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Fermat's principle in elastodynamics

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Abstract. In general anisotropic inhomogeneous elasticity, disturbances propagate along rays which are neither straight nor perpendicular to the wave fronts, but, as in optics, they are still characterized by their rendering the time of travel between two points stationary.

0. Introduction

The principle of stationary time in optics was first formulated as a principle of least time by the great French mathematician Pierre de Fermat (1601–1665). In modern notation the principle states

$$\delta \int_A^B \frac{\mathrm{d}s}{v} = 0, \tag{0.1}$$

where $\mathrm{d}s$ is the length element along the path and v is the speed of propagation. The generalization of this principle to the propagation of weak waves in an elastic solid is marred by the difficulty that in the general anisotropic and inhomogeneous case the rays are not necessarily straight or perpendicular to the wave fronts, but their directions, as well as the speed of propagation, are only implicitly contained in the equations of motion. To the best of our knowledge, the validity of the principle of stationary time in elastodynamics has never been rigorously established for the general case. Hence this paper.

In Section 1 we review the basic equations of propagation and decay of the wave amplitude and we give a first definition of the rays as characteristics of the decay equation. In Section 2 we show the equivalence of this definition with the one derived from the bicharacteristics of the equations of motion, governed by a constrained Hamiltonian system. Such systems, first studied by Dirac, are the subject of Section 3, where the general procedure for obtaining

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the (homogeneous) Lagrangian associated with the given Hamiltonian, is validated. Finally, in Section 4, this procedure is reinterpreted in terms that yield Fermat's principle in a straightforward manner. Some intriguing new results relevant to the eventual expression of the Lagrangian governing the rays in terms of matrix invariants are presented, but not exploited, at the end of Section 2.

1. Preliminaries on elastic waves

The equations of motion for a hyperelastic body can be written as

$$\frac{D}{DX^j}\left(\frac{\partial \omega}{\partial u_{i,j}}\right) - \rho \ddot{u}_i = 0 \tag{1.1}$$

in a Cartesian reference coordinate system X^i $(i = 1, 2, 3)$, where

u_i = displacement vector components,

ρ = density in the reference configuration,

$\omega = \omega(u_{i,j}; X^k)$ = strain energy per unit reference volume,

and where superimposed dots denote time derivatives and commas denote partial differentiation with respect to the Cartesian coordinates.

Because of frame indifference, the dependence of ω on the displacement gradient cannot be arbitrary, but must rather be limited to (smooth) functions of the form

$$\omega = \hat{\omega}(e_{ij}; X^k), \tag{1.2}$$

with

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}). \tag{1.3}$$

A propagating wave-front is a smooth function of the form

$$\phi(X^i) = t, \tag{1.4}$$

where t is time. An acceleration wave propagating on this front is a C^1 solution $u_i = u_i(X^j, t)$ of (1.1) with discontinuous second derivatives on (1.4). Such a solution must satisfy a number of constraints both in terms of the wave-fronts and of the relative strength of the discontinuities.

With the change of variables $X^i, t \to X^i, \phi(X^i) - t$ and using square brackets to denote the jump of the quantity enclosed, the so-called compatibility conditions on the wave front, [1], are obtained as

$$[u_{i,jk}] = [\ddot{u}_i]\phi_{,j}\phi_{,k}, \tag{1.5}$$

$$[\dot{u}_{i,j}] = -[\ddot{u}_i]\phi_{,j}, \tag{1.6}$$

$$\begin{aligned}[\dot{u}_{i,jk}] = &-[\ddot{u}_i]_{,j}\phi_{,k} - [\ddot{u}_i]_{,k}\phi_{,j}\\ &-[\ddot{u}_i]\phi_{,jk} - [\dddot{u}_i]\phi_{,j}\phi_{,k},\end{aligned} \tag{1.7}$$

and

$$[\ddot{u}_{i,j}] = [\ddot{u}_i]_{,j} + [\dddot{u}_i]\phi_{,j}. \tag{1.8}$$

Returning to Eq. (1.1), we rewrite it explicitly as

$$\omega_{klij}u_{k,lj} + \omega_{ij,j} - \rho\ddot{u}_i = 0, \tag{1.9}$$

where we have introduced the notation

$$\omega_{ij} = \partial\omega/\partial u_{i,j}, \tag{1.10}$$

$$\omega_{klij} = \omega_{ijkl} = \partial^2\omega/(\partial u_{k,l}\partial u_{i,j}), \tag{1.11}$$

and where we have used the comma to mean

$$\omega_{ij,k} = \frac{\partial^2\omega}{\partial X^k\,\partial u_{i,j}}, \tag{1.12}$$

rather than a "total" derivative.

