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Solution of Spin Dynamics Equations for ^{3}He Superfluid Phases in a Strong Magnetic Field

I. A. Fomin

L. D. Landau Institute for Theoretical Physics, The Academy of Sciences of the USSR, Moscow, USSR

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A general method of investigating the solutions of spin dynamics equations of superfluid phases of ^{3}He in strong magnetic fields is proposed. The effect of longitudinal spin oscillations upon the frequency of precession is taken into account. The motion of a spin near singular points at which the longitudinal oscillation frequency becomes zero is considered. The formulas derived are applied to A and B phases in the open geometry and to the B phase in the plane-parallel geometry.

1. INTRODUCTION

The spin dynamics of ^{3}He superfluid phases are well described over a wide range of magnetic fields and temperatures by the theory of Leggett,[1,2] and the interpretation of experiments on the spin dynamics of ^{3}He has been reduced to searching for relevant solutions of the Leggett equations. However, at present there is no general investigation of possible solutions of the system of Leggett equations. To obtain information from this system is an independent problem for each particular case. It also should be noted that the spin dynamics of ^{3}He superfluid phases are studied by the NMR method, which is particularly effective when the motion of magnetization is characterized by one or several periods, and the so-called "quasiperiodic" solutions are of particular interest.

In this work we study the solutions of the Leggett equations in the region of strong magnetic fields, i.e., when the spin–orbital energy is small compared to the Zeeman energy. In this case the motion of magnetization is quasiperiodic and the Van der Pol method known in classical mechanics enables us to obtain the general form of solutions of the Leggett equations asymptotic in the given smallness. Such an approach has already been used[3]

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to account for some features of the pulse NMR experiments in ^{3}He A and B phases. But the generalization proposed here seems to be simpler. It also permits investigation of more complicated experimental situations, and subtle features of the phenomena, in particular, the effect of longitudinal oscillations on the shift of the magnetization precession frequency.

2. SOLUTION OF THE EQUATION

According to the Leggett theory, the equations describing time dependence of the total spin **S** for the considered amount of helium and vector **d** in the spin space characterizing the structure of the order parameter (for details see Ref. 1) can be obtained using the Hamiltonian

$$\mathcal{H} = (g^2\mathbf{S}^2/2\chi) - g\mathbf{SH} + U(\mathbf{d}) \tag{1}$$

Here g is the gyromagnetic ratio for ^{3}He nuclei, **H** is the external magnetic field, χ is the paramagnetic susceptibility, and

$$U(\mathbf{d}) \sim \int |\mathbf{n} \cdot \mathbf{d}(\mathbf{n})|^2 \frac{do_{\mathbf{n}}}{4\pi} \tag{2}$$

is the energy of the spin–orbital interaction, arising as a result of the interaction of magnetic dipole moments of He^3 nuclei. The ratio of this energy to the Zeeman energy can be characterized by the parameter $\Omega^2 = \omega_\parallel^2/\omega_L^2$, where $\omega_\parallel$ is the frequency of longitudinal magnetization oscillations and $\omega_L = -g|\mathbf{H}|$ is the Larmor frequency. We shall consider the region of temperature and magnetic field such that $\Omega^2 \ll 1$. This is a rather wide region, since $\Omega^2 \to 0$ at $T \to T_c$ and $\Omega^2 \sim 1/H^2$. Not very large fields are already strong in this sense; thus, at $H = 600$ Oe $\Omega_A^2 < 1/400$ in the whole region of the A phase and $\Omega_B^2 < 1/50$ for the region of the B phase. However, one cannot use too strong a magnetic field ($\gg 1$ kOe) without going beyond the range of applicability of the Leggett theory.

Let us transform the Hamiltonian (1) to dimensionless variables $\mathbf{S}' = -g\mathbf{S}/\chi H$ and $t' = \omega_L t$; then it may be rewritten in the form

$$\mathcal{H}' = \tfrac{1}{2}S'^2 - S'_z + U'(d) \tag{1'}$$

The z axis is oriented in the opposite direction to the field **H**. Below we use dimensionless variables only and the primes will be omitted everywhere.

The Hamiltonian obtained has the form of that of a charged top placed into a magnetic field and in the field $U(\mathbf{d})$, with **S** having the sense of the angular momentum vector of the top measured in the frame of reference rotating with the Larmor frequency around the z axis. In the chosen units **S** coincides also with the angular velocity of the rotation of the top around its own axis. In the case of superfluid He^3 the system of vectors **d** is rotating as a solid "top" with the angular velocity **S**. In the absence of a field U, the vector **S** would perform a regular precession with the Larmor frequency around the z axis and would thus remain constant relative to the system of reference rotating around the z axis with the Larmor frequency. The presence of U leads to a slow variation of **S** due to the smallness of Ω^2. As has been shown in Ref. 3, (see also Appendix A), for a slow variation in **S** relative to the rotating axes, the average orientation of **S** relative to the top is an adiabatic invariant, i.e., it changes more slowly than the vector **S** itself. We have employed this fact, writing the kinetic energy in the Hamiltonian (1), following Leggett, in the form $gs^2/2\chi$ instead of $\frac{1}{2}g\chi_{ik}^{-1}S_iS_k$. In the experiments the system of vectors usually has enough time to be oriented at the initial moment so that the direction of **S** would correspond to maximum susceptibility. As a result, this orientation remains during the large time intervals we are interested in (usually $t \sim 1/\Omega^2$).

The conservation of the orientation of **S** relative to the "top" is a restriction which reduces the number of variables involved in the problem. It is convenient to choose $S \equiv |\mathbf{S}|$ and S_z as canonical momenta and conjugated angles γ and α as coordinates. The angle γ is the angle of rotation around the direction of **S**, and the angle α is the angle of rotation around the z axis. The value of $\mathbf{d}(t)$ at the moment of time t is related to the value at $t=0$ by the rotation matrix $\hat{R}$:

$$\mathbf{d}(t) = \hat{R}(\alpha, \beta, \gamma)\,\mathbf{d}(0) \tag{3}$$

The Eulerian angles α, β, γ are defined according to

$$\hat{R}(\alpha, \beta, \gamma) = \hat{R}_z(\alpha)\hat{R}_y(\beta)\hat{R}_z(\gamma) \tag{4}$$

It is assumed that at $t=0$, $\mathbf{S} \| \mathbf{z}$; then the angle β is an azimuthal angle of the vector **S**, i.e., $\cos\beta = S_z/S$, and the angle α is the polar angle of the same vector. In what follows we derive differential equations for S and S_z, and the chosen initial condition is not essential. Only the orientation of **S** relative to the system of vectors **d** is of importance. After substitution of (3) into the expression for energy $U(\mathbf{d})$ it turns out to be dependent on α, γ, and S_z/S.

