}\right)+o\left(\frac{\hbar}{\tau\Delta}\right) \tag{5}$$

This form of δF_ν suggests that the deviation of the BV quasiparticle distribution function from the local equilibrium distribution is represented by

$$\frac{\partial f^<}{\partial E}\phi = \tau(\xi)\frac{\partial f^<}{\partial E}\beta\xi\frac{\partial \xi}{\partial \mathbf{p}}\cdot\nabla\left(\frac{1}{\beta}\right) \tag{6}$$

The deviation $(\partial f^</\partial E)\phi$, or $\tau(\xi)$, obeys the following Boltzmann equation:

$$\begin{aligned}\frac{\partial f^<}{\partial E_1}\xi_1\frac{\partial \xi_1}{\partial \mathbf{p}_1} = -\int d\mathbf{p}_2\, d\mathbf{p}_3\, d\mathbf{p}_4 \sum_{\nu_2\nu_3\nu_4} \delta_{\mathbf{p}_1}\delta_{+E_1}\left(\frac{1}{2\pi\hbar}\right)^6\frac{\beta}{4}f^<(E_1)f_{\nu_2}^<f_{\nu_3}^>f_{\nu_4}^> \\ \times\left\{[W_1+W_2C_{12}C_{34}-W_3(C_{24}+C_{13})]\xi_1\frac{\partial \xi_1}{\partial \mathbf{p}_1}\tau(\xi_1)\right. \\ \left.+(W_4+W_5C_{34})\frac{\xi_1}{E_1}\frac{\xi_2}{E_2}\nu_2\xi_2\frac{\partial \xi_2}{\partial \mathbf{p}_2}\tau(\xi_2)\right\}\end{aligned} \tag{7}$$

where

$$\delta_p = \delta(\mathbf{p}+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4), \qquad \delta_{\nu E} = \delta(\nu E+\nu_2E_2-\nu_3E_3-\nu_4E_4)$$
$$f_\nu^{\gtrless} = f^{\gtrless}(\nu E)(f^> = 1-f^<)$$
$$C_{ij} = \nu_i\nu_j\gamma_{ij}\Delta^2/E_iE_j \quad (i,j\neq 1), \qquad C_{1j} = \nu_j\gamma_{1j}\Delta^2/E_lE_j$$

and $\gamma_{ij} = (\hat{\mathbf{p}}_i\cdot\hat{\mathbf{p}}_j)$. The quantities W_i are expressed in terms of the scattering amplitudes in the normal state as

$$
\begin{aligned}
W_1 &= (2\pi/\hbar)(T_s^2+3T_t^2)/8 \\
W_2 &= (2\pi/\hbar)[(T_s^2-T_t^2)+4T_{2t}(T_{2t}-T_{2s})]/8 \\
W_3 &= (2\pi/\hbar)[(T_s+T_t)(T_{2s}+T_{2t})+4T_tT_{2t}]/8 \\
W_4 &= (2\pi/\hbar)[(T_s^2+3T_t^2)-2(T_{2s}^2+3T_{2t}^2)]/8 \\
W_5 &= (2\pi/\hbar)[(T_{3s}+T_{3t})(T_{2s}-T_{2t})-4T_{3t}T_{2t}]/8
\end{aligned} \tag{8}
$$

with

$$
\begin{aligned}
T_{2s,2t} &= T_{s,t}(\mathbf{p}_1, -\mathbf{p}_4, \mathbf{p}_3, -\mathbf{p}_2) \\
T_{3s,3t} &= T_{s,t}(\mathbf{p}_1, -\mathbf{p}_3, -\mathbf{p}_2, \mathbf{p}_4)
\end{aligned} \tag{9}
$$

Here $T_s(T_t)$ is the singlet (triplet) scattering amplitude for quasiparticles in the normal state. The quantities $T_{s,t}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$ give the amplitudes for the process in which the incoming quasiparticles have momenta $\mathbf{p}_1$ and $\mathbf{p}_2$, and the outgoing quasiparticles have momenta $\mathbf{p}_3$ and $\mathbf{p}_4$.

