Journal of Low Temperature Physics, Vol. 20, Nos. 1/2, 1975

# Linearized Kinetic Equations and Relaxation Processes of a Superconductor Near $T_c$

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(Received December 31, 1974)

*Starting from the equation of Gor'kov and Eliashberg in a form introduced by Eilenberger, we derive a set of linearized equations for the deviation from the equilibrium value of the quasiparticle distribution function as well as of the order parameter. These equations resemble the Boltzmann equation and the Ginzburg–Landau equation, respectively, and they form a set of coupled equations. Two different modes can be distinguished, depending on whether the order parameter changes in magnitude or in phase. The equations are solved for the case of a stationary quasiparticle injection into a superconductor and the change in the electrochemical potential of the quasiparticles is calculated. Furthermore, we treat the problem of a current flowing perpendicular to a superconducting–normal interface in which a normal current is converted into a supercurrent, and we calculate the extra resistance of the interface.*

## 1. INTRODUCTION

From the very beginning, the BCS theory[1] has been successful in the explanation of dynamic processes such as electromagnetic absorption, attenuation of ultrasound, and, in particular, nuclear spin relaxation. These (and many related) processes are characterized by the fact that they can be explained entirely in terms of quasiparticles and that changes in the order parameter do not need to be taken into account.

Later, many phenomena were found where changes in the order parameter play a prominent role. Vortex motion, for instance, is a well-known example of such a case. Many investigations, both experimental and theoretical, have been devoted to this subject.* In this connection, the question arises of how quasiparticles recombine to Cooper pairs; this problem has been dealt with.[4,5] Also, the relaxation of the pair density is a related problem,

*We cite here only two of the more recent publications,[2,3] in which further references may be found.

which has been investigated theoretically[6] and experimentally.[7] The same is true as regards nonequilibrium stationary states that are obtained by application of intense electromagnetic radiation.[8,9] Finally, we mention the observation[10–12] of finite voltage differences in a superconductor. This, in particular, has stimulated the present work.

The model on which the following calculations are based is a superconductor where the electrons interact, mediated by phonons, and where impurity scattering is large. We will assume that the phonons are in an equilibrium state of definite temperature, and that the deviations of the electronic system from equilibrium are so small that a linearized theory is applicable. This may be so in many cases of practical interest; it also allows the classification of various modes. When dealing with specific situations, we restrict the discussion to the vicinity of the transition temperature. There, we may easily single out the modes that involve the order parameter since the characteristic times become infinite at a second-order phase transition.

In Sections 2–4 we present the theory in a form most convenient for the present purpose (and also, very likely, to many others). Essentially, the theory is based on the work of Eilenberger[13] and Eliashberg.[14] In Sections 5 and 6 we treat the stationary processes in which injected quasiparticles and a normal current are converted into a supercurrent. In Section 7 we solve some simple time-dependent problems and discuss the physical interpretation of the various quantities and results of the theory.

## 2. THE EILENBERGER EQUATIONS

The approximate momentum independence of the electronic self-energies allows us to simplify considerably the Gor'kov equations, which we consider in the modification introduced by Eliashberg.[15] It was noted first by Eilenberger[13] (and independently by Larkin and Ovchinnikov[16]) that these equations can be presented in terms of a contracted matrix Green's function*

$$\hat{G}_{\mathbf{p}}(\omega, \omega'; \mathbf{r}) = \frac{i}{\pi} \int d\xi\, \hat{G}\left(\omega, \omega'; \mathbf{r} + \frac{\mathbf{r}'}{2}, \mathbf{r} - \frac{\mathbf{r}'}{2}\right) \exp(-i\mathbf{p}\mathbf{r}')\, d^3\mathbf{r}' \tag{1}$$

and that this Green's function is normalized

$$\{\hat{G}_{\mathbf{p}}(\mathbf{r})\hat{G}_{\mathbf{p}}(\mathbf{r})\}_{\omega\omega'} = (1/T)\,\delta_{\omega,\omega'} \cdot \hat{1} \tag{2}$$

which we have written in a form using the short-hand notation

$$\{AB\}_{\omega\omega'} = T \sum_{\omega''} A(\omega, \omega'')B(\omega'', \omega') \tag{3}$$

*We follow the notation of Ref. 16. Discrete Matsubara frequencies will be denoted by $\omega$, $\omega'$, etc.

The resulting equations can be simplified further in the dirty limit $T_c\tau_i \ll 1$, which we consider exclusively in the following. Following Usadel,[17] we introduce the angular average

$$\hat{G}(\omega, \omega'; \mathbf{r}) = \int (d\Omega_{\mathbf{p}}/4\pi)\hat{G}_{\mathbf{p}}(\omega, \omega'; \mathbf{r}) \tag{4}$$

and obtain the fundamental equation

$$\begin{aligned}&\{[\omega\hat{\tau}_3 + i\hat{U} + i\hat{\Sigma}_{\text{ph}} + (1/2\tau_s)\hat{\tau}_3\hat{G}\hat{\tau}_3, \hat{G}]\}_{\omega\omega'} \\ &\quad = D\{[\nabla - ie\mathbf{A}\hat{\tau}_3, \hat{G}[\nabla - ie\mathbf{A}\hat{\tau}_3, \hat{G}]]\}_{\omega\omega'}\end{aligned} \tag{5}$$

Here $D = v_0^2\tau_i/3$ is the diffusion coefficient, $\tau_s$ is the spin flip time, $\mathbf{A}$ is the vector potential, and $\hat{\tau}_\alpha$ are the Pauli matrices. The perturbation is denoted by $\hat{U}$, which, for instance, in the case of an electric potential, is given by $e\phi \cdot \hat{1}$. Furthermore, the electron–phonon interaction contributes the self-energy

$$\hat{\Sigma}_{\text{ph}}(\omega, \omega'; \mathbf{r}) = i\pi\lambda T \sum_{\bar{\omega}} B(\bar{\omega})\hat{G}(\omega + \bar{\omega}, \omega' + \bar{\omega}; \mathbf{r}) \tag{6}$$

where

$$B(\omega) = \int (d\Omega_{\mathbf{p}'}/4\pi)D(\mathbf{p} - \mathbf{p}'; \omega) \tag{7}$$

is the angular average of the phonon Green's function normalized such that $B(0) = -1$. The dimensionless interaction constant $\lambda$ is also known to measure the mass enhancement.

In a state of equilibrium, the solution of Eqs. (5) and (6) is of the form

$$\hat{G}^{(e)}(\omega) = \alpha(\omega)\hat{\tau}_3 + \beta(\omega)\hat{\tau}_\theta; \hat{\Sigma}_{\text{ph}}^{(e)}(\omega) = \rho(\omega)\hat{\tau}_3 + \sigma(\omega)\hat{\tau}_\theta \tag{8}$$

where

$$\hat{\tau}_\theta = [\exp(-i\theta\hat{\tau}_3)]\hat{\tau}_1 \tag{9}$$

and $\theta = \theta(\mathbf{r})$ has been chosen such that $\beta$ is real and positive. We will have $\theta = 0$ in most of the cases we consider later. Due to the normalization, we have $\alpha^2 + \beta^2 = 1$. In a homogeneous superconductor ($\theta = 0$) without paramagnetic impurities, $\alpha$ and $\beta$ are of the form

$$\alpha(\omega) = \tilde{\omega}(\tilde{\omega}^2 + \tilde{\Delta}^2)^{-1/2}; \qquad \beta(\omega) = \tilde{\Delta}(\tilde{\omega}^2 + \tilde{\Delta}^2)^{-1/2} \tag{10}$$

where $\tilde{\omega} = \omega + i\rho$ and $\tilde{\Delta} = i\sigma$.

