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A LIMIT THEOREM OF CERTAIN REPAIRABLE SYSTEMS

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Abstract. Many large engineering systems can be viewed (or imbedded) as a series system in time. In this paper, we introduce the structure of a repairable system and the reliabilities of these large systems are studied systematically by studying the ergodicities of certain non-homogeneous Markov chains. It shows that if the failure probabilities of components satisfy certain conditions, then the reliability of the large system is approximately $\exp(-\beta)$ for some $\beta > 0$. In particular, we demonstrate how the repairable system can be used for studying the reliability of a large linearly connected system. Several practical examples of large consecutive-k-out-of-n:F systems are given to illustrate our results. The Weibull distribution is derived under our natural set-up.

Key words and phrases: Reliability, Markov chain, transition probabilities, linear system, repair-system, consecutive-k-out-of-n:F system.

1. Introduction

Today the public requires all engineering systems, such as atomic power plants, aircrafts, automobiles and computers, to be highly reliable. In practice, to avoid deterioration or breakdown of a system, it is routinely checked, maintained and repaired on a regular basis, for example, monthly. As a first approximation to monitor such a repair-maintenance behavior in relation to its reliability, we introduce a model based on a non-homogeneous Markov process as follows.

Assume there are $k+1$ levels of deterioration of the system, say $(1, 2, \ldots, k, k+1)$, where state 1 stands for the system in perfect condition and state $k+1$ stands for the system breaking down or not being able to be repaired. The state $k+1$ is an absorbing state. In addition to the above assumptions, we assume that if the system is in the state i $(1 \le i \le k)$ at the

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time t, then the system can *only* either deteriorate to the state $(i+1)$ or go to the state 1 (if the system is repaired) at the time $t+1$. Mathematically, this system can be described by the following stochastic model.

Let $\{X(t)\}$ be a Markov process defined on a finite state space $S=\{1,2,\ldots,k+1\}$ and a discrete index space $\mathscr{T}=\{0,1,\ldots,n\}$ with the transition matrix

$$(1.1)\qquad M_t(n)=\begin{vmatrix} p_{11}(t,n) & p_{12}(t,n) & 0 & \cdots & 0 & 0 \\ p_{21}(t,n) & 0 & p_{23}(t,n) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{k1}(t,n) & 0 & \vdots & \cdots & \vdots & p_{k,k+1}(t,n) \\ 0 & 0 & \vdots & \cdots & 0 & 1 \end{vmatrix},$$

where $p_{ij}(t,n)$ is the transition probability from state i moving to state j as the index moves from $t-1$ to t; i.e.,

$$p_{ij}(t,n)=P(X(t)=j\,|\,X(t-1)=i)\,.$$

For convenience, the index space $\mathscr{T}$ can be viewed as a discrete time space. However, t is not restricted to an explanation as time. For example, in a large series system, t can be easily interpreted as the t-th component.

The transition matrix of the form (1.1) is deliberately over-simplified to emphasize the essence of the repair-deterioration behavior. Most of our results can be extended to a more general transition matrix at the expense of more complicated calculations. In this paper, we shall concentrate on a model with a transition matrix having the form (1.1). When there is no essential loss of generality, we shall make more assumptions to simplify the calculations and to make our result more transparent.

Let $\pi_0=(\pi_1,\ldots,\pi_{k+1})$, where $\pi_i=P(X(0)=i)$, $i=1,\ldots,k+1$, be initial probabilities at time $t=0$ and $\sum_{i=1}^{k+1}\pi_i\equiv 1$. It follows that the reliability of the system (repair system) at time (index) n is

$$(1.2)\qquad R_n=\pi_0\prod_{t=1}^{n}M_t(n)U_0'\,,$$

where $U_0=(1,\ldots,1,0)$ is a $1\times(k+1)$ row vector. Hence, for large n the reliability R_n of a repairable system can be completely characterized by the ergodicity of the product $\prod_{t=1}^{n}M_t(n)$.