By virtue of the Fermi distribution functions and the momentum and energy delta-functions in the collision integral, we can assume that the scattering amplitudes depend only on angle variables and can convert the integral over the momenta $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_4$ into integrals over the energy variables ξ_2, ξ_3, ξ_4 and the angles.[24]

Performing the angular integrations, we finally obtain

$$
\begin{aligned}
\xi_1\frac{\partial f^<}{\partial E_1} = &-\int d\xi_2\, d\xi_3\, d\xi_4 \sum_{\nu_2\nu_3\nu_4} \pi^2(m^*)^3\left(\frac{1}{2\pi\hbar}\right)^6 \delta_{+E_1}\beta f^<(E_1)f_{\nu_2}^<f_{\nu_3}^>f_{\nu_4}^> \\
&\times\left\{\left[\beta_1+\beta_2\nu_2\nu_3\nu_4\frac{\Delta^4}{E_1E_2E_3E_4}\right.\right. \\
&\left.-\beta_3\left(\nu_2\nu_4\frac{\Delta^2}{E_2E_4}+\nu_3\frac{\Delta^2}{E_1E_3}\right)\right]\xi_1\tau(\xi_1) \\
&\left.+\left(\beta_4+\beta_5\nu_3\nu_4\frac{\Delta^2}{E_3E_4}\right)\frac{\xi_1}{E_1}\frac{\xi_2}{E_2}\nu_2\xi_2\tau(\xi_2)\right\}
\end{aligned} \tag{10}
$$

where m^* is the effective mass at the Fermi surface, the β_i are

$$
\begin{aligned}
\beta_1 &= (2\pi/\hbar)\langle T_s^2+3T_t^2\rangle/8 \\
\beta_2 &= (2\pi/\hbar)\langle[(T_s^2-T_t^2)+4T_{2t}(T_{2t}-T_{2s})]\gamma_{12}\gamma_{34}\rangle/8 \\
\beta_3 &= (2\pi/\hbar)\langle[(T_s+T_t)(T_{2s}+T_{2t})+4T_tT_{2t}]\gamma_{13}\rangle/8 \\
\beta_4 &= (2\pi/\hbar)\langle[(T_s^2+3T_t^2)-2(T_{2s}^2+3T_{2t}^2)]\gamma_{12}\rangle/8 \\
\beta_5 &= (2\pi/\hbar)\langle[(T_{3s}+T_{3t})(T_{2s}-T_{2t})-4T_{3t}T_{2t}]\gamma_{34}\gamma_{12}\rangle/8
\end{aligned} \tag{11}
$$

and the brackets mean the angular average defined by

$$\langle A\rangle = \int \frac{d\theta_1\, d\phi_1}{4\pi} \frac{\sin\theta_1}{\cos(\theta_1/2)} \int \frac{d\tilde{\phi}}{2\pi} A \tag{12}$$

where θ_1 is the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$, ϕ_1 is the angle between the planes spanned by $(\mathbf{p}_1, \mathbf{p}_2)$ and $(\mathbf{p}_3, \mathbf{p}_4)$, and $\tilde{\phi}$ is the azimuthal angle of $\mathbf{p}_2$ around the axis $\mathbf{p}_1$.

It is to be noted that the β_i are the same quantities as the α_i in II for $i=1$, 2, and 3, and are related to Einzel and Wölfle's parameters[9] as $\beta_2/\beta_1=\delta_0$, $\beta_3/\beta_1=\gamma_0$, $\beta_4/\beta_1=\lambda_1^-$, and $\beta_5/\beta_1=\gamma_1$.

3. VARIATIONAL SOLUTION OF THE DIFFUSIVE THERMAL CONDUCTIVITY

In this section we determine a trial function $\tau_v(E)$ to solve the Boltzmann equation (10) variationally. The variational solution of the diffusive thermal conductivity is compared with exact solutions in the two limits $T=T_c$ and $T=0$.

The diffusive thermal conductivity $\kappa(T)$ is expressed as[22]

$$\kappa(T) = \frac{2}{3}\frac{N_F E_F}{m^*}\frac{1}{T}\langle\langle \xi^2 \tau(\xi)\rangle\rangle \tag{13}$$

where E_F is the Fermi energy, N_F is the density of states at the Fermi surface for both spin projections, and $\langle\langle A\rangle\rangle$ denotes the energy average defined by

$$\langle\langle A\rangle\rangle = -\int_{-\infty}^{\infty} d\xi \frac{\partial f^<}{\partial E} A \tag{14}$$

It is difficult to solve directly the Boltzmann equation (10) in the superfluid phase and to calculate the diffusive thermal conductivity from Eq. (13) using its solution. Therefore we calculate the diffusive thermal conductivity by a variational method.