## 3. THE LINEARIZED BOLTZMANN EQUATION

In the presence of a perturbation of frequency $\omega_0$, there is a linear response

$$\delta\hat{G} = \delta\hat{G}(\omega, \omega - \omega_0); \qquad \delta\hat{\Sigma} = \delta\hat{\Sigma}_{\text{ph}}(\omega, \omega - \omega_0)$$

in the Green's function as well as in the self-energy. In a straightforward way, one obtains from Eqs. (2), (5), and (6) a set of linear, inhomogeneous equations for these responses.

Since these equations are given in the Matsubara representation, one must exercise particular care in changing over to the real frequency form. Letting $\omega_0 = -i\Omega + 0$, $\omega = -iE - i\Omega/2$, we denote the real frequency quantities by $\delta\hat{G}_E$ and $\delta\hat{\Sigma}_E$. According to Eliashberg,[14] these quantities can be written as the sum of a retarded, an advanced, and an anomalous part as in the following example:*

$$\delta\hat{G}_E = \delta\hat{G}_E^{\mathrm{R}} \tanh\frac{E-\Omega/2}{2T} - \delta\hat{G}_E^{\mathrm{A}} \tanh\frac{E+\Omega/2}{2T} + \delta\hat{G}_E^{(\mathrm{a})} \tag{11}$$

Then, the anomalous part of Eq. (6) assumes the following form:

$$\begin{aligned}\delta\hat{\Sigma}_E^{(\mathrm{a})} = i\pi\lambda \int \frac{dE'}{4\pi i}&[B^{\mathrm{R}}(-iE+iE') - B^{\mathrm{A}}(-iE+iE')]\\ &\times\left[\coth\frac{E-E'}{2T} - \tanh\frac{E}{2T}\right]\delta\hat{G}_{E'}^{(\mathrm{a})}\\ &+\Omega\cdot\left(4T\cosh^2\frac{E}{2T}\right)^{-1}[\delta\hat{\Sigma}_E^{\mathrm{R}} + \delta\hat{\Sigma}_E^{\mathrm{A}}]\end{aligned} \tag{12}$$

where terms of the order $\Omega^2$ have been omitted, since it will be assumed that the changes evolve sufficiently slowly in time. Furthermore, corrections of the order $\Omega/T$ to the first term of Eq. (12) have been neglected. We emphasize that in the stationary (but nonequilibrium) case $\Omega \to 0$, $\delta\hat{\Sigma}_E^{(\mathrm{a})}$ depends only on $\delta\hat{G}_E^{(\mathrm{a})}$. This means that there is a closed set of equations involving only anomalous quantities.

In many cases, the linearized equations have such a symmetry that it is advantageous to distinguish two modes:

$$\begin{aligned}&(\mathrm{T}) \qquad \hat{U} \propto \hat{1}, \hat{\tau}_{\theta+\pi/2}; \qquad \delta\hat{G}_E = a_E\hat{1} + b_E\hat{\tau}_{\theta+\pi/2}\\ &(\mathrm{L}) \qquad \hat{U} \propto \hat{\tau}_3, \hat{\tau}_\theta; \qquad \delta\hat{G}_E = a_E\hat{\tau}_3 + b_E\hat{\tau}_\theta\end{aligned} \tag{13}$$

Since in the complex plane the change in the order parameter is perpendicular and parallel to its equilibrium value, we will refer to the modes (T) and (L) as the transverse and longitudinal modes, respectively. A further important property is that $a_E^{(\mathrm{T})}$ is an even and $a_E^{(\mathrm{L})}$ is an odd function of energy.

*In principle, a term $(\delta G_E^{\mathrm{R}} + \delta G_E^{\mathrm{A}})$ should be added to the right-hand side of this equation. However, there is no use to add such a term in the case of contracted Green's functions, where allowance already has to be made for corrections that are necessary to compensate the interchange in the order of frequency and momentum integrations. Note that, in the present definition, the external frequency enters in the symmetric form.

The normalization provides a definite relation between $a_E$ and $b_E$. As far as the anomalous parts are concerned, this relation assumes the forms

$$
\begin{aligned}
&(\mathrm{T}) \qquad b_E^{(a)} = -i\mathcal{N}_2(E)a_E^{(a)}/\mathcal{N}_1(E) \\
&(\mathrm{L}) \qquad b_E^{(a)} = +i\mathcal{R}_2(E)a_E^{(a)}/\mathcal{N}_1(E)
\end{aligned}
\tag{14}
$$

The quantities $\mathcal{N}_j$ and $\mathcal{R}_j$ are various combinations of $\alpha$ and $\beta$. For instance, $\mathcal{N}_1(E) = \frac{1}{2}[\alpha^{\mathrm{R}}_{E+\Omega/2} - \alpha^{\mathrm{A}}_{E-\Omega/2}]$. In the following, however, we will assume that $\Omega$ is sufficiently small that it may be neglected in these combinations. Then $\mathcal{N}_j$ and $\mathcal{R}_j$ are the real and imaginary parts of $\alpha$ and $\beta$ as given in the following relations:

$$
\begin{aligned}
\mathcal{N}_1(E) + i\mathcal{R}_1(E) &= \alpha_E^{\mathrm{R}} = -(\alpha_E^{\mathrm{A}})^* \\
\mathcal{N}_2(E) + i\mathcal{R}_2(E) &= \beta_E^{\mathrm{R}} = (\beta_E^{\mathrm{A}})^*
\end{aligned}
\tag{15}
$$

Note that $\mathcal{N}_j$ are even and $\mathcal{R}_j$ are odd functions of the energy variable $E$, and that $\mathcal{N}_1$ is the normalized density of states in a superconductor. For illustration, let us consider the homogeneous case and the case where $1/\tau_S = 0$. Then Eq. (10) holds, and for energies less than the Debye energy, we have with sufficient accuracy

$$\tilde{\omega}_E^{\mathrm{R(A)}} = -i(1+\lambda)E + (-)1/2\tau_E; \qquad \tilde{\Delta}_E = (1+\lambda)\Delta \tag{16}$$

The inelastic collision time $\tau_E$ is approximately independent of $E$ in the range $|E| < T$, which, if $\Delta \ll T$, is the range in which we are most interested. Then the functions $\mathcal{N}_j$ and $\mathcal{R}_j$ take the form shown graphically in Fig. 1.