The canonical equations for the Hamiltonian (1′) have the following form:

$$\begin{aligned} \dot{S} &= -\partial U/\partial\gamma, \qquad & \dot{\gamma} &= S + \partial U/\partial S \\ \dot{S}_z &= -\partial U/\partial\alpha, \qquad & \dot{\alpha} &= -1 + \partial U/\partial S_z \end{aligned} \tag{5}$$

Since $U \sim \Omega^2$, it is seen from the equations that there are two "slow" variables S and S_z and two "fast" variables α and γ in the problem. To obtain the first term in the asymptotic expansion of solutions of system (5) in Ω^2 by the Van der Pol method, system (5) should be averaged over periods of fast motions; this procedure differs for two cases: (1) the "resonance" case, when $|S-1| \ll 1$; we shall consider the region $|S-1| \sim \Omega$; (2) the "nonresonance" case, when $|S-1| \sim 1$.

Let us first consider the resonance case. From system (5) it is evident that from two fast variables α and γ one slow variable $\alpha+\gamma \equiv \phi$ can be formed. Performing a canonical transformation with the generating function $F=\alpha(P+S)+\gamma S$, we pass to new coordinates ϕ, α and to the corresponding momenta $S-1$ and $P=S_z-S$. In these variables the Hamiltonian has the form

$$\mathcal{H}=\tfrac{1}{2}(S-1)^2-P+U(\alpha, \phi, P/S) \tag{6}$$

Now this Hamiltonian is averaged over the period of the fast variable α; as a result we have

$$\bar{\mathcal{H}}=\tfrac{1}{2}(S-1)^2-P+V(\phi, P/S) \tag{7}$$

where $V \equiv \bar{U}$, the bar denotes averaging. Two pairs of motion equations correspond to the averaged Hamiltonian:

$$\dot{S}=-\partial V/\partial\phi, \qquad \dot{\phi}=S-1 \tag{8}$$

$$\dot{P}=0 \qquad \dot{\alpha}=-1+\partial V/\partial P \tag{9}$$

In the right-hand sides of the equation for $\dot{\phi}$ the term $\partial V/\partial S$ is omitted, since, in accordance with the character of the approximation, $\dot{S}$ is known to an accuracy of Ω^2. This means that for times $\sim 1/\Omega^2$ the error in S already will be as much as Ω^2, i.e., of the same order as $\partial V/\partial S$. In other words, within the approximation used in this case V depends on P, and and not on P/S. According to the first equation of system (9), P is the integral of motion. For given P, system (8) has stationary solutions $S=1$, $\phi=\phi_S$, where ϕ_S are the roots of the equation

$$\partial V(\phi, P)/\partial\phi=0 \tag{10}$$

For deviations of ϕ and S from the stationary values, system (8) describes harmonic oscillations with frequency

$$\omega_\parallel^2=\partial^2 V/\partial\phi^2 \tag{11}$$

Stationary solutions with $\partial^2 V/\partial\phi^2<0$ are unstable. Provided ϕ and S have stationary values, then according to the second equation of system (9), $\mathbf{S}$ is precessing with the frequency

$$\omega_\perp=-1+\partial V(\phi_s, P)/\partial P \tag{12}$$

Such conditions can be easily realized experimentally, since the time for the spin to relax to the stationary value is less by a factor of the order of magnitude Ω^2 than that of the spin relaxation to the equilibrium value.[4]

Provided ϕ and S oscillate near their stationary values, then in the right-hand side of the equation for $\dot{\alpha}$ there are, due to the dependence of V on ϕ, terms varying with the frequency of longitudinal oscillations. In this case for the total separation of motions with different frequencies it is necessary to pass to action-angle variables (see, for instance, Ref 9). Since α is the cyclic coordinate, then P should not be substituted. As the other momentum action one should choose

$$\mathcal{J}=\frac{1}{2\pi}\oint(S-1)\,d\phi=\frac{1}{2\pi}\oint\{2[\mathscr{E}+P-V(\phi,P)]\}^{1/2}d\phi \tag{13}$$

In order to find the frequencies characterizing the motion, one should, in accordance with the standard procedure, express the energy $\mathscr{E}$ through $\mathcal{J}$ and P, and differentiate $\mathscr{E}$ over $\mathcal{J}$; this yields $\omega_\parallel$, and, correspondingly, differentiation of $\mathscr{E}$ over P yields $\omega_\perp$. The procedure can be carried out explicitly for small oscillations, when

$$V(\phi,P)\approx V(\phi_s,P)+\frac{1}{2}\frac{\partial^2 V}{\partial\phi^2}(\phi-\phi_S)^2 \tag{14}$$

In this case

$$\mathscr{E}=\mathcal{J}\left(\frac{\partial^2 V}{\partial\phi^2}\right)^{1/2}+V(\phi_S,P)-P \tag{15}$$

and correspondingly,

$$\omega_\parallel=(\partial^2 V/\partial\phi^2)^{1/2} \tag{16}$$

$$\omega_\perp=-1+(\partial V/\partial P)_\phi+\mathcal{J}\,d\omega_\parallel/\partial P \tag{17}$$

A comparison of (17) with (12) shows that because of vibrations there arises a correction to the precession frequency which is proportional to $\mathcal{J}$. This correction, like the second term of (12), is of the order of Ω^2. It also is small due to the smallness of the vibrations. For this correction to be measurable, vibration should exist at least for times larger than $1/\Omega^2$. According to estimates,[10] the longitudinal vibrations decay over a time of the order of $(1/\Omega^2)\omega\tau$. So one must either use the region where $\omega\tau\ll 1$, or support the vibrations by an rf field.