There are exact solutions of the Boltzmann equation (10) in the two limits $T=T_c$[17] and $T=0$,[6,7]

$$\tau(\xi) = \tau_N(0)\left[\frac{\pi^2}{\pi^2+(\beta\xi)^2} - \frac{5}{12}\frac{\alpha_\kappa}{6-\alpha_\kappa} - \frac{187}{2160}\frac{\alpha_k}{20-\alpha_\kappa} \times\left(1-\frac{7}{17}\frac{(\beta\xi)^2}{\pi^2}\right)+\cdots\right], \qquad T=T_c \tag{15}$$

$$\tau(\xi) = \text{const}, \qquad T=0$$

where $\tau_N(0)$ $[=16\pi^2\hbar^6\beta^2/(m^*)^3\beta_1]$ is the relaxation time for a quasiparticle in the normal state at the Fermi surface and $\alpha_\kappa = 2\beta_4/\beta_1$. Inspecting the above limiting forms of $\tau(\xi)$, we adopt the trial function $\tau_v(E)$ given by Eq. (1) in the whole temperature region,

$$\begin{aligned}\tau_v(E) &= \tau_\kappa[1+C_1/[\pi^2+(\beta E)^2]+C_2(\beta E)^2] \\ &\equiv \tau_\kappa X(E)\end{aligned}$$

where τ_κ, C_1, and C_2 are constant parameters.

Accordng to the variational principle,[25] the true diffusive thermal conductivity $\kappa(T)$ is bounded from below by $\kappa_v(T)$ calculated variationally:

$$\kappa(T) \geq \kappa_v(T) = \left[\frac{2}{3}\frac{E_F N_F}{m^*}\frac{1}{T}\langle\langle \xi^2 X(E)\rangle\rangle \tau_\kappa\right]_{\max} \tag{16}$$

where τ_κ is related to C_1 and C_2 implicitly through the relation

$$\begin{aligned}-\langle\langle \xi_1^2 X(E_1)\rangle\rangle \frac{1}{\tau_\kappa} &= \beta_1\langle \xi_1^2 X^2(E_1)\rangle_c \\ &+\beta_2\left\langle \nu_2\nu_3\nu_4\frac{\Delta^4}{E_1E_2E_3E_4}\xi_1^2X^2(E_1)\right\rangle_c \\ &-\beta_3\left\langle\left(\nu_2\nu_4\frac{\Delta^2}{E_2E_4}+\nu_3\frac{\Delta^2}{E_1E_3}\right)\xi_1^2X^2(E_1)\right\rangle_c \\ &+\beta_4\left\langle\frac{\xi_1}{E_1}\frac{\xi_2}{E_2}\nu_2\xi_1\xi_2X(E_1)X(E_2)\right\rangle_c \\ &+\beta_5\left\langle\frac{\xi_1}{E_1}\frac{\xi_2}{E_2}\nu_2\nu_3\nu_4\frac{\Delta^2}{E_3E_4}\xi_1\xi_2X(E_1)X(E_2)\right\rangle_c\end{aligned} \tag{17}$$

The notation $[A]_{\max}$ denotes the maximum value of A as a function of the variational parameters C_1 and C_2, and $\langle B\rangle_c$ is defined by

$$\begin{aligned}\langle B\rangle_c = -\int_{-\infty}^{\infty} d\xi_1\, d\xi_2\, d\xi_3\, d\xi_4 \sum_{\nu_2\nu_3\nu_4} \pi^2(m^*)^3\left(\frac{1}{2\pi\hbar}\right)^6 \delta_{+E_1} \\ \times \beta f^<(E_1)f^<_{\nu_2}f^>_{\nu_3}f^>_{\nu_4}B\end{aligned} \tag{18}$$

The energy integrals in Eq. (17) are calculated numerically by using Gauss' method. In this calculation we use the same expression for δ $(=\beta\Delta)$

of the weak coupling theory as was employed in I:

$$\delta=\left[\frac{8\pi^2}{7\zeta(3)}\right]^{1/2}\frac{(1-t)^{1/2}}{t},\qquad 0.99\le t\le 1.0$$
$$\delta=\frac{\pi}{\gamma}\frac{\Delta(T)}{\Delta(0)}\frac{1}{t},\qquad 0.0\le t<0.99 \tag{19}$$

where $\gamma=1.78107$ is the Euler number and the values of $\Delta(T)/\Delta(0)$ have been given by Mühlschlegel.[26] The reduced diffusive thermal conductivity $\tilde{\kappa}_v(T)$ $[=\kappa_v(T)T/\kappa_v(T_c)T_c]$ is a function of the parameter δ alone. If one uses other expressions for δ (strong coupling effect), one has to replace the value of t by an appropriate value corresponding to the value of the new δ. In the present calculation, however, this effect is not so great.