Considering the definitions (1) and (11), we recognize that $-\frac{1}{4}N(0)(\delta\hat{G}_E)_{11}$ is the change in the quasiparticle density of one spin direction and per unit energy range. Since, at $\Omega = 0$, the contribution $\frac{1}{2}N(0)[a_E^{\mathrm{R}} - a_E^{\mathrm{A}}][-\frac{1}{2}\tanh(E/2T)]$ has to be considered as being due to the change $\frac{1}{2}N(0)[a_E^{\mathrm{R}} - a_E^{\mathrm{A}}]$ in the density of states, we conclude that the anomalous part of $(\delta\hat{G}_E)_{11}$ is proportional to a change $\delta f_E$ in the quasiparticle distribution function. Since

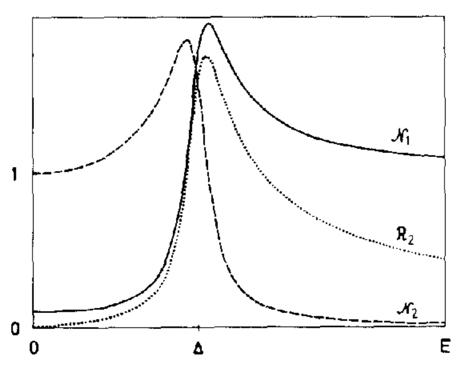


Fig. 1. $\mathcal{N}_1(E)$, the normalized density of states in a superconductor (solid line), $\mathcal{N}_2(E)$ (dashed line), and $\mathcal{R}_2(E)$ (dotted line) as defined by Eqs. (15) for $\Delta\tau_E = 5$.

$N(0)\mathcal{N}_1$ is the unperturbed density of states, we have

$$\delta f_E = -a_E^{(a)}/4\mathcal{N}_1(E) \tag{17}$$

We remark that this change is measured relative to the Fermi level of the equilibrium state.

From the anomalous part of Eq. (5), one can construct a Boltzmann equation for the quasiparticle distribution $\delta f_E$, which, though it differs in detail in the transverse and in the longitudinal modes, is of the same general structure. Due to its importance, we present it here explicitly, replacing $-i\Omega$ by $\partial/\partial t$, in the following form:*

$$\mathcal{N}_1(E)\,\delta \dot{f}_E - K(\delta f) - P_E - Q_E = \dot{h}_E \tag{18}$$

Here the contribution of the perturbation $\frac{1}{4}i\{[\hat{U}, \hat{G}^{(e)}]\}_{11}^{(a)}$ has been denoted by $P_E$. The quantity $h_E$ is essentially the commutator of the last term of Eq. (12) proportional to $(\delta\hat{\Sigma}^R + \delta\hat{\Sigma}^A)$ with $\hat{G}^{(e)}$. It will be discussed in the next section. The collision integral is of the following form:

$$K(\delta f) = -2\pi \int dE' \frac{\mu(E - E')}{\cosh(E/2T)\cosh(E'/2T)\sinh[(E - E')/2T]} \times \mathcal{M}(E, E')\left[\cosh^2\frac{E}{2T}\delta f_E - \cosh^2\frac{E'}{2T}\delta f_{E'}\right] \tag{19}$$

where

$$\mu(E) = -\mu(-E) = (i\lambda/4\pi)[B^R(-iE) - B^A(-iE)] \tag{20}$$

is the effective phonon density of states.† The further quantities depend on the mode, as follows.

(*T*) *Transverse Mode.* Here

$$\mathcal{M}^{(T)}(E, E') = \mathcal{N}_1(E)\mathcal{N}_1(E') + \mathcal{N}_2(E)\mathcal{N}_2(E') \tag{21}$$

and

$$Q_E^{(T)} = -2|\Delta|\mathcal{N}_2(E)\,\delta f_E + \operatorname{div} D\,\mathcal{M}^{(T)}(E, E)\left\{\nabla\delta f_E + \frac{e}{4T\cosh^2(E/2T)}\,\delta\dot{\mathbf{A}}\right\} \tag{22}$$

(*L*) *Longitudinal Mode.* In this case

$$\mathcal{M}^{(L)}(E, E') = \mathcal{N}_1(E)\mathcal{N}_1(E') - \mathcal{R}_2(E)\mathcal{R}_2(E') \tag{23}$$

*Actually, the term $\mathcal{N}_1\delta\dot{f}_E$ carries a factor $(1 + \lambda)$ originating from renormalization effects. Since it is not of particular importance, we have dropped this factor.

†In the Debye model, we have $\mu = A|E|E$, such that the inelastic collision time $\tau_E = [28\pi\zeta(3)T^3A]^{-1}$.

and

$$Q_E^{(L)} = \operatorname{div} D\, \mathcal{M}^{(L)}(E, E)\, \nabla \delta f_E \tag{24}$$

There are also contributions to $Q_E$ quadratically in $\delta \mathbf{A}$, which are important if the electromagnetic field amplitude is sufficiently large or if it is applied to a state with a large static supercurrent velocity. In the latter case, furthermore, there is a possibility of a coupling between transverse and longitudinal modes by a contribution to $Q^{(T),(L)}$, which is of the form

$$\mp \operatorname{div} 2D\, [\nabla \theta + 2e\mathbf{A}] \mathcal{N}_2 \mathcal{R}_2\, \delta f^{(L),(T)}$$

Concerning the properties of the collision integral, we note that it is a symmetric operator with respect to the quantity $[\cosh^2 (E/2T)]\, \delta f_E$ and that it conserves the parity. Furthermore, $\int dE\, K(\delta f) = 0$. In the case $\Delta = 0$, $K(\delta f)$ becomes equal to the collision integral of the linearized electron–phonon Boltzmann equation of a normal metal in which the phonon distribution is fixed to its equilibrium value.

In the longitudinal mode, the stationary Boltzmann equation (18) is rather of a type one expects; the collision integral $K^{(L)}(\delta f)$ has already been discussed by Bardeen *et al.*[18] The most important feature of the Boltzmann equation of the transverse mode is that there is the term $2|\Delta| \mathcal{N}_2(E) \delta f_E$ which acts as a local source or sink of quasiparticles. However, the total number of quasiparticles is still conserved, as we will show later.

## 4. THE LINEARIZED, TIME-DEPENDENT GINZBURG–LANDAU EQUATION

We derive here an equation of motion for the change $\delta\Delta$ in the order parameter, which represents, as we shall see, the regular parts of $\delta\hat{G}$. This equation and the Boltzmann equation (18) form a coupled system in the general, nonstationary case ($\Omega \neq 0$).

Considering the regular quantities, we define, for instance,

$$\delta\hat{G}^{(r)}(\omega) = \begin{cases} \delta\hat{G}^{R}_{E=i\omega} & \text{if} \quad \omega > 0 \\ \delta\hat{G}^{A}_{E=i\omega} & \text{if} \quad \omega < 0 \end{cases} \tag{25}$$

and obtain the following relation:

$$\begin{aligned} \delta\hat{\Sigma}^{(r)}(\omega) = {} & i\pi\lambda T \sum_{\omega'} B(\omega - \omega')\, \delta\hat{G}^{(r)}(\omega') \\ & + i\pi\lambda \int \frac{dE'}{4\pi i} B(\omega + iE') \frac{-\Omega}{4T\cosh^2 (E'/2T)} [\delta\hat{G}^{R}_{E'} + \delta\hat{G}^{A}_{E'}] \\ & + i\pi\lambda \int \frac{dE'}{4\pi i} B(\omega + iE')\, \delta\hat{G}^{(a)}_{E'} \end{aligned} \tag{26}$$

Due to the smooth dependence of $\delta\hat{\Sigma}^{(r)}$ on the frequency (the range of variation being given by the Debye energy), we may evaluate certain frequency sums in the same way as in the BCS theory. In Eq. (26), for instance, we substitute in the first line

$$T\sum_{\omega'} B(\omega - \omega') \rightarrow T \sum_{|\omega'|<\omega_c} (-1)$$

Furthermore, the integrands in the second and third lines contain factors which become small at larger energies $E'$, so that we are allowed to replace $B(\omega + iE')$ there by $(-1)$ without any restriction.