3. APPLICATION TO A AND B PHASES

The motion of spins in a strong magnetic field thus may be described once the averaged dipole energy is known. This energy for each particular case is found directly with the help of (2) and (3) and by the averaging procedure. The form of the averaged energy depends both on the structure of the phase and on the initial conditions; more exactly, it depends on how the axis around which it rotates is oriented relative to the "top." As an example, we present here the formulas for V, $\omega_\parallel$, and $\omega_\perp$ in the A and B phases for the case when at $t=0$ the system is oriented by the external field only and is in equilibrium. This case for $\mathscr{I}=0$ has been already considered[3,5,6] in connection with interpretation of pulse NMR experiments in superfluid ^{3}He.[7,8] In the A phase

$$V_A = -\frac{\Omega_A^2}{8}\left[\left(1+\frac{P}{S}\right)^2+\frac{1}{2}(\cos 2\phi)\left(2+\frac{P}{S}\right)^2\right] \tag{18}$$

and for the stable stationary solution $\phi=0$, $S=1$ in accordance with (16) and (17)

$$\omega_{A\parallel}=\tfrac{1}{2}\Omega_A(P+2) \tag{19}$$

$$\omega_{A\perp}+1=-\tfrac{1}{8}\Omega_A^2(4+3P)+\mathscr{I}\Omega_A/2 \tag{20}$$

In the B phase

$$V_B=\frac{2\Omega_B^2}{15}\left[\frac{1}{2}+\frac{P}{S}+\left(2+\frac{P}{S}\right)\cos\phi\right]^2 \tag{21}$$

$\phi_S=0$ will be a stable stationary point at $-2\leqslant P\leqslant -\frac{5}{4}$ with characteristic frequencies:

$$\omega_{B\parallel}=\Omega_B\left[\frac{8}{15}(P+2)\left|P+\frac{5}{4}\right|\right]^{1/2} \tag{22}$$

$$\omega_{B\perp}+1=\frac{16}{15}\Omega_B^2\left(P+\frac{5}{4}\right)-\mathscr{I}\Omega_B\left(P+\frac{13}{8}\right)\left[\frac{8}{15(P+2)|P+\frac{5}{4}|}\right]^{1/2} \tag{23}$$

and for $-\frac{5}{4}\leqslant P\leqslant 0$, accordingly, $\cos\phi_S=-(P+\frac{1}{2})/(P+2)$ with frequencies

$$\omega_{B\parallel}=\Omega_B(1+\tfrac{4}{5}P)^{1/2} \tag{24}$$

$$\omega_{B\perp}+1=2\mathscr{I}\Omega_B/[5(1+4P/5)^{1/2}] \tag{25}$$

For $\mathscr{I}=0$ these formulas coincide with the corresponding formulas of Ref. 3 taking into account that the definition of P here differs in sign from that employed there. In some formulas of Ref. 3 the terms $\sim\Omega^4$ are retained, which exceeds the accuracy.

The case of more complicated initial conditions is considered in Appendix B.

4. SINGULAR POINTS

As may be seen from the previous section, the frequency of small longitudinal oscillations at some values of P becomes zero. In the neighborhood of such points the dependence of $\omega_\parallel$ and $\omega_\perp$ on $\mathcal{J}$ and P cannot be described any more by formulas (19), (20), (22), and (23) and further investigation is required for determining this dependence. Here we consider only cases which are realized in A and B phases under the same initial conditions as in Section 3.

(a) In the A phase there is only one singular point $P=-2$. At this point the dependence of the potential V_A on ϕ vanishes. Since in this case all derivatives of V_A with respect to ϕ become zero, we are not able to use the expansion of V_A in powers of $\phi-\phi_S$. Fortunately, for the potential V_A determined by formula (18) the problem of determining $\omega_\parallel$ and $\omega_\perp$ may be solved at any values of $\mathcal{J}$ and P. The action is defined by an integral of the type

$$\mathcal{J}=(1/4\pi)\oint(a+b\cos\phi)^{1/2}\,d\phi \tag{26}$$

which is expressed through the complete elliptic integrals, this expression is different for $a>b$ and $a<b$. The difference is caused by a change in the character of the motion; this can be understood most easily when using the phase trajectories, i.e., the trajectories described by a point in coordinates ϕ, $S-1$ with motion of the system (see Fig. 1).

In the case under consideration $a=2(\mathscr{E}+P)+\frac{1}{4}\Omega_A^2(P+1)^2$ and $b=\frac{1}{8}\Omega_A^2(P+2)^2$. At $a=b$ the phase trajectory is a separatrix (solid line in Fig. 1). For $a>b$ the phase trajectories (dot-dash lines) pass outside the separatrix, and ϕ runs continuously through all the values from $-\pi$ to $+\pi$, and the motion is of a rotational type. In this case

$$\mathcal{J}=\mathcal{J}_A E(r)/r \tag{27}$$

where $r=[2b/(a+b)]^{1/2}$ and $\mathcal{J}_A=(\Omega_A/2\pi)(P+2)$. Here, $E(r)$ is the complete elliptic integral of the second kind. The above formula determines implicitly the dependence of energy $\mathscr{E}$ on $\mathcal{J}$ and P. Differentiating (27) with respect to $\mathcal{J}$ and P, taking account of the definitions of a and b, we get

$$\omega_{\parallel A}^{>}=\frac{\pi\Omega_A}{2}\,\frac{P+2}{rK(r)} \tag{28}$$

$$\omega_{\perp A}^{>}+1=-\frac{\Omega_A^2}{8}\left[(3P+4)+\frac{2(P+2)}{r^2}\left(\frac{E(r)}{K(r)}-1\right)\right] \tag{29}$$

The system of equations (27)–(29) determines parametrically the dependence of $\omega_{\parallel A}^{>}$ and $\omega_{\perp A}^{>}$ on $\mathcal{J}$ and P for $\mathcal{J}>\mathcal{J}_A$. The straight line $\mathcal{J}=\mathcal{J}_A$ is