Let us now discuss the diffusive thermal conductivity in the two limits $T=T_c$ and $T=0$. At $T=T_c$, the exact theoretical expression of the diffusive thermal conductivity $\kappa_{\text{exact}}(T_c)$ was derived by Højgaard Jensen *et al.*[17] and Brooker and Sykes,[18,19]

$$\kappa_{\text{exact}}(T_c)T_c=\frac{4p_{\text{F}}^5}{3\pi^2(m^*)^2\hbar^3}\frac{1}{N_{\text{F}}^2\beta_1}K \tag{20}$$

where p_{F} is the Fermi momentum and the quantity K is given by[17]

$$K=\frac{1}{3}(12-\pi^2)+4\alpha_\kappa\sum_{n\ \text{even}}\frac{2n+1}{n^2(n+1)^2}\frac{1}{n(n+1)-\alpha_\kappa} \tag{21}$$

For the relevant value of α_κ (i.e., $3.0\le\alpha_\kappa\le 4.0$; see Table II), the variational result $\kappa_v(T_c)$ agrees very well with $\kappa_{\text{exact}}(T_c)$ within 1% error. This indicates that our trial function is a very good solution at the transition temperature.

In the limit $T=0$, $\tau(\xi)=\text{const}$ is an exact solution of the Boltzmann equation (10), as was noted by Pethick *et al.*[6,7] Since our trial function includes this exact solution, $\kappa_v(T)$ tends to the correct diffusive thermal conductivity $\kappa(0)$:

$$\kappa(T)T|_{T=0}=\frac{4p_{\text{F}}^5}{3\pi^2(m^*)^2\hbar^3}\frac{1}{N_{\text{F}}^2\beta_1}\left(\frac{3}{8}\pi+\frac{3}{8}\pi\frac{\beta_2}{\beta_1}-\frac{2}{8}\pi\frac{\beta_3}{\beta_1}\right)^{-1} \tag{22}$$

From Eqs. (20)–(22), we obtain the following exact expression of $\tilde{\kappa}(0)$,[6–8]

$$\tilde{\kappa}(0)=\frac{1}{K}\left(\frac{3}{8}\pi+\frac{3}{8}\pi\frac{\beta_2}{\beta_1}-\frac{2}{8}\pi\frac{\beta_3}{\beta_1}\right)^{-1} \tag{23}$$

Equations (20)–(23) can be used for the examination of a numerical calculation of the diffusive thermal conductivity. As we shall see below, our numerical results of $\kappa_v(T)$ are consistent with these limiting values.

4. DIFFUSIVE THERMAL CONDUCTIVITY IN THE *s-p-d*-WAVE APPROXIMATION

In this section, we calculate the diffusive thermal conductivity using the transition probability estimated in the *s-p-d*-wave approximation. A new set of Landau parameters[20] as well as old ones[2] are used to estimate the scattering amplitudes in the collision integral.

In order to calculate the diffusive thermal conductivity from Eqs. (16) and (17), we have to estimate the values of β_i in Eq. (17). Using the properties of the Fermi distribution functions as well as the momentum and energy conservation laws in the collision integral, we can assume that the scattering amplitudes depend only on the angles θ_i and ϕ_i as

$$\begin{aligned} T_{s,t} &= T_{s,t}(\theta_1, \phi_1) \\ T_{2s,2t} &= T_{s,t}(\theta_2, \phi_2) \\ T_{3s,3t} &= T_{s,t}(\theta_3, \phi_3) \end{aligned} \tag{24}$$

where the angles θ_i and ϕ_i are defined as follows: θ_1 is the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$, and ϕ_1 is the angle between two planes spanned by $(\mathbf{p}_1, \mathbf{p}_2)$ and $(\mathbf{p}_3, \mathbf{p}_4)$. The angles θ_2 and ϕ_2 are defined similarly to θ_1 and ϕ_1 by interchange of $\mathbf{p}_2 \leftrightarrow -\mathbf{p}_4$ and the angles θ_3 and ϕ_3 by interchange of $\mathbf{p}_2 \leftrightarrow -\mathbf{p}_3$. Employing the potential scattering model for the scattering amplitudes of the normal quasiparticles, we obtained the following form for $T_{s,t}(\theta_i, \phi_i)$ in the *s-p-d*-wave approximation:[11,27–29]

$$\begin{aligned} N_F T_s(\theta_i, \phi_i) &= S_0 + S_1 \cos\theta_i + S_2[\tfrac{1}{2}(3\cos^2\theta_i - 1) \\ &\quad + \tfrac{3}{4}(\cos\theta_i - 1)^2(\cos^2\phi_i - 1)] \\ N_F T_t(\theta_i, \phi_i) &= [T_1(\cos\theta_i - 1) + \tfrac{3}{2}T_2(\cos^2\theta_i - 1)]\cos\phi_i \end{aligned} \tag{25}$$