Thus it is consistent both to put

$$(\delta\hat{\Sigma}^{(r)})_{12} = -i(1 + \lambda)\,\delta\Delta \tag{27}$$

and also to neglect the diagonal elements of $\delta\hat{\Sigma}^{(r)}$, if we are not interested in normalization effects proportional to the factor $(1 + \lambda)$. Hence Eq. (5) allows us to express $\delta\hat{G}^{(r)}$ as a function of $\delta\Delta$. Furthermore, this expression depends on $\Omega$ only quadratically, which means that we are allowed to neglect this dependence. In the next step, $\delta\hat{G}^{(r)}$ thus obtained will be inserted in the self-consistency relation (26); and from the off-diagonal component one obtains the following equation:*

$$\frac{\pi}{8T}[\delta\dot{\Delta} + 2ie\psi\Delta] = \delta(\Delta, \Delta^*, \mathbf{A})\frac{-1}{N(0)}\frac{\delta}{\delta\Delta^*}F_{\mathrm{GL}} \tag{28}$$

where $F_{\mathrm{GL}}$ is the Ginzburg–Landau free energy functional, and where the operation $\delta(\Delta, \Delta^*, \mathbf{A})$ means linearization with respect to $\Delta$, $\Delta^*$, and $\mathbf{A}$. Furthermore,

$$2e\psi = (8T/\pi|\Delta|)\int dE'\,\beta^{\mathrm{R}}_{E'}\,\delta f_{E'} \tag{29}$$

The real and the imaginary parts of $\psi$ are connected with the transverse and longitudinal modes, respectively. It can be shown that $\psi$ is transformed by a gauge transformation in the same way as the electric potential $\phi$.

Equation (28) is reminiscent of a time-dependent Ginzburg–Landau equation, which has been proposed previously by Schmid.[19] There the quantity $\psi$ was introduced phenomenologically, whereas here it is directly connected to the change in the quasiparticle distribution.

We proceed to calculate $h_E$, which couples Eq. (18) to Eq. (28). The result is

$$\begin{aligned} h_E^{(\mathrm{T})} &= \frac{\mathcal{N}_2(E)}{4T\cosh^2(E/2T)}\,ie^{i\theta}\,\delta\Delta^{(\mathrm{T})} \\ h_E^{(\mathrm{L})} &= \frac{\mathcal{R}_2(E)}{4T\cosh^2(E/2T)}\,e^{i\theta}\,\delta\Delta^{(\mathrm{L})} \end{aligned} \tag{30}$$

*Here we neglect second-order perturbations, as considered, e.g., in the next section.

Also, we calculate the current density in the linear approximation. As far as a general expression appropriate to the dirty limit is concerned, we refer to Ref. 16. The terms arising from retarded and advanced quantities yield the supercurrent density, in which the dependence on $\Omega$ can be neglected, whereas the anomalous parts form the normal current density. Explicitly, one finds

$$\delta \mathbf{j}_s = \delta(\Delta, \Lambda^*, \mathbf{A})\mathbf{j}_{\mathrm{GL}} \tag{31}$$

where $\mathbf{j}_{\mathrm{GL}}$ is the supercurrent density of Ginzburg and Landau; and

$$\delta \mathbf{j}_n = -\sigma_0 \int dE\, \mathcal{M}^{(\mathrm{T})}(E, E)\left[\frac{1}{4T\cosh^2(E/2T)}\delta\dot{\mathbf{A}} + \frac{1}{e}\nabla\delta f_E\right] \tag{32}$$

Again, we have omitted terms that arise when a static supercurrent is present. As far as the charge density is concerned, we obtain

$$\delta\rho = 2eN(0)\left\{\int dE\, \mathcal{N}_1(E)\,\delta f_E - e\phi\right\} \tag{33}$$

Note that the Boltzmann equation and the time-dependent Ginzburg–Landau equation together imply the continuity equation $\delta\dot{\rho} + \operatorname{div}\delta\mathbf{j} = 0$.

## 5. SHIFT OF THE CHEMICAL POTENTIAL BY QUASIPARTICLE INJECTION

The methods developed in the preceding sections are applied here to a type of experiment performed first by Clarke[11] and discussed by Tinkham and Clarke.[20] A quasiparticle current $I_N$ is injected via a tunnel junction from a normal metal ($N$) into a superconductor ($S$), and creates there a nonequilibrium quasiparticle distribution. If the area of the junction is sufficiently large, and if the thickness of the superconductor in the region adjacent to it is small enough, we can consider this distribution to be spatially uniform, and neglect any normal current flow in the superconductor. Furthermore, we are interested in a stationary situation, which means that the Boltzmann equation (18) is decoupled from the Ginzburg–Landau equation (28). The converse of this statement is not true. It follows from the Ginzburg–Landau equation that the quasiparticle distribution causes a supercurrent to flow such that charge neutrality is maintained.

The change in the quasiparticle distribution is detected by a probe ($P$) consisting of a normal metal coupled by a second tunnel junction to the superconductor. In particular, the voltage $V_P$ is measured that is required to reduce the current to the probe $I_P$ to zero.

The difference in the electrochemical potentials $\mu_N - \mu_S = eV_N$ between normal metal and superconductor shows up in the tunneling Hamiltonian as an extra time dependence. We start from the following transformation for the field operators of normal metal and superconductor:

$$\begin{aligned}\psi_j(t) = &\ \exp\left[i\left(H - \sum_{j'} \mu_{j'} N_{j'}\right)t\right] \psi_j(0) \\ &\times \exp\left[-i\left(H - \sum_{j'} \mu_{j'} N_{j'}\right)t\right] \exp(-i\mu_j t) \\ &= \tilde{\psi}_j(t) \exp(-i\mu_j t)\end{aligned}$$

where $H$ is the full Hamiltonian of the decoupled system. We recognize that the most convenient operators $\tilde{\psi}(t)$ can be used if allowance is made for an extra time-dependent factor. As usual, we neglect the momentum dependence of the transfer matrix element, for which we write $T_{SN}$ and $T_{NS}$ for short. Then, in the system of notations used, for instance, in connection with Eq. (5), the tunneling Hamiltonian corresponds to the perturbation

$$\hat{H}_T = \hat{H}_{SN} + \hat{H}_{NS} = T_{SN} \exp(-ieV_N \hat{\tau}_3 t) + T_{NS} \exp(ieV_N \hat{\tau}_3 t) \tag{34}$$