Taking jumps of Eq. (1.9) and using the compatibility conditions yields the propagation condition

$$(\omega_{klij}\phi_{,l}\phi_{,j} - \rho\delta_{ki})[\ddot{u}_k] = 0, \tag{1.13}$$

which shows that the wave-fronts must satisfy the determinantal equation

$$\det(\omega_{klij}\phi_{,l}\phi_{,j} - \rho\delta_{ki}) = 0. \tag{1.14}$$

In other words, the wave-fronts are characteristics of the equations of motion.

Taking the time derivative of Eq. (1.9)

$$\omega_{mnklij} u_{k,lj} \dot{u}_{m,n} + \omega_{klij} \dot{u}_{k,lj} + \omega_{klij,j} \dot{u}_{k,l} - \rho \dddot{u}_i = 0 \tag{1.15}$$

and assuming that the material is at rest before the arrival of the wave, we obtain for the jump of Eq. (1.15)

$$\begin{aligned} &-\omega_{mnklij}\phi_{,l}\phi_{,j}\phi_{,n}[\ddot{u}_k][\ddot{u}_m] \\ &-(\omega_{klij}+\omega_{kjil})\phi_{,j}[\ddot{u}_k]_{,l} \\ &-(\omega_{klij}\phi_{,lj}+\omega_{klij,j}\phi_{,l})[\ddot{u}_k] \\ &-(\omega_{klij}\phi_{,l}\phi_{,j}-\rho\delta_{ki})[\dddot{u}_k]=0, \end{aligned} \tag{1.16}$$

where ω_{mnklij} denotes third derivatives of ω in the style of Eq. (1.11).

To a solution ϕ of the determinantal equation (1.2) there corresponds a vector field $[\ddot{u}_i]$ on the wave-front satisfying Eq. (1.13). The direction of this vector field is determined by Eq. (1.13), but its magnitude will abide by a P.D.E. which we presently derive. Let

$$[\ddot{u}_i] = ab_i, \tag{1.17}$$

where b_i is a unit vector. Multiplying Eq. (1.16) by b_i and enforcing (1.13) and the symmetry of ω_{klij}, we obtain the 1st order P.D.E.

$$A_l a_{,l} + Ba^2 + Ca = 0, \tag{1.18}$$

with

$$A_l = (\omega_{klij} + \omega_{kjil})\phi_{,j} b_k b_i, \tag{1.19}$$

$$B = \omega_{mnklij}\phi_{,l}\phi_{,j}\phi_{,n} b_k b_m b_i, \tag{1.20}$$

and

$$C = (\omega_{klij} + \omega_{kjil})\phi_{,j} b_{k,l} b_i + (\omega_{klij}\phi_{,lj} + \omega_{klij,j}\phi_{,l}) b_k b_i. \tag{1.21}$$

Given a solution of (1.14) and initial conditions for a, integration of (1.18) fully describes the decay or growth of the wave amplitude. The characteristics of (1.18) are given by

$$\frac{dX^1}{A_1} = \frac{dX^2}{A_2} = \frac{dX^3}{A_3} \tag{1.22}$$

and, being the lines along which the disturbance naturally propagates, are called rays.

2. Rays and bicharacteristics

The rays can be obtained in a different, more illuminating, manner as projections of the characteristic strips of the non-linear first-order P.D.E. (1.14), also called bi-characteristics, for obvious reasons. To see this we start from Eq. (1.14) rewritten, after dividing by ρ^3, as

$$H(p_j, X^i) \equiv \det(E_{klij}p_l p_j - \delta_{ki}) = 0, \tag{2.1}$$

where

$$p_i = \phi_{,i} \tag{2.2}$$

and

$$E_{ijkl} = \frac{1}{\rho}\,\omega_{ijkl}. \tag{2.3}$$

Since the unknown function, ϕ, does not appear explicitly in (2.1), the characteristic strips are given by the Hamiltonian system

$$\frac{dX^i}{ds} = \frac{\partial H}{\partial p_i}, \tag{2.4}$$

$$\frac{dp_i}{ds} = -\frac{\partial H}{\partial X^i}, \tag{2.5}$$

plus the equation

$$\frac{d\phi}{ds} = p_i \frac{dX^i}{ds}, \tag{2.6}$$

which can be integrated after the system (2.4), (2.5) has been solved.