Using the forward scattering sum rule

$$\sum_l T_l = 0 \tag{26}$$

we have eliminated T_0 in Eq. (25). Landau parameters $F_l^{s,a}$ are related to T_l and S_l as $T_l = A_l^s + A_l^a$ and $S_l = A_l^s - 3A_l^a$ with

$$A_l^{s,a} = F_l^{s,a}/[1 + F_l^{s,a}/(2l+1)] \tag{27}$$

Substituting the above expressions for $T_{s,t}$ into Eq. (11), we find the following expression for β_i in terms of S_0, S_1, S_2, T_1, and T_2 as

$$\begin{aligned} \tilde{\beta}_i &\equiv N_F^2 \frac{\hbar}{\pi} \beta_i \\ &= \sum_{m,n} [E^i_{S_m S_n} S_m S_n + E^i_{T_m T_n} T_m T_n + E^i_{S_m T_n} S_m T_n] \end{aligned} \tag{28}$$

TABLE I

Values of $E^i_{S_mS_n}$, $E^i_{T_mT_n}$, and $E^i_{S_mT_n}$

i	4	5
$E_{S_0S_0}$	1/6	7/30
$E_{S_0S_1}$	3/10	−11/210
$E_{S_0S_2}$	11/210	−1/30
$E_{S_1S_1}$	−1/42	−29/630
$E_{S_1S_2}$	−3/70	−5/462
$E_{S_2S_2}$	−191/2310	185/2002
$E_{T_1T_1}$	−24/35	52/63
$E_{T_1T_2}$	8/35	100/231
$E_{T_2T_2}$	216/385	68/143
$E_{S_0T_1}$	0	−16/105
$E_{S_0T_2}$	0	−16/105
$E_{S_1T_1}$	0	−88/315
$E_{S_1T_2}$	0	−8/165
$E_{S_2T_1}$	0	8/1155
$E_{S_2T_2}$	0	24/585

Since the β_i are the same quantities as the α_i in II for $i = 1, 2$, and 3, as was mentioned in Section 2, we tabulate the coefficients $E^i_{S_mS_n}$, $E^i_{T_mT_n}$, and $E^i_{S_mT_n}$ in Table I only for $i = 4$ and 5 (see also Table III of II).

As is shown above, we need the values of the Landau parameters to estimate the β_i. Recently new heat capacity data for ^{3}He between 1 and 10 mK at pressures from 0 to 32 bar have been reported.[20,30,31] Using this new heat capacity data at 20.94 bar,[20] we calculate the Landau parameters F^s_0, F^a_0, and F^s_1 : $F^s_0 = 42.35$, $F^a_0 = -0.8111$, and $F^s_1 = 8.077$ (set I). On the other hand, values of the Landau parameters tabulated in Wheatley's review[2] are $F^s_0 = 59.78$, $F^a_0 = -0.7357$, and $F^s_1 = 12.51$ at 21 bar (set II). We calculate the diffusive thermal conductivity using these two sets of Landau parameters (sets I and II).

First we use the values of set I, which yield $A^s_0 = 0.9769$, $A^a_0 = -4.2938$, and $A^s_1 = 2.1875$. Since the values of F^s_2, F^a_1, and F^a_2 are not known explicitly, we treat A^s_2, A^a_1, and A^a_2 as free parameters under the constraint given by Eq. (2);

$$\kappa_{\text{exact}}(T_c) = \kappa_{\text{exp}}(T_c)$$

where κ_{exact} is the exact theoretical value of the diffusive thermal conductivity [see Eqs. (20) and (21)], and κ_{exp} is the experimental value, $\kappa_{\text{exp}}(T_c)T_c = 15.3$ erg/sec cm at 21 bar.[2] We have varied A^s_2, A^a_1, and A^a_2 over the range $-10 < A^{s,a}_2, A^a_1 < 10$ under the constraint given by Eq. (2), and found the maximum and minimum values of the reduced diffusive