To first order, $\hat{H}_T$ does not lead to a change in the distribution function of the superconductor. Hence we retain the second-order contribution, which arises effectively from the perturbation

$$\hat{U} = -i\pi N_N(0) \hat{H}_{SN} \hat{G}_N \hat{H}_{NS} \tag{35}$$

Since $\hat{G}_N(\omega) = \hat{\tau}_3 \operatorname{sgn} \omega$, we have in the Matsubara representation

$$\begin{aligned}\hat{U}(\omega, \omega') = &-i\pi |T_{NS}|^2 N_N(0) \\ &\times \tfrac{1}{2}[(\hat{1} + \hat{\tau}_3) \operatorname{sgn}(\omega - \omega_n) - (\hat{1} - \hat{\tau}_3) \operatorname{sgn}(\omega - \omega_{n'})]\end{aligned} \tag{36}$$

where $\omega - \omega' = \omega_n + \omega_{n'}$. When performing the analytical continuation, we have to put $\omega_n = -ieV_N + 0$ and $\omega_{n'} = ieV_N + 0$. From this we find that the inhomogeneous term $P_E$ of the Boltzmann equation assumes the form

$$P_E^{(T)} = \frac{1}{2} \pi N_N(0) |T_{NS}|^2 \mathcal{N}_1(E) \left[\tanh \frac{E + eV_N}{2T} - \tanh \frac{E - eV_N}{2T}\right] \tag{37}$$

in the transverse mode, and

$$\begin{aligned}P_E^{(L)} = &\ \frac{1}{2} \pi N_N(0) |T_{NS}|^2 \mathcal{N}_1(E) \\ &\times \left[2 \tanh \frac{E}{2T} - \tanh \frac{E + eV_N}{2T} - \tanh \frac{E - eV_N}{2T}\right]\end{aligned} \tag{38}$$

in the longitudinal mode. Note that we have normalized $|T_{NS}|^2$ such that it refers to the unit volume of the superconductor. Hence $I_N = 2N(0)\mathcal{V} \int dE\, P_E$, where $\mathcal{V}$ is the volume of the superconductor in which quasiparticles are injected.

When calculating the current flowing from the superconductor to the probe, we encounter the quantities $G^>(t,t') = -i\langle\tilde{\psi}(t)\tilde{\psi}^+(t')\rangle$ and $G^< = i\langle\tilde{\psi}^+(t')\tilde{\psi}(t)\rangle$. After Fourier transformation, the corrections to the equilibrium value of $G^{\gtrless}$ can be related to the various parts of $\delta\hat{G}_E$ as follows:

$$\delta G_E^{\gtrless} = \tfrac{1}{2}(\delta\hat{G}_E^{(a)})_{11} \pm f(\mp E)[\delta\hat{G}_E^R - \delta\hat{G}_E^A]_{11} \tag{39}$$

From this we obtain in linear approximation

$$I_P = \frac{1}{e}G_{nn}^{(SP)} \int dE\, \mathcal{N}_1(E)\left[\frac{eV_P}{4T\cosh^2(E/2T)} + \delta f_E\right] \tag{40}$$

Here the probe has been assumed to be in equilibrium, and $G_{nn}^{(SP)}$ is the normal-state conductance of the junction between superconductor and probe. We remark that only the transverse mode contributes to the current.*

A stationary solution of the Boltzmann equation (18) which is in the transverse mode and spatially uniform (and where the phase $\theta = 0$) is obtained as follows. Integrating this equation with respect to the energy, we find that

$$2\Delta \int dE\, \mathcal{N}_2(E)\, \delta f_E = \int dE\, P_E = \frac{1}{2N(0)\mathscr{V}} I_N \tag{41}$$

From this we conclude that, at a given current, $\delta f_E$ becomes arbitrarily large as $\Delta$ goes to zero. Hence the proper form of $P_E$ will be irrelevant in this limit. An inspection of the Boltzmann equation shows that

$$\delta f_E = \frac{1}{4T\cosh^2(E/2T)} \frac{\mathcal{N}_1(E)}{\mathcal{N}_1(E) + 2\Delta\tau_E\mathcal{N}_2(E)}\eta \tag{42}$$

In deriving this relation, we have neglected the contribution $\mathcal{N}_2(E)\mathcal{N}_2(E')$ to $\mathcal{M}(E,E')$, and also we have assumed that the collision rate

$$\frac{1}{\tau_E} = \int dE' \frac{2\pi\mu(E-E')}{\sinh[(E-E')/2T]} \frac{\cosh(E/2T)}{\cosh(E'/2T)} \mathcal{N}_1(E') \tag{43}$$

is not changed appreciably if the factor $\mathcal{N}_1(E')[\mathcal{N}_1(E') + 2\Delta\tau_{E'}\mathcal{N}_2(E')]^{-1}$ is inserted in the integrand. Introducing the integrals

$$J_1 = (1/\eta)\int dE\, \mathcal{N}_1(E)\, \delta f_E; \qquad J_2 = (1/\eta)\int dE\, \mathcal{N}_2(E)\, \delta f_E \tag{44}$$

where $\delta f$ is given by Eq. (42), we define a relaxation time

$$\tau_R = J_1/2\Delta J_2 \tag{45}$$

*Furthermore, one can show that the zero-current voltage $V_P$ at $I_P = 0$ should not change noticeably if the probe becomes superconducting.

If $\Delta \ll T$, we have $J_1 = 1$. Then we obtain from Eq. (40) and Eq. (41)

$$eV_P = \tau_R I_N / 2N(0)\mathscr{V} \tag{46}$$

This relation agrees with Refs. 11 and 20. A further meaning of $\tau_R$ will be discussed in Section 7.

In a homogeneous superconductor, where $1/\tau_s = 0$ and where $\mathcal{N}_j$ take the form shown in Fig. 1, we find

$$J_2 = (\pi/8T\tau_E)[1 + (1/2\Delta\tau_E)^2]^{-1/2} \tag{47}$$

provided that $\Delta \ll T$ and that $\tau_E$ is independent of $E$ in the range of interest. The largest contribution to $J_2$ comes from the energy range from above $\Delta$ to several $\Delta$'s. This is even more the case if paramagnetic impurities are added. Under this circumstance, $\mathcal{N}_1$ can be replaced by 1, and $\mathcal{N}_2$ by the expression

$$\mathcal{N}_2(E) = \frac{\Delta\Gamma}{E^2 + \Gamma^2}; \qquad \Gamma = \frac{1}{\tau_s} + \frac{1}{2\tau_E} \tag{48}$$

Strictly, this form is valid only in the gapless case. However, since the center energy range is of no importance, it also applies to the present situation. If

$$T \gg \Delta(\Gamma\tau_E)^{1/2} \tag{49}$$

we can put $\cosh^2(E/2T) = 1$ in Eq. (42), and obtain $J_2$, from which we deduce the following relaxation time:

$$\tau_R = \frac{4T}{\pi\Delta}\left(\frac{\tau_E}{2\Gamma}\right)^{1/2}\left(1 + \frac{\Gamma}{2\Delta^2\tau_E}\right)^{1/2} \tag{50}$$

Note that this result includes Eq. (47).