Let us denote by Z_{ki} the argument of det in Eq. (2.1), namely,

$$Z_{ki} = E_{klij}p_l p_j - \delta_{ki}. \tag{2.7}$$

Equations (2.4) and (2.5) can then be rewritten as

$$\frac{\mathrm{d}X^i}{\mathrm{d}s} = \frac{\partial \det}{\partial Z_{mn}}(E_{mlni} + E_{minl})p_l, \tag{2.8}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}s} = -\frac{\partial \det}{\partial Z_{mn}} E_{mlnj,i}p_l p_j. \tag{2.9}$$

The derivative of the determinant with respect to an entry is the co-factor of that entry. For a (symmetric) singular matrix, any row (or column) of cofactors is proportional to its eigenvector associated with the zero eigenvalue (assumed simple). It follows, then, in our case, that on the wave-front

$$\frac{\partial \det}{\partial Z_{mn}} = k b_m b_n, \tag{2.10}$$

where k is a scalar. Introducing this result in (2.8), we obtain

$$\frac{\mathrm{d}X^i}{\mathrm{d}s} = k(E_{mlni} + E_{minl})p_l b_m b_n, \tag{2.11}$$

which is equivalent to (1.22). This shows that the rays are the projections of the bicharacteristics onto the X-space (see [2]).

An interesting explicit result can be obtained for the extra equation (2.6). For the case of 2 dimensions we obtain, after some manipulations of Eq. (2.8), the identity

$$p_i \frac{\mathrm{d}X^i}{\mathrm{d}s} = 4 \det(\mathbf{Z}) + 2\,\mathrm{tr}(\mathbf{Z}), \tag{2.12}$$

where tr is the trace. Since Eq. (2.1) is to be satisfied, we conclude that

$$\frac{\mathrm{d}\phi}{\mathrm{d}s} = 2\,\mathrm{tr}(\mathbf{Z}). \tag{2.13}$$

For 3 dimensions the corresponding results are

$$p_i \frac{\mathrm{d}X^i}{\mathrm{d}s} = 6 \det(\mathbf{Z}) + (\mathrm{tr}(\mathbf{Z}))^2 - \mathrm{tr}(\mathbf{Z}^2), \tag{2.14}$$

and

$$\frac{d\phi}{ds} = (\mathrm{tr}(\mathbf{Z}))^2 - \mathrm{tr}(\mathbf{Z}^2). \tag{2.15}$$

Equations (2.13) and (2.15) could have also been obtained directly from (2.11) by noting that the constant k in (2.10) is just the product of the non-zero eigenvalues of $\mathbf{Z}$.

These explicit forms of $p_i(dX^i/ds)$ in terms of the invariants of $\mathbf{Z}$ appear to be new results. Since, as shown in the following sections, with the appropriate change of variables, $p_i(dX^i/ds)$ is the Lagrangian associated with the rays, Eqs. (2.13) and (2.15) may have some relevance for future investigations.

3. Hamiltonian constraints and homogeneous Lagrangians

The bicharacteristics satisfy Hamiltonian equations of motion,

$$\frac{dX^i}{ds} = \frac{\partial H}{\partial p_i} \tag{3.1}$$

and

$$\frac{dp_i}{ds} = -\frac{\partial H}{\partial X^i}, \tag{3.2}$$

where the Hamiltonian H is given by Eq. (2.1), and they also satisfy the constraint equation

$$H = 0, \tag{3.3}$$

which is equivalent to the original partial differential equation describing the propagation of waves in an elastic medium. Moreover, the Hamiltonian function H is not uniquely determined by the physics (geometry) of the situation. Only the zero level of H has a physical significance. In particular, Eq. (1.14) has the same physical content as Eq. (2.1). Their corresponding Hamiltonians, both equally valid, differ by a factor of ρ^3. Their corresponding characteristic strips are identical to each other, modulo a reparametrization.

Constrained Hamiltonians were first studied by P.A.M. Dirac [3]. They arise from Lagrangian systems with infinite dimensional symmetry groups. In particular, the constraint given by the vanishing of the Hamiltonian, called a Hamiltonian constraint, appears for Lagrangians which are homogeneous functions (of degree 1) of the velocity; such Lagrangians are invariant under the infinite dimensional group of transformations given by reparametrizations of the motions. If $L(\mathbf{v}, \mathbf{X})$ is a homogeneous Lagrangian, then, by Euler's formula,

$$\frac{\partial L}{\partial v_i} v_i - L = 0, \tag{3.4}$$

so that the Hamiltonian H corresponding to L vanishes identically. Hence, it is possible that the constrained Hamiltonian system describing the bicharacteristics corresponds to a homogeneous Lagrangian. Our aim is to find this Lagrangian following the method given in [4], which is an extension of the method leading to Jacobi's principle of least action [5].