TABLE II

Values of $\tilde{\kappa}_v(0)$, A_2^s, A_1^a, A_2^a, and the $\tilde{\beta}_i$

Condition[b]	$\tilde{\kappa}_v(0)$	A_2^s	A_1^a	A_2^a	$\tilde{\beta}_1$ $(\tau_N(0)T_c^2)^a$	$\tilde{\beta}_2$	$\tilde{\beta}_3$	$\tilde{\beta}_4$ (α_κ)	$\tilde{\beta}_5$
A	0.56	1.4	−1.8	−0.706	61.7 (0.21)	23.4	21.3	95.8 (3.1)	25.8
B	0.39	2.3	−4.1	−0.709	77.1 (0.17)	36.9	17.7	144.3 (3.7)	19.6
C	0.52	0.0	−1.2	−0.513	30.8 (0.30)	11.6	5.6	46.7 (3.0)	9.7
D	0.34	1.0	−3.3	−0.384	41.0 (0.22)	19.9	1.6	78.8 (3.8)	3.2

[a] Unit of $\tau_N(0)T_c^2$ is μsec mK^2.

[b] A: Maximum of $\tilde{\kappa}_v(0)$ under $\kappa_{exact}(T_c) = \kappa_{exp}(T_c)$ with set I ($A_0^s = 0.9769$, $A_0^a = -4.2938$, and $A_1^s = 2.1875$).

B: Minimum of $\tilde{\kappa}_v(0)$ under $\kappa_{exact}(T_c) = \kappa_{exp}(T_c)$ with set I.

C: Maximum of $\tilde{\kappa}_v(0)$ under $\kappa_{exact}(T_c) = \kappa_{exp}(T_c)$ with set II ($A_0^s = 0.9835$, $A_0^a = -2.7843$, and $A_1^s = 2.4197$).

D: Minimum of $\tilde{\kappa}_v(0)$ under $\kappa_{exact}(T_c) = \kappa_{exp}(T_c)$ with set II.

thermal conductivity at $T = 0$, $\tilde{\kappa}_v(0)$. The Landau parameters A_2^s, A_1^a, and A_2^a corresponding to the maximum value of $\tilde{\kappa}_v(0)$ are given in row A of Table II with values of $\tilde{\kappa}_v(0)$ and β_i. The same parameters for the minimum value of $\tilde{\kappa}_v(0)$ are given in row B of Table II.

Using these values, we have calculated the reduced diffusive thermal conductivity $\tilde{\kappa}_v(T) = \kappa_v(T)T/\kappa_v(T_c)T_c$, which is shown in Fig. 1. We have found that the curves denoted as A and B in Fig. 1 give good upper and lower bounds for $\tilde{\kappa}_v(T)$ in the whole temperature range. The curves of $\tilde{\kappa}_v(T)$ for other values of Landau parameters (A_2^s, A_1^a, and A_2^a) lie between these upper and lower bounds over the whole temperature range as long as we impose the constraint (2).

In the same way, we also calculate $\kappa_v(T)$ using the values of set II, which yield $A_0^s = 0.9835$, $A_0^a = -2.7843$, and $A_1^s = 2.4197$. The Landau parameters A_2^s, A_1^a, and A_2^a for $\tilde{\kappa}_v(0) =$ maximum (minimum) are given in row C (row D) of Table II with values of $\tilde{\kappa}_v(0)$ and β_i. In Fig. 2 we show the results of $\tilde{\kappa}_v(T)$ using these values. Again the curves C and D give good upper and lower bounds for $\tilde{\kappa}_v(T)$ in the whole temperature range.

From Figs. 1 and 2, we find that the upper and lower bounds are fairly close to each other. We can therefore conclude that the values of Landau parameters do not affect significantly the behavior of $\tilde{\kappa}_v(T)$ in the *s-p-d*-wave approximation as long as we impose the condition given by Eq. (2).