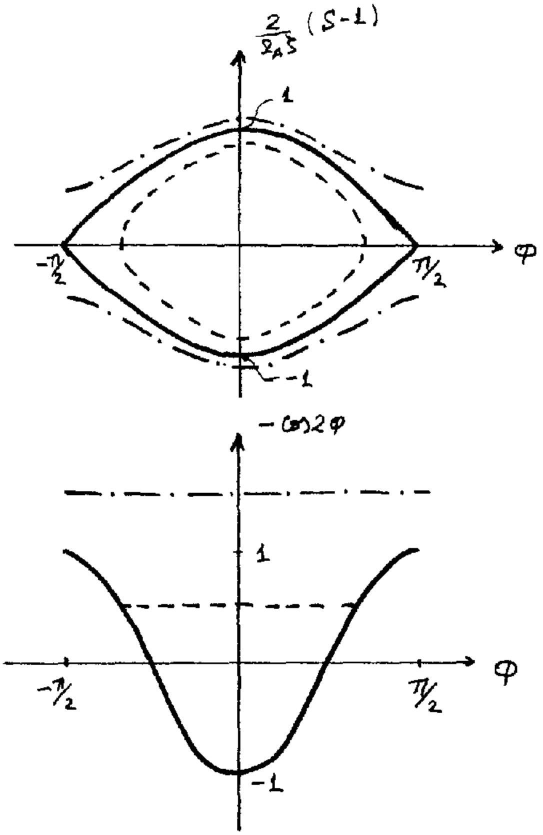

Fig. 1. Potential energy (bottom) and phase trajectories (top) in the vicinity of singular point $P=-2$.

the line of singularities $\omega_{\|A}$ and $\omega_{\perp A}$ in the plane $\mathcal{J}$, P. In order to elucidate the character of singularities one should expand the equations (27)–(29) in the vicinity of the singular line in the small value $z=(1-r^2)/4$ (the coefficient $\frac{1}{4}$ is chosen for convenience). The principal terms of the expansion are of the form

$$\mathcal{J}/\mathcal{J}_A - 1 = z \ln 4e/z \tag{30}$$

$$\omega_{\|A}^{>} = \frac{\pi\Omega_A(P+2)}{\ln(4/z)} \tag{31}$$

$$\omega_{\perp A}^{>} + 1 = -\frac{\Omega_A^2}{8}\left[P + \frac{4(P+2)}{\ln(4/z)}\right] \tag{32}$$

From the above it is clear that for $\mathcal{J} \to \mathcal{J}_A$, $\omega_{\|A}^{>}$ and the term in $\omega_{\perp A}^{>}$ proportional to $\mathcal{J}$ tend zero as $1/\ln(\mathcal{J}/\mathcal{J}_A - 1)$. When the motion takes place far from the separatrix ($\mathcal{J} \gg \mathcal{J}_A$, r $\ll 1$)

$$\omega_{\|A}^{>} \approx 4\mathcal{J} \tag{33}$$

$$\omega_{\perp A}^{>} + 1 \approx -\tfrac{1}{4}\Omega_A^2(P+1) \tag{34}$$

For $a < b$, ϕ oscillates in the range less than π; the phase trajectories are closed and lie inside the separatrix (dashed line in Fig. 1). In this case instead of system (27)–(29) we have ($u = 1/r$)

$$\mathcal{J}/2\mathcal{J}_A = (u^2 - 1)K(u) + E(u) \tag{35}$$

$$\omega_{\|A}^{<} = \tfrac{1}{4}\pi\Omega_A(P+2)/K(u) \tag{36}$$

$$\omega_{\perp A}^{>} + 1 = -\tfrac{1}{8}\Omega_A^2[P + 2(P+2)E(u)/K(u)] \tag{37}$$

This system defines $\omega_\|$ and $\omega_\perp$ at $\mathcal{J}$ in the range $0 \leq \mathcal{J} \leq 2\mathcal{J}_A$. It should be noted that since this range overlaps with the region described by formulas (27)–(29), two values of $\omega_\|$ and $\omega_\perp$ correspond to each value of $\mathcal{J}$ in the range $\mathcal{J}_A < \mathcal{J} < 2\mathcal{J}_A$ at given P. For $\mathcal{J} \to 0$ the system (35)–(37) leads to formulas coinciding with formulas (19) and (20) for small vibrations. In the vicinity of the singularity $\mathcal{J} = 2\mathcal{J}_A$ the expansion similar to (30)–(32) has the form

$$1 - \mathcal{J}/2\mathcal{J}_A = z \ln 4e/z \tag{38}$$

$$\omega_{\|A}^{<} = \frac{\pi\Omega_A(P+2)}{2\ln(4/z)} \tag{39}$$

$$\omega_{\perp A}^{<} + 1 = -\frac{\Omega_A^2}{8}\left[P + \frac{4(P+2)}{\ln(4/z)}\right] \tag{40}$$

The main terms of the expansion in the small value z are retained. Without violating the accuracy of the above formulas, one may add terms of higher order with coefficients chosen so that for $\mathcal{J} \to 0$ correct values of the frequencies $\omega_{\|A}^{<}$ and $\omega_{\perp A}^{<}$ are obtained. Thus interpolation formulas describing relevant frequencies in the whole range $0 \leq \mathcal{J} \leq 2\mathcal{J}_A$ can be obtained. Good formulas are obtained, provided that in (38), $\ln(4e/z)$ is substituted by $\ln[4e(1+z)/z]$, and, accordingly, in (39) and (40), $\ln(4/z)$ by $(d/dz)\{z \ln[4e(1+z)/z]\}$.

(b) In the B phase at $\mathcal{J} = 0$ there are two singular points, $P = -2$ and $P = -\frac{5}{4}$. The point $P = -2$ has the same character as in the A phase. Expanding V_B in the vicinity of this point in $\zeta \equiv P + 2$, we get

$$V_B = -\tfrac{2}{5}\Omega_B^2\zeta(1 + \cos\phi) \tag{41}$$

For this potential the action has, to an accuracy of the coefficient and with the definitions of a and b, the same form as in A phase:

$$\mathcal{J}=(1/2\pi)\oint(a'+b'\cos\phi)^{1/2}\,d\phi, \tag{42}$$

where $a'=2\mathscr{E}+2\zeta(1+\frac{2}{5}\Omega_B^2)$ and $b'=\frac{4}{5}\Omega_B^2\zeta$. Proceeding in the same way as before, we obtain for $a'>b'$

$$\mathcal{J}=\mathcal{J}_B E(r)/r \tag{43}$$

Now $r=[2b'/(a'+b')]^{1/2}$ and $\mathcal{J}_B=(4\Omega_B/\pi)(\frac{2}{5}\zeta)^{1/2}$. Correspondingly,

$$\omega_{\|B}^{>}=\frac{\pi\Omega_B}{rK(r)}\left(\frac{2}{5}\zeta\right)^{1/2} \tag{44}$$

$$\omega_{\perp B}^{>}+1=\frac{4\Omega_B^2}{5r^2}\left[(1-r^2)-\frac{E(r)}{K(r)}\right] \tag{45}$$