If $p_{i,i+1}(t,n)$ is greater than a constant $1>d>0$ for all t and n, then the reliability R_n of a repairable system tends to zero as $n\to\infty$, which is of no practical interest. In this paper, we are mainly interested in the case where the reliability R_n tends to a non-zero constant as $n \to \infty$.

Even at its simple form (1.1), our model covers many well-known examples. A typical example of this kind is the so-called consecutive-k-out-of-n:F system (see Chiang and Niu (1981), Bollinger (1982), Tong (1985) and Hwang (1986)), which contains n components linearly connected and fails if and only if k consecutive components fail. The reliability of the system has been studied by, for example, Bollinger (1982), Aki (1985), Fu (1985, 1986) and Hwang (1986). For the case where all the components have the same failure probability $1/n^{1/k}$, Fu (1986) had imbedded this system into a Markov chain defined on the state space $\mathscr{S} = \{1, \ldots, k+1\}$ and the index space $\mathscr{T} = \{0, 1, \ldots, n\}$ with the transition matrix

$$M_t(n) = \begin{vmatrix} 1 - \frac{\lambda}{n^{1/k}} & \frac{\lambda}{n^{1/k}} & 0 & \cdots & 0 & 0 \\ 1 - \frac{\lambda}{n^{1/k}} & 0 & \frac{\lambda}{n^{1/k}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 - \frac{\lambda}{n^{1/k}} & 0 & 0 & \cdots & 0 & \frac{\lambda}{n^{1/k}} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{vmatrix}, \tag{1.3}$$

for every $1 \le t \le n$, where n is the total number of components of the system. Chao and Lin (1984), Fu (1985) and Papastavridis (1987), under different methods, have shown that the reliability of a large consecutive-k-out-of-n:F system converges to a non-zero constant $\exp(-\lambda^k)$. Their techniques are indirect. However, in this paper we shall take a more general and direct approach to this problem.

There are many systems other than the one mentioned above which can be imbedded into this type of Markov chain, especially linearly connected systems. Since the transition probabilities given in (1.1) are allowed to depend on the index (time) t and the final index n (or the total number of components of the system), the Markov chain is non-homogeneous. Therefore, the ergodicity of the product $\prod_{t=1}^{n} M_t(n)$ becomes a convergence problem for a double-array. The standard method (see Seneta (1981)) to study the ergodicity of homogeneous Markov chains is usually based on the concept of Birkoff's coefficient of contraction. A transition matrix similar to (1.3), in the context of demography, is called a Leslie matrix (or more generally a "Renewal-type matrix"):

$$L = \begin{vmatrix} p_1 & q_1 & 0 & 0 & \cdots & 0 & 0 \\ p_2 & 0 & q_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_k & 0 & \vdots & \vdots & \cdots & 0 & q_k \\ p_{k+1} & 0 & \vdots & \vdots & \cdots & 0 & 0 \end{vmatrix}, \tag{1.4}$$

whose ergodicity had been studied, for example, by Lopez (1961) and Seneta (1981). Unlike the transition probabilities p_i and q_i in the Leslie matrix (1.4), the transition probabilities in the matrix $M_t(n)$ of (1.1) depend on the indexes t and n. The techniques used in Lopez (1961) and Seneta (1981) cannot be directly carried over to study the ergodicity of the product $\prod_{t=1}^{n} M_t(n)$.

In this paper, the ergodicity of the product $\prod_{t=1}^{n} M_t(n)$ of a non-homogeneous Markov chain will be studied directly via Chapman-Kolmogorov equations and exponential bounds. Our results provide a general method for studying the reliabilities of certain large engineering systems, particularly large, linearly connected systems.