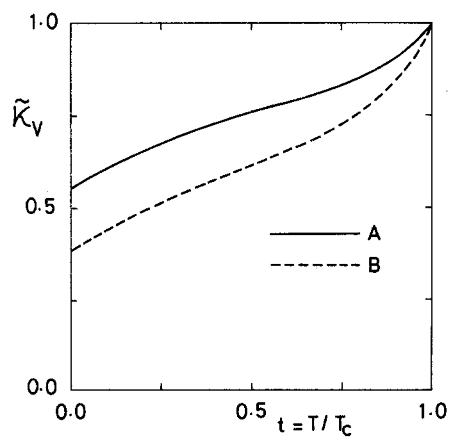

Fig. 1. Reduced diffusive thermal conductivity $\tilde{\kappa}_v = \kappa_v(T)T/\kappa_v(T_c)T_c$ as a function of reduced temperature $t = T/T_c$ (new Landau parameter results). The solid curve A is for $\tilde{\kappa}_v(0) =$ maximum and the dashed curve B is for $\tilde{\kappa}_v(0) =$ minimum.

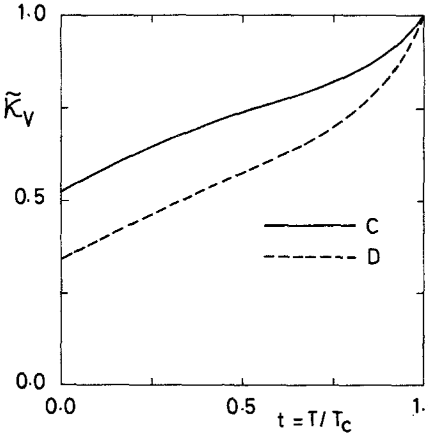

Fig. 2. Reduced diffusive thermal conductivity $\tilde{\kappa}_v = \kappa_v(T)T/\kappa_v(T_c)T_c$ as a function of reduced temperature t (old Landau parameter results). The solid curve C is for $\tilde{\kappa}_v(0)$ = maximum and the dashed curve D is for $\tilde{\kappa}_v(0)$ = minimum.

In particular, for any possible Landau parameters, $\tilde{\kappa}_v(T)$ decreases monotonically with decreasing temperature. It has already been shown in III that the behavior of the reduced shear viscosity $\tilde{\eta}(T) = \eta(T)/\eta(T_c)$ does not much depend on the values of the Landau parameters in the s-p-d-wave approximation as long as we impose the condition $\eta_{\text{exact}}(T_c) = \eta_{\text{exp}}(T_c)$, where η_{exact} is the exact theoretical value of the shear viscosity[17–19] and η_{exp} is the experimental value.[2]

Finally, we calculate $\kappa_v(T)$ in the s-wave and s-p-wave approximations using the values of set II for comparison. In Fig. 3, we show the results of $\tilde{\kappa}_v(T)$ in the s-wave, s-p-wave, and s-p-d-wave approximations together with the result by Wölfle and Einzel.[32] There are considerable differences in the behavior of $\tilde{\kappa}_v(T)$ among the three approximations. This indicates that large-l components of scattering amplitudes play a significant role in the diffusive thermal conductivity as well as in the shear viscosity.[11]

The reduced diffusive thermal conductivity $\tilde{\kappa}_v(T)$ in the s-p-d-wave approximation decreases monotonically as the temperature decreases and approaches the low-temperature limiting value $\tilde{\kappa}_v(0)$ $[0.3 \lesssim \tilde{\kappa}_v(0) \lesssim 0.5]$. This behavior of $\tilde{\kappa}_v(T)$ is very different from that found by Wölfle and Einzel,[32] which first increases with decreasing temperature and then decreases toward the value $\tilde{\kappa}(0) \simeq 0.74$. In order to examine the validity of the approximate kernel of the collision integral used by Wölfle and Einzel,[8]

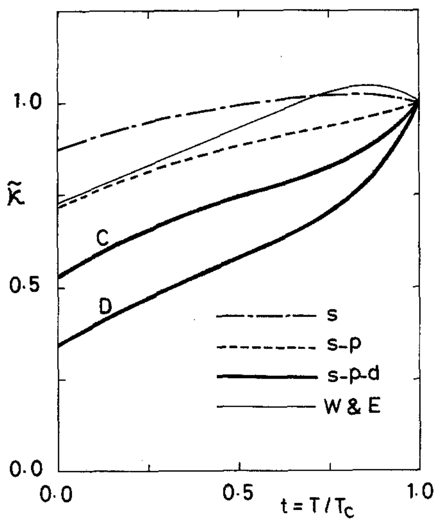

Fig. 3. Reduced diffusive thermal conductivity $\tilde{\kappa}_v = \kappa_v(T)T/\kappa_v(T_c)T_c$ in the three approximations for the scattering amplitude, as a function of reduced temperature t. The solid curves C and D are the results in the s-p-d-wave approximation with Landau parameters of rows C and D in Table II, respectively. Dashed curve is the result in the s-p-wave approximation, and the dash-dot curve is the result in the s-wave approximation with set II of Landau parameters. The thin solid curve is the reduced diffusive thermal conductivity calculated by Wölfle and Einzel.[32]