For $a'<b'$ and $u=1/r$

$$\mathcal{J}=2\mathcal{J}_B[(u^2-1)K(u)+E(u)] \tag{46}$$

$$\omega_{\|B}^{<}=\frac{\pi\Omega_B}{2K(u)}\left(\frac{2}{5}\zeta\right)^{1/2} \tag{47}$$

$$\omega_{\perp B}^{<}+1=-\frac{4\Omega_B^2}{5}\frac{E(u)}{K(u)} \tag{48}$$

The difference in the dependences of $\omega_{\|B}$ and $\omega_{\perp B}$ on ζ as compared to those in the A phase is connected with the different dependences of $\mathcal{J}_B$ and $\mathcal{J}_A$ on ζ. While $\mathcal{J}_B\sim\sqrt{\zeta}$, $\mathcal{J}_A\sim\zeta$. The character of the singularity near the separatrix is the same as in the A phase and we shall not present the corresponding formulas here. For comparison with the nonresonance case it would be useful to have a formula for $\mathcal{J}\gg\mathcal{J}_B$:

$$\omega_{\|B}^{>}\approx\mathcal{J} \tag{49}$$

$$\omega_{\perp B}^{>}+1\approx-2\Omega_B^2/5 \tag{50}$$

(c) At the point $P=-\frac{5}{4}$ the coefficient of expansion of the potential V_B in small deviations from the stationary value $\phi=0$ changes sign. For $P<-\frac{5}{4}$, $\phi=0$ is the minimum of V_B; for $P>-\frac{5}{4}$, $\phi=0$ becomes a maximum, and in the vicinity of this value two minima arise. For an analytical study of this transition V_B should be expanded in ϕ to terms following the quadratic ones near $P=-\frac{5}{4}$. Having retained the terms that are principal with respect to ϕ and $\kappa\equiv P+\frac{5}{4}$, we get

$$V_B(\phi,\kappa)=\Lambda[V_0(\kappa)-\alpha\kappa\phi^2+\phi^4] \tag{51}$$

where the value $\Lambda = 3\Omega_B^2/160$ is chosen so that the coefficient of ϕ^4 is equal to unity, $V_0(\kappa) = (16\kappa/3)^2$, $\alpha = 32/3$. In the case $\mathcal{J} = 0$ the derivatives of $\omega_\parallel$ and $\omega_\perp$ with respect to P have a singularity at $\kappa = 0$, analogously to the second-order phase transition in the Landau theory. This case has been considered in detail in Ref. 3.

If $\mathcal{J} \neq 0$, then substituting (51) into (13), we get

$$\mathcal{J} = (1/\pi)(\Lambda/2)^{1/2} \oint [(\phi_1^2 - \phi^2)(\phi^2 - \phi_2^2)]^{1/2}\, d\phi \tag{52}$$

where ϕ_1^2 and ϕ_2^2 are the roots of the equation

$$(\phi^2)^2 - \alpha\kappa\phi^2 - \xi + V_0(\kappa) = 0 \tag{53}$$

with $\xi = (\mathscr{E} + P)/\Lambda$. The singularity in the dependence of $\omega_\parallel$ and $\omega_\perp$ on $\mathcal{J}$ and P arises, as in the two previous cases, when the phase trajectory coincides with the separatrix.

The situation $\phi_2^2 = 0$, $\phi_1^2 = \alpha\kappa$ corresponds to motion along the separatrix, as may be seen from Fig. 2. In this case the integral in (52) can be easily evaluated and yields the equation of the line in the $\mathcal{J}, \kappa$ plane where the frequencies $\omega_\parallel$ and $\omega_\perp$ have a singularity:

$$\mathcal{J} = \frac{(2\Lambda)^{1/2}}{3\pi} (\alpha\kappa)^{3/2} \equiv \mathcal{J}_0(\kappa) \tag{54}$$

The value of $\mathcal{J}_0(\kappa)$ correponds to the area of one loop of the figure-eight in Fig. 2. The singularity on the line (54) arises when approaching the separatrix from inside, i.e., from the value $\xi < V_0(\kappa)$. On approaching the separatrix from the region $\xi > V_0(\kappa)$, i.e., from the phase trajectories involving both loops of the figure-eight, the singularity arises on the line $\mathcal{J} = 2\mathcal{J}_0(\kappa)$. No possible motions correspond to values of $\mathcal{J}$ lying within the range $\mathcal{J}_0(\kappa) < \mathcal{J} < 2\mathcal{J}_0(\kappa)$ at fixed κ. A region of forbidden values of $\mathcal{J}$ thus arises for $P > -\frac{5}{4}$, similar to the separation region at the first order phase transition. The analytical dependence of $\omega_\parallel$ and $\omega_\perp$ on $\mathcal{J}$ and κ in the neighborhood of the point $P = -\frac{5}{4}$ can be investigated, as has been done above for the point $P = -2$. The integral in formula (52) is expressed through complete elliptic mtegrals K and E. For $\phi_2^2 > 0$ when the phase trajectories pass inside one of the loops of the separatrix

$$\mathcal{J} = \frac{\mathcal{J}_0}{(2 - q^2)^{1/2}} \left[E(q) - 2\frac{1 - q^2}{2 - q^2} K(q) \right] \tag{55}$$

where $q^2 = (\phi_1^2 - \phi_2^2)/\phi_1^2$. Differentiating (55) with respect to $\mathcal{J}$ and κ,