2. Main results

For given initial probability $\pi_0 = (\pi_1, \dots, \pi_{k+1})$ and $i = 1, 2, \dots, n$, let

$$a(i) = \pi_0 \prod_{t=1}^{i} M_t(n) = (a_1(n, i), \dots, a_{k+1}(n, i)), \tag{2.1}$$

where $a_j(n, i)$ is the probability of the system in the state j at the time (or index) $t = i$ and it can be written as

$$a_j(n, i) = P(X(i) = j) = \pi_0 \prod_{t=1}^{i} M_t(n) U_j', \tag{2.2}$$

$j = 1, \dots, k + 1$, where $U_j = (0, \dots, 0, 1, 0, \dots, 0)$ is a $1 \times (k + 1)$ row unit vector with 1 at the j-th coordinate. It follows from (1.2) that the reliability of the system is

$$R_n = \pi_0 \prod_{t=0}^{n} M_t(n) U_0' = \sum_{j=1}^{k} a_j(n, n). \tag{2.3}$$

The most general version of our result is presented in Corollary 2.2. For convenience, we consider the following simple model that $p_{j1}(n, t) = p_n$ and $p_{j,j+1}(n, t) = q_n = 1 - p_n$ which are independent of index t but depend on n. Then the transition probability matrix (1.1) of the process is reduced to

$$M_t(n) = M(n) = \begin{vmatrix} p_n & q_n & 0 & \cdots & \cdots & 0 \\ p_n & 0 & q_n & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_n & 0 & \vdots & \cdots & 0 & q_n \\ 0 & \vdots & \vdots & \cdots & 0 & 1 \end{vmatrix}, \tag{2.4}$$

for every $1 \leq t \leq n$.

It follows from Chapman-Kolmogorov equations that the following recursive equations hold: for $1 \leq i \leq n$,

$$\begin{aligned} a_1(n,i) &= p_n(a_1(n,i-1) + \cdots + a_k(n,i-1)), \\ a_2(n,i) &= q_n a_1(n,i-1), \\ &\vdots \\ a_k(n,i) &= q_n a_{k-1}(n,i-1), \\ a_{k+1}(n,i) &= q_n a_k(n,i-1) + a_{k+1}(n,i-1). \end{aligned} \tag{2.5}$$

To prove our main results we need the following lemmas.

LEMMA 2.1.
(i) $M(n)U_0' = U_0' - q_n U_k' = (U_0' - U_k') + p_n U_k'$,
(ii) $M(n)U_1' = p_n U_0'$,
(iii) $M(n)U_j' = q_n U_{j-1}', j = 2,\ldots,k$.

PROOF. The above results follow directly from the structure of $M(n)$ and definitions of $U_j', j = 0, 1,\ldots, k$.

LEMMA 2.2.
(i) $a(i)U_0' = a(i-1)U_0' - q_n a(i-1)U_k'$,
(ii) $a(i)U_0' \leq a(j)U_0'$ *for all* $n \geq i \geq j \geq 0$,
(iii) $a(i-k+1)U_1' \leq a(i)U_0'$ *for all* $i \geq k$.

PROOF. Result (i) of this lemma is a direct conclusion of Lemma 2.1(i) and the definition of $a(i)$ given by (2.1). Since $q_n a(i-1)U_k' \geq 0$, the result (ii) follows directly from the first part of result (i). Finally, result (iii) follows from

$$
\begin{aligned}
a(i)U_0' &= a(i-1)(U_0' - U_k') + p_n a(i-1)U_k' \\
&\geq a(i-1)(U_0' - U_k') \\
&= a(i-2)(U_0' - U_k' - U_{k-1}') + p_n a(i-2)U_{k-1}' \\
&\geq a(i-2)(U_0' - U_k' - U_{k-1}') \qquad (2.6) \\
&\vdots \\
&= a(i-k+1)(U_0' - U_k' - \cdots - U_2') + p_n a(i-k+1)U_2' \\
&\geq a(i-k+1)U_1' .
\end{aligned}
$$

This completes the proof.

The inequalities (ii) and (iii) are very critical to prove our main results. We would like to give the intuitive implications for these two fundamental inequalities:

(A) The inequality of Lemma 2.2(ii) implies that the reliability R_t of the system is a decreasing function of the time (index).