In the case of regular Lagrangians, the inverse Legendre transformation is obtained by solving Eq. (3.1) for p_i in terms of

$$v^i = \frac{\mathrm{d}X^i}{\mathrm{d}s}. \tag{3.5}$$

The Lagrangian corresponding to the Hamiltonian H is given by

$$L(\mathbf{v}, \mathbf{X}) = p_i(\mathbf{v}, \mathbf{X}) v^i - H(\mathbf{p}(\mathbf{v}, \mathbf{X}), \mathbf{X}). \tag{3.6}$$

In the case under consideration, the equation

$$v^i = \frac{\partial H}{\partial p_i} \tag{3.7}$$

need not have a solution $p(\mathbf{v}, \mathbf{X})$ satisfying the constraint condition $H(\mathbf{p}(\mathbf{v}, \mathbf{X}), \mathbf{X}) = 0$. However, since the parametrization of the bicharacteristics is not yet fixed, and the change of parametrization corresponds to a rescaling of the velocity vector, given a vector $\mathbf{v}$ at $\mathbf{X}$, we try to find $\mathbf{p}$ such that

$$\frac{\partial H}{\partial p_i}(\mathbf{p}, \mathbf{X}) = \lambda v^i \tag{3.8}$$

and

$$H(\mathbf{p}, \mathbf{X}) = 0, \tag{3.9}$$

where λ is a nonzero factor of proportionality. In a neighbourhood of $(\mathbf{p}, \mathbf{X})$ satisfying Eq. (3.9) we can solve Eq. (3.8) for $\mathbf{p}$ in terms of $\mathbf{v}$, obtaining

$\mathbf{p} = \mathbf{p}(\mathbf{v}, \mathbf{X})$, provided that

$$\det \frac{\partial^2 H}{\partial p_i\, \partial p_i} \neq 0.^* \tag{3.10}$$

Substituting $\mathbf{p}(\mathbf{v}, \mathbf{X})$ in Eq. (3.6), and taking into account the constraint condition (3.9), we obtain the Lagrangian

$$L(\mathbf{v}, \mathbf{X}) = p_i(\mathbf{v}, \mathbf{X}) v^i. \tag{3.11}$$

Since the proportionality factor λ in Eq. (3.8) is a homogeneous function of $\mathbf{v}$ of degree -1,

$$\lambda(a\mathbf{v}) = a^{-1}\lambda(\mathbf{v}), \tag{3.12}$$

we have

$$\frac{\partial \lambda}{\partial v^k} v^k = -\lambda. \tag{3.13}$$

Hence, we obtain from Eq. (3.8)

$$\frac{\partial^2 H}{\partial p_i\, \partial p_j} \frac{\partial p_i}{\partial v^k} v^k = \lambda v^i + v^i \frac{\partial \lambda}{\partial v^k} v^k = 0, \tag{3.14}$$

which implies that $\mathbf{p}(\mathbf{v}, \mathbf{X})$ is a homogeneous function of $\mathbf{v}$ of degree 0. Therefore, the obtained Lagrangian is a homogeneous function of $\mathbf{v}$ of degree 1.

Consider now the variational principle

$$\delta \int L(\mathbf{v}, \mathbf{X})\, \mathrm{d}s \equiv \delta \int p_i(\mathbf{v}, \mathbf{X}) v^i\, \mathrm{d}s = 0, \tag{3.15}$$

where $p_i(\mathbf{v}, \mathbf{X})$ is obtained by solving the system (3.8), (3.9). The Euler equation associated with this principle is

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L}{\partial v^i}\right) = \frac{\partial L}{\partial X^i}, \tag{3.16}$$

* The danger that this determinant might be identically zero for all tensors E_{ijkl} is easily dispelled by checking a particular case, such as a two-dimensional isotropic material with non-vanishing Poisson's ratio. The general form of E_{ijkl} (and its connection to repeated eigenvalues) for which the determinant in (3.10) vanishes, merits further study.

but, by virtue of (3.14) and (3.10), and the symmetry of $\partial p_i/\partial v^j$ implied by (3.8),

$$\frac{\partial L}{\partial v^i} = \frac{\partial p_j}{\partial v^i} v^j + p_i = p_i, \tag{3.17}$$

and, using (3.8) and (3.3),

$$\frac{\partial L}{\partial X^i} = v^j \frac{\partial p_j}{\partial X^i} = \lambda^{-1} \frac{\partial H}{\partial p_j} \frac{\partial p_j}{\partial X^i} = -\lambda^{-1} \frac{\partial H}{\partial X^i}, \tag{3.18}$$

so that (3.16) reads

$$\frac{\mathrm{d}p_i}{\mathrm{d}s} = -\lambda^{-1} \frac{\partial H}{\partial X^i}. \tag{3.19}$$

Hence, the equations of motion corresponding to the homogeneous Lagrangian (3.11) are equivalent, up to a parametrization, to the original Hamiltonian equations (3.1) and (3.2).

