# DISCUSSION: 'Some Observations on Sih's Strain Energy Density Approach for Fracture Prediction', by I. Finnie and H. O. Weiss.*

*G. C. Sih*
*Institute of Fracture and Solid Mechanics, Lehigh University*
*Bethlehem, Pennsylvania 18015 USA*
*tel: 215/691-7000 ext. 376*

The authors performed two fracture experiments on 5 in × 9 in × 0.004 in beryllium sheets. Each of the specimens contained in a 2 in long crack, with a slit opening of approximately 0.005 in, which is inclined at an angle $\beta$=45deg with the applied tension. The purpose of their work was to compare the fracture angles $\theta_o$ at crack initiation with predictions based on

(1) the maximum tensile stress (or $\sigma_\theta$) criterion [1], and

(2) the strain energy density (or S) criterion [2,3].

Beryllium was chosen for its low value of Poisson's ration, $\nu \simeq 0$. They reported difficulties in measuring $\theta_o$ since material separation starts from a rounded notch rather than a perfectly sharp crack as assumed in the mathematical analyses [1-3]. Needless to say, the value $\theta_o$ also depends on the distance $r_o$ from the crack tip at which it is measured. Previous work on a notch-sensitive strain energy density theory [4,5] has shown that a small change in $r_o$ could lead to large variations of $\theta_o$, depending on the loading angle $\beta$.

As indicated, the strain energy density criterion [2,3] includes a Poisson ratio $\nu$ dependence; even in the range of $\nu$ = 0.25 to 0.3, the fracture angle $\theta_o$ predicted by the S criterion may differ substantially from that of maximum stress [1]. For example, in the case of pure shear, if $\nu$ = 0.3, the S criterion gives $\theta_o$ = -82.3deg, while the $\sigma_\theta$ criterion yields $\theta_o$ = -70.5deg. The case at hand has the angle of inclination fixed at $\beta$ = 45deg, for which the $\sigma_\theta$ criterion predicts the angle $\theta_o$ = -53deg, regardless of the value of $\nu$. Experimental data [5,6] on polymethylmethacrylate (PMMA) plates with $\nu \simeq 0.3$ showed that the average fracture angle $\theta_o$ is between -53deg, to -54deg for $\beta$ = 45deg. The average value of $\theta_o$ = -50.67deg, reported by the authors for beryllium with $\nu \simeq 0$ and $\beta$ = 45deg, seems to imply that material with a lower Poisson's ratio attains a smaller fracture angle. This trend of $\theta_o$ decreasing with $\nu$ is predicted by the S criterion [2,3]; since fracture angles at crack initiation are known to vary for different materials, Poisson's ratio could be an influencing factor not accounted for in [1].

Now suppose that the crack in the beryllium specimen is modeled by a very narrow elliptical notch having a ratio of b/a=0.001, where a and b are the semi-major and semi-minor axes of the ellipse. In addition, if a small Poisson's ratio of $\nu \simeq 0.01$ is assumed, the strain energy density theory [5] predicts a fracture angle of $\theta_o$=-40.20deg at

---

**International Journal of Fracture* 10 *(1974)* a distance of $r_o = 0.06a$, measured from the crack front. The ideal case of $b/a = 0^o$ and $\nu = 0$, used by the authors, gives $\theta_o = -34.5$deg. Hence, the value of $\theta_o$ can be affected by small perturbations of the parameters $b/a$, $\nu$, and $r_o/a$.

It is apparent that the crack configuration in the specimen must be carefully prepared to conform with the shape assumed in the mathematical analysis before an accurate account of theory and experiment can be made, and that a clear distinction between what is calculated and what can be measured must be made before claiming agreement or disagreement of theory with experiment.