(B) The inequality of Lemma 2.2(iii) shows that the reliability of the system at time i is at least greater than or equal to the probability that the system is at the state 1 at the time $(i-k+1)$.

LEMMA 2.3. *For $n \geq i \geq k$,*

$$p_n q_n^{k-1} a(i)U_0' \leq a(i)U_k' \leq q_n^{k-1} a(i)U_0' . \qquad (2.7)$$

PROOF. Since

$$a(i)U_k' = q_n^{k-1} a(i-k+1)U_1' = p_n q_n^{k-1} a(i-k)U_0' ,$$

the inequalities follow immediately from the inequalities of Lemma 2.2(ii) and (iii), respectively.

THEOREM 2.1. *For any given π_0 satisfies $\pi_0 U_0' \equiv 1$ and if $q_n = \lambda / n^{1/k}$, $\lambda > 0$, then*

(i) $\lim_{n\to\infty} R_n = e^{-\lambda^k}$, *and*

(ii) $\lim_{n\to\infty} a(n)U_j' = \begin{cases} e^{-\lambda^k}, & \text{if } j = 1 , \\ 0, & \text{if } j = 2,\dots,k . \end{cases}$

PROOF. Without loss of generality, we assume $\pi_0 = (1,0,\dots,0)$. It follows that we have the initial conditions $a(i)U_0' \equiv 1$ and $a_k(n,i)U_k' = 0$ for all $n > k > i = 0$ and $a_k(n,k) = q_n^{k-1}$. Lemma 2.1(i) yields that the reliability of the system is

$$R_n = \pi_0 M^n(n) U_0' = a(n) U_0' = a(n-1) U_0' - q_n a(n-1) U_k'$$
$$= \left(1 - \frac{q_n a(n-1) U_k'}{a(n-1) U_0'}\right) a(n-1) U_0'$$
$$= \prod_{i=1}^{n} \left(1 - \frac{q_n a(n-i) U_k'}{a(n-i) U_0'}\right) a(0) U_0' .$$

Note that $a(0) = \pi_0$ and $a(0)U_0' = 1$. Hence

$$R_n = \prod_{i=1}^{n} \left(1 - \frac{q_n a(n-i) U_k'}{a(n-i) U_0'}\right). \tag{2.8}$$

Since $q_n = 1/n^{1/k}$, it follows from the initial conditions, Lemma 2.3 and equation (2.8) that

$$\left(1 - \frac{\lambda^k}{n}\right)^{n-k+1} \le R_n \le \left(1 - \frac{\lambda^k}{n} + \frac{\lambda^{k+1}}{n^{(k+1)/k}}\right)^{n-k} . \tag{2.9}$$

The result (i) follows immediately from the above inequalities by taking the limit.

Note, that by Lemmas 2.1 and 2.2, the following inequalities hold:

(2.10) $p_n R_{n-1} \le a(n) U_1' \le R_n$,

(2.11) $q_n^j p_n a(n-j-1) U_0' \le a(n) U_j' \le q_n a(n-j) U_0', \quad j = 1, \ldots, k$.

Since $q_n = \lambda / n^{1/k}$ the result (ii) immediately follows from result (i) and inequalities (2.10) and (2.11). This completes the proof.

The condition $q_n = \lambda / n^{1/k}$ is vital to our results. In other words, to have non-trivial reliability for a large system, the failure probabilities of components in the system should be inversely proportional to the k-th root of the size of the system. Furthermore, if the condition is satisfied, then the large system is either at state 1 with probability $\exp\{-\lambda^k\}$ or at state $(k+1)$ with probability $1 - \exp\{-\lambda^k\}$.

THEOREM 2.2. *If $q_n = \lambda / n^{1/k}$, then*

$$\lim_{n \to \infty} M^n(n) = \begin{vmatrix} e^{-\lambda^k} & 0 & \cdots & 0 & 1 - e^{-\lambda^k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e^{-\lambda^k} & 0 & \cdots & 0 & 1 - e^{-\lambda^k} \\ 0 & 0 & \cdots & 0 & 1 \end{vmatrix} . \tag{2.12}$$

PROOF. Since the state $k+1$ is an absorbing state, this theorem follows directly from the fact that Theorem 2.1(ii) holds for all π_0 which satisfies $\pi_0 U_0' \equiv 1$.