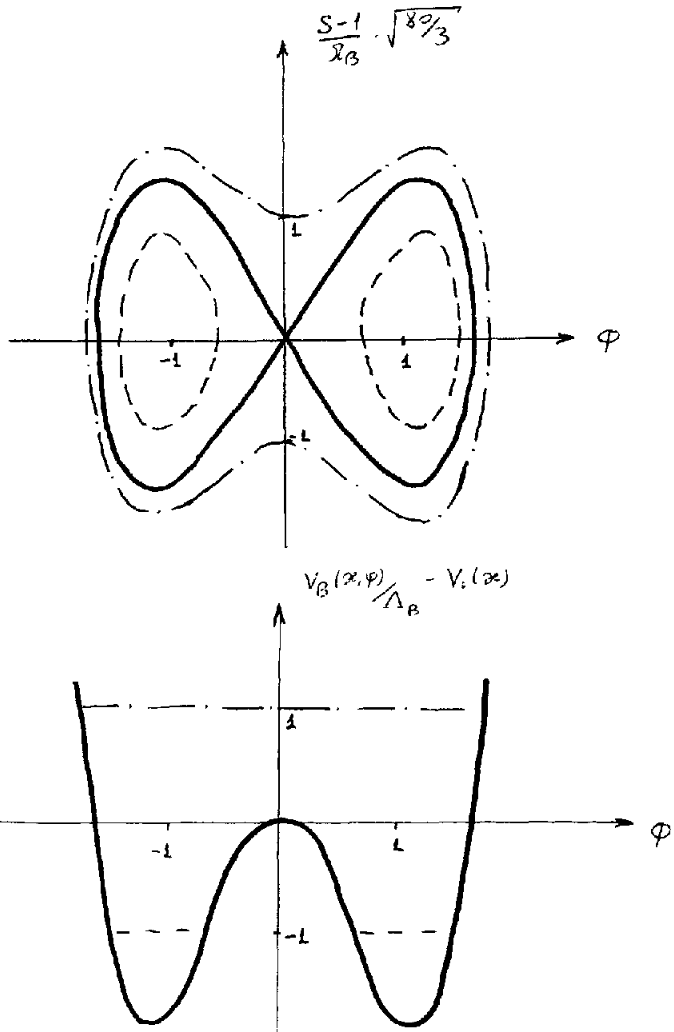

Fig. 2. Potential energy and phase trajectories in the vicinity of singular point $P = -\frac{5}{4}$.

allowing for the fact that $\phi_1^2+\phi_2^2=\alpha\kappa$ and $\phi_1^2\phi_2^2=V_0(\kappa)-\xi$, we get

$$\omega_{\parallel}^{<}=\frac{\pi}{K(q)}\left(\frac{2\Lambda\alpha\kappa}{2-q^2}\right)^{1/2} \tag{56}$$

$$\omega_{\perp}^{<}+1=\Lambda\left[\frac{dV_0}{d\kappa}-\frac{\alpha^2\kappa}{2-q^2}\frac{E(q)}{K(q)}\right] \tag{57}$$

For $q \to 0$ formulas (56) and (57) produce the same values of $\omega_\parallel$ and $\omega_\perp$ as do (24) and (25), and when approaching the separatrix $y \equiv q'^2 = 1 - q^2 \to 0$:

$$1 - \mathcal{J}/\mathcal{J}_0 = \frac{3y}{4} \ln \frac{16e}{y} \tag{58}$$

$$\omega_\parallel^< = 2\pi \frac{(2\Lambda\alpha\kappa)^{1/2}}{\ln(16/y)} \tag{59}$$

$$\omega_\perp^< + 1 = \Lambda\left[\frac{dV_0}{d\kappa} - \frac{2\alpha^2\kappa}{\ln(16/y)}\right] \tag{60}$$

For $\phi_2^2 < 0$

$$\mathcal{J} = \frac{2\mathcal{J}_0}{(2r^2-1)^{1/2}}\left[\frac{1-r^2}{2r^2-1}K(r) + E(r)\right] \tag{61}$$

where $r^2 = \phi_1^2/(\phi_1^2 - \phi_2^2)$, and

$$\omega_\parallel^> = \frac{\pi}{K(r)}\left[\frac{\Lambda\alpha\kappa}{2(2r^2-1)}\right]^{1/2} \tag{62}$$

$$\omega_\perp^> + 1 = \Lambda\left[\frac{dV_0}{d\kappa} - \alpha^2\kappa\frac{(r^2-1)K(r) + E(r)}{(2r^2-1)K(r)}\right] \tag{63}$$

On approaching the separatrix $r^2 \to 1$ these formulas transform into $(y \equiv 1 - r^2)$

$$\frac{\mathcal{J}}{2\mathcal{J}_0} - 1 = \frac{3}{4} y \ln \frac{16e}{y} \tag{64}$$

$$\omega_\parallel^> = \frac{\pi(2\Lambda\alpha\kappa)^{1/2}}{\ln(16/y)} \tag{65}$$

$$\omega_\perp^> + 1 = \Lambda\left[\frac{dV_0}{d\kappa} - \frac{2\alpha^2\kappa}{\ln(16/y)}\right] \tag{66}$$

As follows from formulas (58)–(60) and (64)–(66), on approaching the separatrix from both inside and outside, $\omega_\parallel$ and the singular part of $\omega_\perp$ become zero logarithmically. In the limit of high energies $(\xi - V_0)/(\alpha\kappa)^2 \gg 1$, we get from the system (61)–(63)

$$\omega_\parallel^> = \left[\frac{\pi}{K(1/\sqrt{2})}\right]^{4/3}\left(\frac{3\mathcal{J}}{4}\right)^{1/3} \tag{67}$$

$$\omega_\perp^> + 1 = \Lambda\left[\frac{dV_0}{d\kappa} - \alpha\left(\frac{E(1/\sqrt{2})}{K(1/\sqrt{2})} - \frac{1}{2}\right)\left(\frac{3\pi\mathcal{J}}{\sqrt{2}K(1/\sqrt{2})}\right)^{2/3}\right] \tag{68}$$

In particular, $\omega_\parallel$ and $\omega_\perp$ will have such values at $P = -\frac{5}{4}$.

The existence of singular points considered above also helps in understanding qualitatively the reason for the formation of the domain walls in the pulsed NMR experiments.[12,13] In these experiments the tipping of magnetization at the angle larger than some critical value leads, after relaxation, to formation of the domain walls. This method of domain-wall formation can be understood by considering the longitudinal vibrations excited when the tipping ac field is turned on and off.