In the case of inelastic phonon scattering, Tinkham[20] has obtained the result $\tau_R = 0.57[\Delta(0)/\Delta(T)]\tau_E$, whereas here we have $\tau_R = 0.73[\Delta(0)/\Delta(T)]\tau_E$. The two expressions are rather close, but the ratio $\tau_R/\tau_E$ they predict does not agree so well with measured values of $\tau_R$[11] and independent measurements of $\tau_E$.[21] However, one should keep in mind that $\tau_E$ tends to decrease in an impure metal.[22]

We emphasize that the result of this paper is derived rigorously from the microscopic theory, and there has been no need to make assumptions on the interpretation of the relevant quantities. On the other hand, Tinkham's identification of states with $k > k_F$ and $k < k_F$ as particle and hole states, respectively, is correct only in the normal metal. According to Gor'kov,[23] the particle and the hole states have positive and negative energies, respectively, to which the probabilities $u_k^2$ and $v_k^2$ are assigned.

We mention that another difference in the quasiparticle concept of both theories should have no influence on the final result. Here the number of quasiparticles coincides with that of the electrons, whereas in Ref. 20 their number is variable and vanishes at $T = 0$. These two concepts can easily be related by defining in the latter case a particle and a hole distribution of positive energy which are connected with the present distribution function as follows:

$$f_p(E) = f_E; \qquad f_h(E) = 1 - f_{-E}; \qquad E > 0 \tag{51}$$

Evidently,

$$2N(0) \int dE\, \mathcal{N}_1\, \delta f_E = 2N(0) \int_0^\infty dE\, \mathcal{N}_1\, \delta(f_p - f_h)$$

is Tinkham's branch imbalance $Q$.

Consider now the case $\Delta^2 \ll \Gamma/\tau_E$. Then $\tau_R = 2T/\pi\Delta^2$ is independent of $\Gamma$ and $\tau_E$, and this result holds if $\tau_R \gg \tau_E$. Furthermore, the quantity $\psi$ defined by Eq. (29) is, in the transverse mode, equal to the electric potential $\phi$. This follows from the form of $J_2$, and also from the condition of approximate charge neutrality, which means that $\delta\rho$ of Eq. (33) is zero. Therefore we conclude that the time-dependent Ginzburg–Landau equation of Ref. 19 applies to the case under consideration.*

## 6. RESISTIVITY OF A SUPERCONDUCTING NORMAL INTERFACE

An extra resistance arises when a current passes perpendicularly an interface where a normal metal and a superconductor are in metallic contact.[10,12] Since the current may be assumed to be steady, all time derivatives vanish in the Boltzmann equation. The electromagnetic potentials, being proportional to $\dot{\phi}$ and $\dot{\mathbf{A}}$, also disappear.

However, the presence of an electric potential becomes manifest as a boundary condition for the distribution $\delta f_E$ in the normal metal. There we have $-K(\delta f) = D\,\nabla^2 \delta f_E$. This homogeneous equation has a solution of the form $\delta f_E = [4T\cosh^2(E/2T)]^{-1}\eta, \nabla^2\eta = 0$. Evidently, the charge density is zero if $\eta = e\phi$. Therefore we require that in the normal metal

$$\delta f_E = \frac{1}{4T\cosh^2(E/2T)}\, e\phi; \qquad \nabla^2\phi = 0 \tag{52}$$

If there is a proximity effect, this boundary condition holds in a region far off the interface. In the following, we assume that the superconductor and the normal metal occupy the half-spaces $x > 0$ and $x < 0$, respectively, and that

*This result reduces the range of applicability of the model considered in Ref. 23, where the case $\Delta \ll \Gamma \ll T$ has been studied without taking into account inelastic phonon scattering.

the current is flowing in the $x$ direction. Then in the normal metal $\phi = -E_0 x + \phi_0$.

Since by Eq. (52), $\delta f_E$ is an even function, the response of the superconductor is in the transverse mode. Hence

$$-K^{(T)}(\delta f) = Q_E^{(T)} = -2\Delta \mathcal{N}_2(E)\,\delta f_E + D\frac{d}{dx}\mathcal{M}^{(T)}(E, E)\frac{d}{dx}\,\delta f_E \tag{53}$$

where we have used Eq. (22) for $Q_E^{(T)}$, which holds since there is no supercurrent ($\theta = \mathbf{A} = 0$) in the unperturbed state. This equation is similar in structure to one discussed in the preceding section and the second term of $Q_E^{(T)}$ plays here the role of $P_E^{(T)}$ there. Therefore, close to the transition temperature, the energy dependence of $\delta f_E$ is the same as the one given by Eq. (42). From this, we deduce the following differential equation for the quantity $\eta$:

$$2\Delta J_2 \eta = D\frac{d}{dx}\int dE\,\frac{\mathcal{M}^{(T)}(E, E)}{4T\cosh^2(E/2T)}\,\frac{d}{dx}\,\frac{\mathcal{N}_1}{\mathcal{N}_1 + 2\Delta\tau_E\mathcal{N}_2}\,\eta \tag{54}$$

In the above expression, $\mathcal{N}_2(E)$ has, for energies sufficiently large as compared to $\Delta$, the same asymptotic behavior as it has in the case of electron scattering at paramagnetic impurities, as we will show below. Furthermore, it will be assumed that $\mathcal{N}_2(E)$ can be replaced by an expression of the form (48) in the whole energy range. Even if there is locally an energy gap, this assumption can be justified by the fact that the center energy range contributes less to the integral $J_2$. Then we may substitute $1/\tau_R$ for $2\Delta J_2$, where $\tau_R$ is given by Eq. (50). As far as the integral on the right-hand side of Eq. (54) is concerned, we realize that, if inequality (49) is satisfied, the region $\Delta(\Gamma\tau_E)^{1/2} < E < T$ contributes most to this integral. There, $\mathcal{N}_1 = 1, \mathcal{N}_2 = 0$, and we obtain for Eq. (54) the following form:

$$(1/\tau_R)\eta = D\eta'' \tag{55}$$

where $\eta'' = d^2\eta/dx^2$.

In equilibrium, the spatial dependence of the order parameter can be found from the Ginzburg–Landau equation. Close to the transition temperature, where the coherence length $\xi_{GL}$ is large, we may assume that $\Delta$ vanishes continuously at the interface. Hence,

$$\Delta(x) = \begin{cases} \Delta(T)\tanh(x/2^{1/2}\xi_{GL}) & \text{if} \quad x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{56}$$

Now, from Eqs. (2) and (5), it follows that

$$\tilde{\omega}\beta - \Delta\alpha = \tfrac{1}{2}D(\beta'/\alpha)', \qquad \alpha^2 + \beta^2 = 1 \tag{57}$$

where $\alpha = \alpha(\omega; x)$, etc., and where (neglecting renormalization), $\tilde{\omega} = \omega + (1/2\tau_E)\,\mathrm{sgn}\,\omega$. Consider the case $|\omega| \gg \Delta$, such that $\alpha = \mathrm{sgn}\,\omega$. Then $\beta$ is given by

$$\beta = (2|\tilde{\omega}|D)^{-1/2} \int dx' \exp\left[-(2|\tilde{\omega}|/D)^{1/2}|x - x'|\right]\Delta(x')$$

$$= \frac{\Delta}{|\tilde{\omega}|} + \frac{D\Delta''}{2\tilde{\omega}^2} \tag{58}$$

We perform the analytical continuation $\omega \to -iE \pm 0$ and, considering Eq. (48), we realize that the identification $\Gamma = 1/2\tau_E - D\Delta''/2\Delta$ can be made.