For a slightly more general case, we assume the transition probabilities $p_{j,j+1}(t,n)$ defined in (1.1) are independent of t but depend on n and the state j given as follows

$$p_{j,j+1}(t,n) = \lambda_j / n^{1/k}, \quad \lambda_j > 0, \quad j = 1,2,\ldots,k \,. \tag{2.13}$$

COROLLARY 2.1. *If the transition probabilities* $p_{j,j+1}(t,n)$ *defined in* (1.1) *satisfy* (2.13), *then*

$$\lim_{n\to\infty} R_n = e^{-\lambda^*}, \quad \text{where} \quad \lambda^* = \prod_{j=1}^{k} \lambda_j, \quad \text{and} \tag{2.14}$$

$$\lim_{n\to\infty} M^n(n) = \begin{vmatrix} e^{-\lambda^*} & 0 & \cdots & 0 & 1-e^{-\lambda^*} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e^{-\lambda^*} & 0 & \cdots & 0 & 1-e^{-\lambda^*} \\ 0 & 0 & \cdots & 0 & 1 \end{vmatrix} . \tag{2.15}$$

PROOF. The results (2.14) and (2.15) immediately follow from the proofs of Theorems 2.1 and 2.2 with some trivial modifications. Hence, we omit the detailed proof here.

This model covers the results of Papastavridis and Lambiris (1987), and Fu and Hu (1987) on the large consecutive-k-out-of-n:F system, with failure probabilities of components being Markov dependent.

For more general cases, we assume that the transition probabilities $p_{j,j+1}(t,n)$ defined in (1.1) depend on t, n and j, and satisfy

$$p_{j,j+1}(t,n) = \lambda_{tj} / n^{1/k}, \quad \lambda_{tj} > 0 \,, \tag{2.16}$$

for $t = 1,\ldots,n$ and $j = 1,\ldots,k$. The following general results also hold.

COROLLARY 2.2. *If the conditions* (2.16) *and*

$$\lambda = \lim_{n\to\infty} \sum_{j=1}^{\infty} \frac{1}{j} \sum_{t=1}^{n} \lambda_t^j < \infty \,, \tag{2.17}$$

where $\lambda_t = \prod_{j=1}^{k} \lambda_{tj}$, *are satisfied then*

(i) $\lim_{n\to\infty} R_n = e^{-\lambda}$, *and*

$$\text{(ii)} \quad \lim_{n\to\infty} \prod_{t=1}^{n} M_t(n) = \begin{vmatrix} e^{-\lambda} & 0 & \cdots & 0 & 1-e^{-\lambda} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e^{-\lambda} & 0 & \cdots & 0 & 1-e^{-\lambda} \\ 0 & 0 & \cdots & 0 & 1 \end{vmatrix} .$$

This corollary can be proved by the same method used in Theorems 2.1 and 2.2 together with the following well-known lemma.

LEMMA 2.4. (i) *If* $0 < U_{t,n} < 1$ *for all* n *and* $1 \le t \le n$, *then the product* $\prod_{t=1}^{n}(1 - U_{t,n})$ *and the sum* $\sum_{t=1}^{n} U_{t,n}$ *both converge or diverge together.*

(ii) *If* $0 < U_{t,n} < 1$ *for all* n *and* $1 \le t \le n$, *and*

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} \frac{1}{j} \sum_{t=1}^{n} U_{t,n}^{j} = \theta < \infty , \tag{2.18}$$

then

$$\lim_{n\to\infty} \prod_{t=1}^{n} (1 - U_{t,n}) = e^{-\theta} . \tag{2.19}$$

Condition (2.18), the Cesaro summability condition, is satisfied if, for example, λ_{tj} are bounded. This model does cover many large linear systems, in particular, the large consecutive-k-out-of-n:F systems studied by Chao and Lin (1984), Fu (1985, 1986), Papastavridis (1987) and Chrysaphinou and Papastavridis (1988).