A more serious problem appears in the very nature of metal fracture, in which the granular structure of the material prevents an accurate assessment of the angle $\theta_o$. In addition, fracture of metallic plate specimens is basically a three-dimensional phenomenon and cannot be adequately explained by a two-dimensional analysis which ignores the thickness effect. Too often, agreement between theory and experiment is purely coincidental. A classical example is the plasticity strip model for predicating the yield zone size in thin metal sheets or plates. The so-called observed 'plasticity' ahead of the crack is due to necking of the material in the thickness direction while the analysis considers no such effect and is strictly a planar one. A refined two-dimensional elastic-plastic stress analysis will reveal that the material along the line of expected crack extension will not distort sufficiently to cause yielding. Therefore, any fracture criterion which excludes thickness deformation is not expected to predict the fracture behavior of metal specimens with any degree of accuracy. A more suitable material for testing two-dimensional theories of crack propagation is PMMA which is less sensitive to changes in plate thickness*. Figures 1 to 3 illustrate the projected fracture paths for three PMMA plate specimens with cracks oriented at angles $\beta$ = 30deg, 45deg and 60deg to the direction of applied load. Photographs of actual crack trajectories, each representing one quarter of a cut specimen, are placed next to the mathematically predicted paths. The agreement is extremely good, out to a distance of more than half the crack length. One can even observe the changes in the local convexity of the crack trajectories as $\beta$ changes from 30deg to 60deg. The curves in Figure 1 to 3 are computed from the strain energy density theory [5] using an exact two-dimensional stress solution of an inclined crack. Because the time interval between

---

*In metal alloys, a certain amount of energy is consummed in developing the commonly observed shear lips which represent portions of the material that have been distorted or necked. The actual process of material separation on a macroscale takes place in those portions of the material with less distortion (or more dilatation). Depending on the size of the shear lips and material properties, it is often difficult to distinguish between brittle and ductile fracture. A clear understanding of metal fracture behavior is further handicapped by the lack of definitions of ductility and brittleness.

crack initiation and complete separation of the PMMA specimen is so short, the material has no time to react to the new crack geometry at each instant of time during crack propagation. This is a reason why the mathematical path of minimum strain energy density computed from the original crack configuration follows so closely to the actual crack trajectory.

The addition of a finite radius $\rho$ of the notch tip produces significant perturbations on the initial direction of material separation. Fracture experiments [7] on elliptical notches subjected to off-axis loading have indicated that the fracture trajectory turns toward the major axis of the elliptical notch before it blends into a path of fracture coinciding with that of a line crack with no tip radius. (Severe surface deformation in metals would obscure much of this local behavior.) This peculiar feature of crack propagation indeed has been predicted by the strain energy density theory [5]. The region in which notch fracture differs from that of a line crack has been referred to as the 'core' region by Sih [4].

With regard to fracture initiation under compression, no attempts [2,3] have been made to show detailed agreement of the strain energy density theory with experiment. The simplified approach of assuming fracture to initiate from a more vulnerable macrocrack serves the purpose of illustrating qualitatively that the loading angle $\beta$ can have a significant effect on the critical compressive stress to trigger unstable macrofracture. It also explains why the compressive failure stress can exeeed Griffith's prediction of eight times the tensile failure stress as it normally does in the case of rocks and other brittle materials. A more refined quantative comparison of theory and experiment remains to be accomplished. In rocks, one must distinguish the initiation of fracture on a microscale from that of fully developed macrofracture. Minute cracks can grow slowly in a stable fashion as the load is increased. Fragmentation of the specimen will eventually take place. This basic feature of stable crack growth* prior to global instability prevails in all materials but is more easily observed in rocks. Such a phenomenon has yet to be fully understood. The single crack model is certainly justified if it is applied to predict the onset of unstable crack propagation under compression. The size of the crack must be selected to coincide with the largest fully developed macroflaw in a region near the expected path of fracture. Experiment on inclined cracks in glass plates fractured in compression agrees well with the strain energy density theory [2].

A few remarks to conclude this discussion are in order. The problem of being able to recognize a useful fracture criterion and then to apply it appropriately with consistency presents the greatest conceptual difficulties. A fracture criterion, as postulated to form a theory, should be unprejudiced by elementary experiences and need not be obvious to one's intuition whatsoever. In fact, it represents that part of the fundamental concepts which cannot be further reduced logically by reason. A postulate in itself is neither correct nor true but, if it serves any

---

*This is analagous to the redundancy in a structure.

purpose at all, it must be useful in predicting physical events. It is an advantage and yet unfortunate that many of the present day failure theories depend on data taken from uniaxial tension tests. In the study of fracture, which is inherently a two-if not three-dimensional problem one is constantly hindered by preconceived ideas carried over from the one-dimensional theory of uniaxial tension in which the maximum value of the axial stress is the only component that can be used to determine failure. In a multiaxial stress field the presence of the other stress components cannot be ignored and must be included in a criterion of fracture. In principle, the maximum stress criterion, which assumes failure to occur when only one of the stress components reaches a maximum, cannot be expected to yield accurate results when another component is of approximately the same magnitude. Although the $\sigma_\theta$ criterion appears to apply under certain conditions, the advantage of the S theory is that it applies not only over these same conditions, but over a much wider range of physical situations as well, i.e., not discrediting the former, but providing a more consistent framework for the prediction of material failure.