For convenience, the dimensionless variable $t = x/2^{1/2}\xi_{\mathrm{GL}}$ is introduced. Thus Eq. (55) assumes the form

$$\ddot{\eta} = A\,(\tanh^2 t)\left[\frac{1 + B^2\cosh^2 t}{\sinh^2 t + A^2C^2(1 + B^2\cosh^2 t)}\right]^{1/2}\eta \tag{59}$$

where $A = [\pi^5/14\zeta(3)]^{1/2}(T_c\tau_E)^{-1/2}$; $B = (2/1.76\pi)[\Delta(0)/\Delta(T)]A$; and $C = 7\zeta(3)/\pi^4$.

At $x = 0$, the distribution $\delta f_E$ and its derivative are continuous. Hence $\phi_0/E_0 = -\eta(0)/\eta'(0)$. On the other hand, we have to require $\eta \to 0$ as $x \to \infty$. This fixes the ratio $\phi_0/E_0 = w_s$, which is the thickness of a slab of normal metal having a resistance equal to the extra resistance $R_e$ of the interface. Therefore

$$w_s = R_e/\rho_N = -\eta(0)/\eta'(0) \tag{60}$$

where $\rho_N$ is the specific resistivity of the normal metal.

Approximate solutions of Eq. (59) may be found in various limiting cases. We have also integrated this equation numerically. In a rather large temperature range, say $0.005 \lesssim (T_c - T)/T_c \lesssim 0.2$, we found that $w_s$ can be represented as follows:

$$w_s = L\xi_{\mathrm{GL}}(0)[(T_c - T)/T_c]^{-\rho} \tag{61}$$

where $L$ and $\rho$ depend on $A$ as follows:

| | | | | |
|---|---|---|---|---|
| $A$ | 0.05 | 0.25 | 1.0 | 3.0 |
| $\rho$ | 0.38 | 0.38 | 0.37 | 0.46 |
| $L$ | 25 | 7.0 | 3.0 | 1.5 |

Roughly, the values $A = 0.25$ ($T_c = 3.8$ K; $\tau_E = 3 \times 10^{-10}$ sec) and $A = 1.0$ ($T_c = 7.2$ K; $\tau_E = 10^{-11}$ sec) might be appropriate to tin and lead, respectively. We recognize that for $A < 1$ the value of the exponent $\rho$ is approximately constant.

On the other hand, if $A$ is sufficiently large, or if $(T_c - T)/T_c$ is very small, Eq. (59) assumes the form*

$$\ddot{\eta} = C^{-1}(\tanh^2 t)\eta \tag{62}$$

This equation can be solved exactly in terms of hypergeometric functions; in particular $\eta(0)/\dot{\eta}(0)$ is given by a combination of $\Gamma$ functions. It turns out that

$$w_s = 1.21\xi_{\mathrm{GL}}(T) \tag{63}$$

Hence, we have the exponent $\rho = 0.5$ in this limit.

As far as the experimental results are concerned, we found in analyzing the data[12] that $\rho$ varies in the range 0.2–0.5. In the most recent results, $\rho = 0.22$ and 0.56 for pure lead and for lead with 3.5 per cent bismuth, respectively.

There are several reasons that may explain this lack of agreement. First, the experimental results depend critically on the preparation of the junction, since there is a tendency for the formation of an oxide layer. Second, the theoretical results have been derived under rather restrictive assumptions. These are: dirty limit, neglect of proximity effect, inequality (49). Further, the values for the exponent $\rho$ are comparatively small, such that even small side effects may produce a noticeable change in its relative magnitude. It should be noted, however, that the exponents of the present theory agree better with the experimental results than the exponent $\rho = 5/6$ that Pippard *et al.*[12] obtained from a phenomenological theory.

We will discuss shortly the experiments done by Yu and Mercereau,[10] who measured the electric potential $\phi$ in the superconductor a distance $X$ off the interface. Using $e\phi = \eta$, we may calculate $\phi$ on the basis of Eq. (59). Again, we find in a rather broad temperature range that $\phi \propto [(T_c - T)/T_c]^{-\kappa}$. For instance, for $A = 1$, we find $\kappa = 0.6$ and 1.1 for $X/\xi_{\mathrm{GL}}(T = 0) = 5$ and 20, respectively. Though the experiment demonstrates convincingly the existence of an electric field in a superconductor, the measured quantity depends on several parameters, such that a reasonable quantitative interpretation is difficult.

The equation (28) involving the order parameter is, in the transverse mode, equivalent to the equation of continuity and hence it allows us to determine the supercurrent, which increases as the quasiparticle current decreases. At this point we wish to emphasize the peculiar fact that the supercurrent is stationary in spite of the presence of an electric field. In a phenomenological way, one may say that the electrochemical potential of the Cooper pair is constant though the electric potential varies in space.

*Observing that $\eta$ is equal to $e\phi$ (if $\Delta \ll T$), we can derive this form also from the equations given in Ref. 19.

## 7. DISCUSSION AND CONCLUSION

Rather to elucidate the meaning of the various quantities introduced in the preceding sections than for reasons of practical application, we investigate the time-dependent relaxation of a perturbance in a homogeneous superconductor. Anticipating some results of the following discussion, we consider the change in the distribution function as due to changes in the electrochemical potential $\delta\mu_E$ and in the temperature $\delta T_E$ of the quasiparticles, which depend on energy, space, and time. In particular

$$\begin{aligned} \delta\mu_E &= [4T\cosh^2(E/2T)]\,\delta f_E^{(T)} \\ \delta T_E &= [4T^2E^{-1}\cosh^2(E/2T)]\,\delta f_E^{(L)} \end{aligned} \tag{64}$$

Considering the relation (32) for the normal current, we recognize that $-\delta\dot{\mathbf{A}} + (1/e)\nabla\mu_E$ is the electrochemical field driving the quasiparticle of a given energy level, as it should be. As far as the quantity $\psi$ of Eq. (29) is concerned, we note that $e\psi = \delta\mu$, provided that $\delta\mu$ is independent of $E$ (and $\Delta \ll T$). Quite generally, the real part of $e\psi$ has the meaning of an average electrochemical potential of the levels in the range $|E| < \Delta$. For an illustration, consider the distribution of Eq. (42), where $\delta\mu_E$, being constant in the range $|E| > \Delta$, drops to very small values for energies near the center. Thus $\psi = \phi/2\Delta\tau_E$ is very small. The reason for this behavior is that in the center energy range, quasiparticles are converted rapidly into Cooper pairs, i.e., into a state where $\delta\mu = 0$, and vice versa.

On the other hand, when $\delta T_E = \delta T$ is constant, we obtain, from Eq. (29), $2ie\psi = -(8/\pi)\,\delta T$. Then from the Ginzburg–Landau equation (28) it follows that the magnitude of $\Delta$ changes by an amount that one expects from a static change $\delta T$ in temperature.