4. Fermat's principle

In this last section we elucidate the foregoing analysis in terms that correspond to our physical situation. We observe that, although Eqs. (1.4) and (2.6) imply that the Lagrangian (3.11) integrated along a ray does measure the time of travel, it does not follow that its integral over an arbitrary path can be interpreted as time of travel along that path. It would, therefore, be premature to conclude that the variational equation (3.15) is a statement of stationary time. To see that this is indeed the case, we first note that, at a given point P, to each possible direction of propagation of the wave front (i.e., to each unit vector $\mathbf{n}$ emanating from P) there corresponds a unique largest speed v of propagation along $\mathbf{n}$ and a unique ray direction $\mathbf{t}$.

The speed of propagation is obtained by setting

$$\phi_{,i} = \frac{n_i}{v}, \tag{4.1}$$

as it follows from Eq. (1.4), and solving for v^2 from the determinantal equation (1.14), which can be seen now as the characteristic polynomial of the eigenvalue problem (1.13). The ray direction is obtained from Eq. (2.8) by setting

$$p_i = n_i/v \tag{4.2}$$

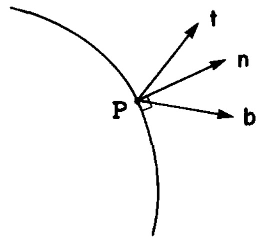

Fig. 1.

and normalizing the result to a unit vector **t**. The three unit vectors **n** (normal to wave front), **b** (acceleration jump) and **t** (ray) are algebraically related by Eq. (2.11). This triad (Fig. 1), in general, gets distorted as **n** rotates, in accordance with the anisotropic properties of the elasticity tensor ω_{ijkl}.

If we should follow the disturbance along **t**, rather than along **n**, its speed of propagation would be

$$v_t = \frac{v}{\mathbf{n} \cdot \mathbf{t}}. \tag{4.3}$$

Let us assume that the vector **t** is given. To find the corresponding **n** we observe that

$$\frac{\mathrm{d}X^i}{\mathrm{d}s} = \mu t^i, \tag{4.4}$$

where μ is a scalar depending on the parameter s. Therefore, according to Eq. (2.4), we may write

$$\mu t^i = \frac{\partial H(p_k, X^j)}{\partial p_i}, \tag{4.5}$$

which is cubic in **p**. A solution of this equation will have the form

$$p_i = p_i(\mu), \tag{4.6}$$

and the value of μ can now be adjusted so that the constraint (2.1) is satisfied, i.e.,

$$H(p_j(\mu), X^i) = 0. \tag{4.7}$$

The **p**'s so obtained are the only vectors to both satisfy the constraint and to produce derivatives of H aligned with **t**. The procedure just described is equivalent to the one described mathematically in Section 3.

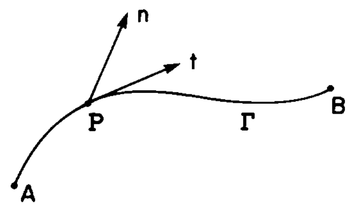

Fig. 2.

Let a curve Γ be given in E_3 as a ray candidate. At each point P along Γ the unit tangent $\mathbf{t}$ gives rise, by the procedure outlined above, to a vector $\mathbf{n}$ and a speed v. The total time of travel T between two points A and B (Fig. 2) is

$$T = \int_A^B \frac{\mathrm{d}l}{v_t}, \tag{4.8}$$

where l measures length along Γ. Using Eqs. (4.3) and (4.2), we write

$$T = \int_A^B \mathbf{p} \cdot \mathbf{t} \, \mathrm{d}l, \tag{4.9}$$

or, for any parametrization s of Γ,

$$T = \int_A^B p_i \frac{\mathrm{d}X^i}{\mathrm{d}s} \, \mathrm{d}s, \tag{4.10}$$

where p_i has been obtained point by point along Γ by the elimination procedure described, and so it is a function of t_i or, through μ, of $\mathrm{d}X^i/\mathrm{d}s$. Equation (4.10) is thus seen to correspond exactly to the Lagrangian (3.11), from which it follows that the time of travel between two points is made stationary by the rays.

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