Chrysaphinou and Papastavridis (1988) proved that the lifetime of a large consecutive-k-out-of-n:F system has a Weibull distribution. Their result can also be immediately deduced from our main result.

Let T_n be the lifetime of a large consecutive-k-out-of-n:F system. Assume all the components are independent and have the same failure probabilities. Further, we assume that the probability of the component failure time no longer than θ is

$$q_n = \lambda\theta / n^{1/k} + o(1/n^{1/k}) . \tag{2.20}$$

In other words, the above assumption says that, for fixed θ, the failure probability of a component depends on the size of the system and tends to zero with a rate $\lambda\theta/n^{1/k}$. It follows immediately from the Theorem 2.1 that for $0 < \theta < \infty$,

$$\lim_{n\to\infty} P(T_n < \theta) = 1 - e^{-(\lambda\theta)^k} . \tag{2.21}$$

This yields the result of Barlow and Proschan ((1975), p. 230) for $k = 1$ and also the general result of Chrysaphinou and Papastavridis (1988). We would like to point out that the scale of the lifetime of system T_n in our case differs from the scale used by Chrysaphinou and Papastavridis by a factor of $n^{1/k}$. The scale of T_n is the same as the failure time θ of the component in our case, but their scale of T_n differs from the failure time of the component by a factor of $n^{1/k}$.

REFERENCES

Aki, S. (1985). Discrete distributions of order k on a binary sequence, *Ann. Inst. Statist. Math.*, **37**, 205-224.

Barlow, R. E. and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing*, Holt, Reinhard and Winston, New York.

Bollinger, R. C. (1982). Direct computation for consecutive-k-out-of-n:F system, *IEEE Trans. Reliability*, **R-31**, 444-446.

Chao, M. T. and Lin, G. D. (1984). Economical design of large consecutive-k-out-of-n:F system, *IEEE Trans. Reliability*, **R-33**, 411-413.

Chiang, D. T. and Niu, S. C. (1981). Reliability of consecutive-k-out-of-n:F system, *IEEE Trans. Reliability*, **R-33**, 411-413.

Chrysaphinou, O. and Papastavridis, S. (1988). Asymptotic distribution of a consecutive-k-out-of-n:F system, Tech. Report, University of Athens, Greece.

Fu, J. C. (1985). Reliability of a large consecutive-k-out-of-n:F system, *IEEE Trans. Reliability*, **R-34**, 127-130.

Fu, J. C. (1986). Reliability of consecutive-k-out-of-n:F system with $(k-1)$-step Markov dependence, *IEEE Trans. Reliability*, **R-35**, 602-606.

Fu, J. C. and Hu, B. (1987). On reliability of a large consecutive-k-out-of-n:F system with $(k-1)$-step Markov dependence, *IEEE Trans. Reliability*, **R-36**, 75-77.

Hwang, F. K. (1986). Simplified reliabilities for consecutive-k-out-of-n:F system, *SIAM J. Algebraic Discrete Methods*, **7**, 258-264.

Lopez, A. (1961). Problems in stable population theory, Tech. Report, Office of Populations Research, Princeton University, Princeton.

Papastavridis, S. (1987). A limit theorem for the reliability of a consecutive-k-out-of-n:F system, *Adv. in Appl. Probab.*, **19**, 746-748.

Papastavridis, S. and Lambiris, M. (1987). Reliability of a consecutive-k-out-of-n:F system for Markov-dependent components, *IEEE Trans. Reliability*, **R-36**, 78-80.

Seneta, E. (1981). *Non-Negative Matrices and Markov Chains*, 2nd ed., Springer, New York-Berlin.

Tong, Y. L. (1985). A rearrangement inequality for the longest run with an application to network reliability, *J. Appl. Probab.*, **22**, 386-393.