For the sake of clarity let us consider the A phase. The potential $V_A(\phi, P)$ has two minima in the interval $(-\pi, \pi)$. The vibration of ϕ can take place at either minimum. Since the tipping is usually slow, the adiabatic invariant $\mathcal{J}$, corresponding to vibrations, is conserving, unless it is near the critical value $\mathcal{J}_A(P)$ [cf.(27)]. $\mathcal{J}_A$ decreases as the tipping angle increases, so at some angle $\mathcal{J}_A(P) = \mathcal{J}$. At angles larger than this value the angle ϕ can no longer be localized at one minimum, but grows monotonously. After relaxation takes place, magnetization can relax, due to spatial inhomogeneities, to different minima of $V_A(\phi, P)$ in different spatial regions, thus forming the domain walls. So, the critical tipping angle is defined by the intensity of the longitudinal vibrations excited at the turning on and off of the ac field. All of the above considerations are valid for the B phase as well, when the tipping angle is less than 104°.

It should be noted that the relaxation time for the longitudinal oscillations is of the same order as the time of tipping of magnetization. Thus, the consideration above gives only the lower boundary for the critical angle. To describe quantitatively the formation of the domain walls, one should also consider the oscillations of magnetization with spatial dependence, which has not yet been done.

5. NONRESONANCE CASE

Let us now consider the case of $(S-1) \sim 1$. To realize these conditions, we could, for instance, change the dc $\mathbf{H}$ abruptly by a factor of the order of $|\mathbf{H}|$. The relaxation of the magnetization to the equilibrium value takes place[4,11] for times of the order of $1/\Omega^4$. During this period the system will be far from resonance. As may be seen from (8), the variable ϕ will also be "fast," and and the Hamiltonian (7) should be averaged over it. Designating $\bar{V}(\phi, P/S) \equiv W(P/S)$, we get

$$\bar{\mathcal{H}} = \tfrac{1}{2}(S-1)^2 - P + W(P/S) \tag{69}$$

and the equations of motion

$$\begin{aligned} \dot{S} &= 0 \qquad \dot{\phi} = S - 1 \\ \dot{P} &= 0 \qquad \dot{\alpha} = -1 + \partial W/\partial P \end{aligned} \tag{70}$$

The longitudinal oscillations of the magnetization are now beyond the limits of accuracy of the approximation and only the precession frequency shift can be determined.

Averaging (18), we obtain

$$W_{\mathrm{A}} = -\tfrac{1}{8}\Omega_{\mathrm{A}}^2[1+(P/S)]^2 \tag{71}$$

$$\omega_{\perp\mathrm{A}}+1 = -(\Omega_{\mathrm{A}}^2/4S)\cos\beta \tag{72}$$

and, accordingly, averaging of (21) yields

$$W_{\mathrm{B}} = \tfrac{2}{5}\Omega_{\mathrm{B}}^2[(P/S)+\tfrac{1}{2}(P/S)^2] \tag{73}$$

$$\omega_{\perp\mathrm{B}}+1 = \tfrac{2}{5}(\Omega_{\mathrm{B}}^2/S)\cos\beta \tag{74}$$

As it should be, (72) coincides with (34), and (74) with (50) for $(P+2) \ll 1$.

APPENDIX A

Let us prove the statement used in Section 2 that if the angular velocity of a rigid body rotation $\boldsymbol{\omega}$ changes slowly, i.e., $\lambda \equiv (1/\omega^2)|d\boldsymbol{\omega}/dt| \ll 1$, then the average for the rotation period of the orientation of $\boldsymbol{\omega}$ relative to the body is the adiabatic invariant. Suppose $\boldsymbol{\nu}$ is the unit vector in the direction of $\boldsymbol{\omega}$, ν_k is its projection upon the fixed basis, ν_i' is the projection upon the basis connected with the body, and R_{ik} is the matrix of transformation from the rotating coordinate system to the fixed one; then

$$\nu_i' = R_{ik}\nu_k \tag{A1}$$

and

$$\frac{d\nu_i'}{dt} = \frac{dR_{ik}}{dt}\nu_k + R_{ik}\frac{d\nu_k}{dt} \tag{A2}$$

It can be easily seen that $(dR_{ik}/dt)\nu_k = -\boldsymbol{\omega}\times\boldsymbol{\nu} = 0$. Then (A2) is averaged over the period of the rigid body rotation. Since $\dot{\boldsymbol{\nu}} \perp \boldsymbol{\nu}$, then the average of the second term on the right-hand side of Eq. (A2) is of higher than first order in λ. Usually it is λ^2, but in some cases, for instance, in describing longitudinal oscillations in the resonance case (see Section 2), the correction is of the order of $\lambda^{3/2}$. Thus $d\nu_i'/dt \ll \lambda$. Q.E.D.

APPENDIX B

As another example of applying the formula derived in Section 2, we consider the case when in the B phase the vector $\mathbf{n}$ at $t=0$ is not parallel to the external magnetic field. Experimentally, this can be achieved by placing helium-B into the space between plane-parallel plates and choosing the distance between the plates and the field intensity such that **n** is everywhere homogeneously oriented.[10] It turns out that there is a region of fields and distances between plates when the surface energy, though it orientates **n**, is still small compared to the dipole energy and may be neglected in calculating the frequencies characterizing the motion of the spin. In this case

$$V = (2/15)\Omega_B^2[u(1+u)+D] \tag{B1}$$

where

$$u = \tfrac{1}{2}(2-\tfrac{5}{4}\sin^2\chi)(1+\cos\beta)\cos\phi + (1-\tfrac{5}{4}\sin^2\chi)\cos\beta - 1 \tag{B2}$$

$$D = \tfrac{5}{4}(\sin^2\chi)(1-\cos\beta)[2(2+u)-(35/32)(\sin^2\chi)(1-\cos\beta)] \tag{B3}$$

χ is the angle between **H** and **n** at $t=0$, and β and ϕ have the same meaning as in the text [see formulas (3) and (6)]. As has been already noted, in the resonance case we may put $S=1$ in the argument and instead of the variable P use directly $\cos\beta$. The stationary values of ϕ according to (10) and (B1) are found from the equation

$$\left(1+2u+\frac{\partial D}{\partial u}\right)\frac{\partial u}{\partial \phi} = 0 \tag{B4}$$

When the term in parentheses is set equal to zero

$$\cos\phi = \frac{1-2\cos\beta}{1+\cos\beta}\,\frac{2(2-5\sin^2\chi)}{8-5\sin^2\chi} \tag{B5}$$