(T) Consider Eqs. (28) and (29). In a transverse mode that is homogeneous we obtain first

$$\delta\dot{\Delta} = -i(8T/\pi)\int dE'\,\mathcal{N}_2(E')\,\delta f_{E'} \tag{65}$$

and then we substitute this relation by means of Eq. (30) in the Boltzmann equation (18). Thus we obtain

$$\mathcal{N}_1(E)\,\delta\dot{f}_E = K^{(T)}(\delta f) - 2\Delta\mathcal{N}_2(E)\,\delta f_E + \frac{2}{\pi}\mathcal{N}_2(E)\int dE'\,\mathcal{N}_2(E')\,\delta f_{E'} \tag{66}$$

Since $\int dE\,\mathcal{N}_2(E) = \pi\Delta$, it follows from Eq. (66) that $\int dE\,\mathcal{N}_1(E)\,\delta\dot{f}_E = 0$. Another property of this equation is that there exists a stationary solution where $\delta f$ is of the type (64) with $\delta\mu_E = \delta\mu$. Then it follows from Eq. (65) that the order parameter rotates uniformly in the complex plane. Putting $\Delta + \delta\Delta = \Delta e^{-i\theta}$, we find indeed $\dot{\theta} = 2\,\delta\mu$.

Consider now the decay of a distribution $\delta f$ given initially by Eq. (42). It differs from the stationary solution discussed above in that a fraction of quasiparticles of about $\Delta/T$ is missing from the center region $|E| < \Delta$. Hence quasiparticles will be scattered in from the outer region at a rate of the order $(\Delta/T) \cdot 1/\tau_E$. As a consequence, the stationary distribution found above is reached in a time roughly given by $\tau_E$.

Such reasoning applies only to a homogeneous superconductor that is strictly closed. In a real situation, however, the superconductor is most probably part of an open system. For illustration, consider the system of Section 5. In this case, the electrochemical potential of the Cooper pairs is kept at zero reference level. This means that $\delta\dot{\Delta} = 0$, and that the last term of Eq. (66) vanishes. Then an initial disturbance of the form (42) will decay with the characteristic time given by Eq. (50), which is $\tau_R^{(T)} = (4/\pi)T\tau_E/\Delta$ in the situation $(1/\tau_s = 0)$ considered here. Such a behavior follows from the arguments used in Section 5 in calculating the distribution $\delta f_E$ and the relaxation time.

(L) In the longitudinal and homogeneous mode, Eq. (28) assumes the form

$$\frac{\pi}{8T}\delta\dot{\Delta} + \int dE'\,\mathscr{R}_2(E')\,\delta f_{E'} = -\frac{7\zeta(3)}{4}\frac{\Delta^2}{\pi^2 T^2}\delta\Delta \tag{67}$$

Since the decay time is of the order $\tau_E(T/\Delta)$, the time derivative can be neglected. Proceeding in the same way as above, we obtain the following Boltzmann equation:*

$$\mathcal{N}_1(E)\,\delta\dot{f}_E = K^{(\mathrm{L})}(\delta f) - \frac{4\pi^2 T^2}{7\zeta(3)\Delta^2}\frac{1}{4T\cosh^2(E/2T)} \times \mathscr{R}_2(E)\int dE'\,\mathscr{R}_2(E')\,\delta\dot{f}_{E'} \tag{68}$$

Due to the large factor $T^2/\Delta^2$, the time derivative of the last term dominates the other one. Assuming that $\delta f_E \propto \mathscr{R}_2(E)/\mathcal{N}_2(E)\cosh^2(E/2T)$, we recognize that in the limit $1/\Delta\tau_E \ll 1$, only the scattering out term of $K^{(\mathrm{L})}$ is important. Then the decay time is given by†

$$\tau_R^{(\mathrm{L})} = \frac{\pi^3}{7\zeta(3)}\frac{T}{\Delta}\tau_E \tag{69}$$

Though the processes are fairly different, the relaxation times of the transverse and of the longitudinal modes are remarkably close. Characteristic to

*Note that there is no stationary mode of Eq. (68) with $\delta T_E = \delta T$, since the phonon temperature is kept fixed.

†This relaxation time was derived first in Ref. 6, where a different method was used. Note that in Ref. 6 there is a misprint such that a factor $8/7\zeta(3)$ is missing in the final result.

both is the divergence $\propto 1/\Delta$ for $T \to T_c$. At the phase transition, quite generally, a divergence should occur for any mode in which the order parameter is involved essentially.

The transverse mode can be excited by electron injection in a tunneling experiment or by driving a current in the direction of a spatial change of the order parameter. Essentially, these are processes that tend to change the particle density. The longitudinal mode is generally associated with a change in the quasiparticle energy. Therefore it is excited by most perturbations. A superposition of a dc and an ac current, for instance, has been proposed in Ref. 6; this method has been used by Peters and Meissner[7] successfully in measuring $\tau_R^{(L)}$. Irradiation of the sample by electromagnetic waves provides other means of excitation. In the case of stationary irradiation, the distribution function obeys the equation

$$0 = K^{(L)}(\delta f) + \frac{W_0}{4N(0)\nu} \times \left\{\left[\tanh\frac{E}{2T} - \tanh\frac{E-\nu}{2T}\right]\mathcal{M}^{(L)}(E, E-\nu) + (\nu \leftrightarrow -\nu)\right]\right\} \quad (70)$$

where the strength of the electromagnetic field of frequency $\nu$ is such that the power $W_0$ would be dissipated in a corresponding normal metal. Such an equation, and in particular its generalization, which includes all nonlinear contributions, has been derived and investigated by Eliashberg[14,25] and Eliashberg and co-workers.[8,26] Several interesting experiments[9] have been done in this direction, without, however, comparing the results with the theory just mentioned. Apparently without knowledge of the previous theoretical work, some authors[27] have treated the same subject on a semi-microscopic basis.

The earliest investigation on the conversion of quasiparticles in Cooper pairs was that of Ginsberg and Schrieffer.[4,5] They termed a process that changes quasiparticles into pairs a recombination process, and it has been thought that it is identical to a transition of a quasiparticle from a state above the gap to one below it. Now, in a normal metal, this last process corresponds to the recombination of a particle with a hole. Clearly, in just half of the electron–phonon collisions the electrons cross the Fermi surface, and hence the recombination time thus defined is equal to $2\tau_E$ at the transition temperature, which is certainly not the recombination time of quasiparticles to Cooper pairs.*

*Since the word recombination has been used in such a definite way in the past, we avoid this term as much as possible. We do not exclude the possibility that for temperatures $T < \Delta$, the times calculated will agree within an order of magnitude with experiment. On the other hand, it will be difficult to separate thermalization and order parameter relaxation processes sufficiently at these low temperatures.

At this point, one should caution against identifying too strictly the quasiparticles as defined here with "normal" electrons. The number of quasiparticles and the number of Cooper pairs, which one expects to be proportional to $|\Delta^2|$, are not always correlated. In the transverse mode, the latter quantity is constant, although the number of quasiparticles may change. The situation is reversed in the longitudinal mode. On the other hand, in the transverse mode a quasiparticle current may vanish and emerge at the same place as a supercurrent.

In conclusion, it can be said that the formalism developed here provides the most convenient tool for the investigation of linear nonstationary processes in superconductors in which inelastic collisions play an important role.

## ACKNOWLEDGMENTS

We would like to thank Dr. U. Brandt and K. Jacoby for advice in connection with numerical calculations.

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