we have made a variational calculation of $\tilde{\kappa}(T)$ using the same values of β_i as Wölfle and Einzel: $\beta_2/\beta_1 = 0.303$, $\beta_3/\beta_1 = 0.097$, $\beta_4/\beta_1 = 0.9$, and $\beta_5/\beta_1 = 0.0$.[32] We have found that the result of $\tilde{\kappa}(T)$ is in good agreement with their result. Therefore, this discrepancy seems to be due to the difference in the values of β_i used in each calculation. In particular, the value of α_κ ($= 1.8$) used by them is much smaller than the present value, which falls between 3.0 and 4.0 (see Table II).

5. SUMMARY AND DISCUSSION

We have calculated the diffusive thermal conductivity of superfluid ^{3}He-B, solving variationally the Boltzmann equation. The transition probability is calculated in the s-p-d-wave approximation. In this calculation, we have used the new set of Landau parameters calculated from recent heat capacity data[20] as well as the old ones given in Wheatley's review[2] (A_0^s, A_0^a, and A_1^s). Since the values of the Landau parameters A_2^s, A_1^a, and A_2^a are not known explicitly, we have treated A_2^s, A_1^a, and A_2^a as free parameters under the constraint given by Eq. (2). We have varied A_2^s, A_1^a, and A_2^a over the range $-10 \le A_2^{s,a}, A_1^a \le 10$, and found the values of A_2^s, A_1^a, and A_2^a corresponding to the maximum and minimum values of $\tilde{\kappa}_v(0)$, from which the upper and lower bounds of the reduced diffusive thermal conductivity $\tilde{\kappa}_v(T)$ have been found in the whole temperature range.

The upper and lower bounds are fairly close to each other. This indicates that the values of the Landau parameters do not much affect the behavior of the reduced diffusive thermal conductivity $\tilde{\kappa}_v(T)$ as long as we impose the condition given by Eq. (2).

The reduced thermal conductivity $\tilde{\kappa}_v(T)$ calculated in the *s-p-d*-wave approximation decreases monotonically with decreasing temperature, in contrast to that of Wölfle and Einzel.[32] It seems that this discrepancy is mainly due to the difference between the values of α_κ used in each calculation.

Finally, we discuss the values of $\tau_N(0)$ $[=16\pi^2\hbar^6\beta^2(m^*)^3\beta_1]$, α_κ $(=2\beta_4/\beta_1)$, and α_η. The quantity α_η, which was denoted α $(=-2\alpha_4/\alpha_1)$ in II, is an important parameter in a calculation of the shear viscosity.

We estimate typical values of A_2^s, A_1^a, and A_2^a using the following conditions: $\kappa_{\text{exact}}(T_c)=\kappa_{\text{exp}}(T_c)$, $\eta_{\text{exact}}(T_c)=\eta_{\text{exp}}(T_c)$, and the truncated forward scattering sum rule. The truncated sum rule is written as

$$A_0^s+A_0^a+A_1^s+A_1^a+A_2^s+A_2^a=0 \tag{29}$$

For A_0^s, A_0^a, and A_1^s, we have used the values of set II, since these values have been used in the other estimations of $\tau_N(0)T_c^2$, α_κ, and α_η. The estimated values of A_2^s, A_1^a, and A_2^a are as follows: $A_2^s=1.7$, $A_1^a=-1.4$, and $A_2^a=-0.98$. Using these values, we have calculated $\tau_N(0)T_c^2$, α_κ, and α_η: $\tau_N(0)T_c^2=0.29\ \mu\text{sec mK}^2$, $\alpha_\kappa=3.2$, and $\alpha_\eta=1.5$.

The values of $\tau_N(0)T_c^2$ and α_η are in good agreement with those calculated in II $[\tau_N(0)T_c^2=0.28\ \mu\text{sec mK}^2$ and $\alpha_\eta=1.5]$ and $\tau_N(0)T_c^2$ is also in agreement with the values estimated from experimental results.[3] The value α_κ is consistent with the values tabulated in Table II. The reduced diffusive thermal conductivity $\tilde{\kappa}_v(T)$ decreases monotonically with decreasing temperature and tends to $\tilde{\kappa}(0)\simeq 0.45$.

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