This solution exists for $|\cos\phi| \leq 1$. The lines in the plane $\sin^2\chi$, $\cos\beta$ on which $\cos\phi = 1$

$$1+\tfrac{5}{4}\sin^2\chi + [4-(25/4)\sin^2\chi]\cos\beta = 0 \tag{B6}$$

and $\cos\phi = -1$

$$1-\tfrac{5}{4}\sin^2\chi + \tfrac{5}{4}\sin^2\chi\cos\beta = 0 \tag{B7}$$

divide the region of variation of the variables $|\cos\beta| \leq 1$ and $\sin^2\chi \leq 1$ into three parts. The solution (B5) exists and is stable in region 1 in Fig. 3. In region 2 the solution $\cos\phi = 1$ is stable, and in region 3, $\cos\phi = -1$. Both these values correspond to $\partial u/\partial\phi$ becoming zero. In region 1 using (B1) and (B5), we get

$$\omega_{\parallel(1)}^2 = \tfrac{1}{5}\Omega_B^2[1-\tfrac{5}{4}(\sin^2\chi)(1-\cos\beta)] \times [1+4\cos\beta+\tfrac{5}{4}\sin^2\chi-(25/4)\sin^2\chi\cos\beta] \tag{B8}$$

$$\omega_{\perp(1)}+1 = -\tfrac{1}{2}\Omega_B^2(\sin^2\chi)[1-(\tfrac{5}{4}\sin\chi)^2(1-\cos\beta)] \tag{B9}$$

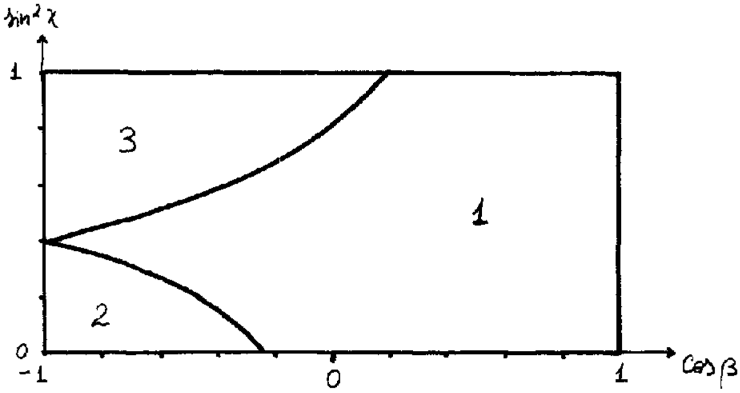

Fig. 3. Regions of different analytical dependences of $\omega_{\|}$ and $\omega_{\perp}$ for the B phase in the plane-parallel geometry.

In region 2,

$$\omega^2_{\|(2)} = -(\Omega_B^2/15)(2-\tfrac{5}{4}\sin^2\chi)(1+\cos\beta)$$
$$\times[1+4\cos\beta+\tfrac{5}{4}\sin^2\chi-(25/4)\sin^2\chi\cos\beta] \qquad \text{(B10)}$$

$$\omega_{\perp(2)}+1=(4\Omega_B^2/15)[(1+4\cos\beta)-\tfrac{5}{16}(\sin^2\chi)(7+40\cos\beta)$$
$$+\tfrac{5}{4}(\tfrac{5}{4}\sin^2\chi)^2(1+7\cos\beta)] \qquad \text{(B11)}$$

and in region 3

$$\omega^2_{\|(3)}=\tfrac{1}{5}\Omega_B^2(2-\tfrac{5}{4}\sin^2\chi)(1+\cos\beta)[\tfrac{5}{4}(\sin^2\chi)(1-\cos\beta)-1] \qquad \text{(B12)}$$

$$\omega_{\perp(3)}+1=\tfrac{1}{4}\Omega_B^2(\sin^2\chi)[1-\tfrac{5}{8}(\sin^2\chi)(1-\cos\beta)] \qquad \text{(B13)}$$

At the boundaries of the regions determined by Eqs. (B6) and (B7) the corresponding frequencies of longitudinal oscillations $\omega_{\|}$ become zero, and the precession frequency shifts $\omega_{\perp}+1$ for the neighboring regions coincide.

The boundaries between the regions and the straight line $\cos\beta=-1$ are singular lines in the sense of Section 4. The singularities arising on these lines can be studied in the same way as was done in Section 4. However, this has not been carried out because of the cumbersome nature of the calculations.

For $\chi=0$ the formulas obtained transform to formulas (22)–(25) with $\mathcal{J}=0$, and for $\cos\beta=1$ into the formulas for the already investigated case of continuous NMR in the plane-parallel geometry.

REFERENCES

1. A. J. Leggett, *Ann. Phys. (N.Y.)* **85**, 11 (1974).
2. A. J. Leggett, *Rev. Mod. Phys.* **47**, 331 (1975).
3. I. A. Fomin, *Zh. Eksp. Teor. Fiz.* **71**, 791 (1976) [*Sov. Phys.—JETP* **44**, 416 (1976)].
4. A. J. Leggett and S. Takagi, *Phys. Rev. Lett.* **34**, 1424 (1975).
5. W. F. Brinkman and H. Smith, *Phys. Lett.* **51A**, 449 (1975).
6. W. F. Brinkman and H. Smith, *Phys. Lett.* **53A**, 43 (1975).
7. D. D. Osheroff and L. R. Corrucini, in *Proc. of 14th Int. Conf. Low Temp. Phys.* (North-Holland, Amsterdam, 1975), Vol. 1, p. 100.
8. D. D. Osheroff and L. R. Corruccini, *Phys. Lett.* **51A**, 447 (1975).
9. L. D. Landau and E. M. Lifshitz, *Mechanics* (Nauka, Moscow, 1973) [English transl., Pergamon Press (1960)].
10. W. F. Brinkman, H. Smith, D. D. Osheroff, and E. I. Blount, *Phys. Rev. Lett.* **33**, 624 (1974).
11. A. J. Leggett and S. Takagi, preprint (1976).
12. J. Kokko, M. A. Paalanen, R. C. Richardson, and Y. Takano, preprint (1977).
13. R. W. Giannetta, E. N. Smith, and D. M. Lee, *Phys. Lett.* **62A**, 335 (1977).