An analogy exists for the case of yielding. It is now common knowledge that the yield strength of a material as determined from a simple tension test cannot be used to describe the yielding of a material element in a triaxial state of stress. Hence, the $J_2$ or Von Mises' yield criterion which depends on all six stress components, was *postulated* to handle such a situation. Loosely speaking, the strain energy density criterion in fracture mechanics plays the same role as the $J_2$-criterion in plasticity. In general, it should be kept in mind that material elements undergo both volume change (dilatation) and shape change (distortion) *simultaneously*. The S-criterion weights both of these effects automatically and predicts that when the applied tensile load is not too closely aligned along the crack plane, the fracture follows the path where the material elements experience more volume change than shape change.

*Acknowledgement:* A portion of the work discussed here is supported by the Air Force Office of Scientific Research under Grant AFOSR-74-2586 with the Institute of Fracture and Solid Mechanics, Lehigh University. The valuable assistance of Dr. M. E. Kipp who provided some of the numerical results in this discussion is gratefully acknowledged. The photographs used to make the figures were provided by Dr. T. T. Wang, Bell Telephone Laboratories, Murray Hill, New Jersey.

## *REFERENCES*

[1] F. Erdogan and G. C. Sih, *Journal of Basic Engineering* 85 (1963) 519-527.

[2] G. C. Sih, Methods of Analysis and Solutions to Crack Problems, in *Mechanics of Fracture I*, Noordhoff International Publishing, Leyden (1973) 21-45.

[3] G. C. Sih, Strain Energy Density Factor Applied to Mixed Mode Crack Problems, *International Journal of Fracture* (in press).

[4] G. C. Sih, Application of Strain Energy Density Theory to Fundamental Fracture Problems, *Proceedings of the 10th Anniversary Meeting of the Society of Engineering Science* (in press).

[5] M. E. Kipp and G. C. Sih, The Strain Energy Density Failure Criterion Applied to Notched Elastic Solids, Institute of Fracture and Solid Mechanics, Lehigh University, Technical Report (1973).

[6] J. G. Williams and P. D. Ewing, *International Journal of Fracture Mechanics* 8 (1972) 441-446.

*4 March 1974 (rev 14 March 1974)*

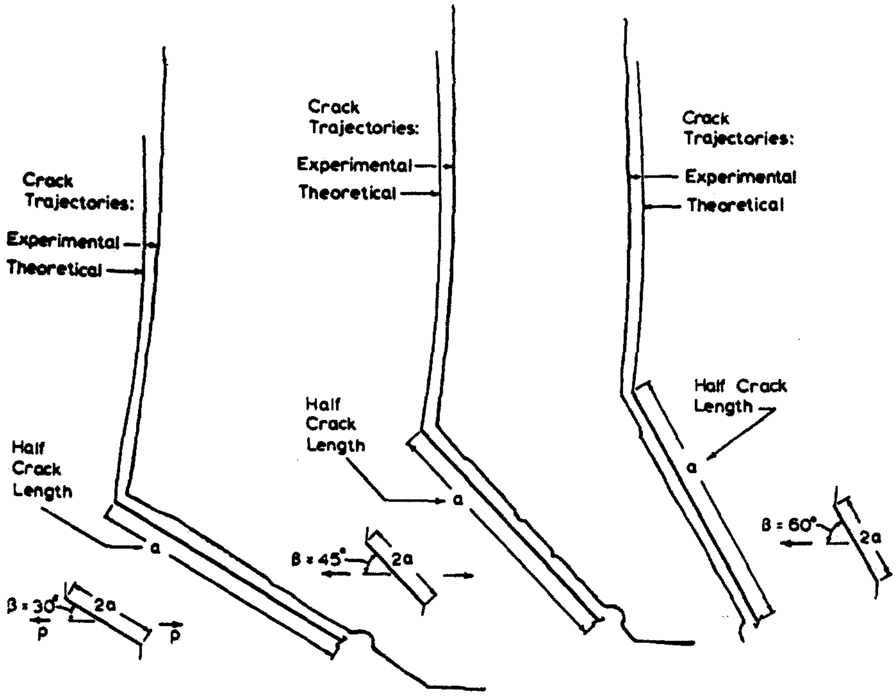

Figure 1 Figure 2 Figure 3