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Expected Utility: An Anniversary and a New Era

PETER C. FISHBURN
AT&T Bell Laboratories

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Abstract

During the past generation, expected utility theory has been widely accepted as the normative standard for decision making under risk and under uncertainty. However, it is now known that reasonable people often violate its assumptions, and a number of generalizations of the theory have been developed to accommodate some of the more common violations. This essay recalls the origins of expected utility in the early 1700s, notes its axiomatizations on the basis of preference comparisons in the mid-1900s, describes violations of those axioms uncovered since then, outlines new theories stimulated by the violations, and suggests where the field might be headed in the next few decades.

Daniel Bernoulli, Oskar Morgenstern, John von Neumann, Frank P. Ramsey, and Leonard J. Savage are remembered in one corner of the scientific world for their common interest in rational choice in the face of uncertainty. Each of these brillant people proposed (jointly for von Neumann and Morgenstern) a theory for coherent and consistent choice among alternative courses of action with uncertain outcomes. Their theories, known collectively as expected utility theory, have had a profound impact on economic and statistical thought and theory during the past generation.

This year marks the 250th anniversary of the publication of Bernoulli's *Specimen Theoriae Novae de Mensura Sortis* and a full generation since the appearance of Savage's *The Foundations of Statistics.* Although expected utility theory remains the preeminent model for rational decision making under risk and uncertainty, a revolution in the foundations of decision has been in progress for several years. Sparked by Maurice Allais's research in the early 1950s, which challenged the tenability of independence axioms used by von Neumann and Morgenstern and by Savage, and fueled by a host of subsequent experiments that demonstrate persistent and systematic violations of expected utility, a handful of new theories have been proposed as normative alternatives to expected utility.

The new theories have been designed to accommodate predictable violations of the traditional models without giving up too much of the mathematical elegance and analytical power of expected utility. Thus, they aim to retain implications of expected utility with regard to risk-taking, stochastic dominance, and other factors while allowing preference patterns that many people consider reasonable but that contradict expected utility.

This article recounts new developments against the background of traditional expected utility and its empirical violations. The emphasis is on mathematically interesting aspects of new theories, conveyed largely through their quantitative representations of preference. The next two sections build the background. We then consider alternatives to the von Neumann-Morgenstern theory for preference under risk and follow this with alternatives to Savage's theory for preference under uncertainty. The paper concludes with a brief summary and thoughts about the future.

1. The linear heritage

It was widely believed in the early years of the development of probability theory that risky monetary ventures ought to be evaluated by their expected returns—the more the better. The first major challenge to this view appeared in 1738 at the hand of Daniel Bernouli, a member of the Swiss family of distinguished mathematicians. Bernoulli (1738) proposed two theses. First, a person's subjective value or utility of wealth does not increase linearly in the amount but rather increases at a decreasing rate. More precisely, the utility of the next increment to wealth is inversely proportional to the amount already on hand, thus giving rise to a logarithmic function for the utility of wealth. This thesis views utility as an intensively measurable quantity that has nothing to do with probability or risk.

Bernoulli's second thesis says that a risky prospect on levels of wealth ought to be evaluated by its expected utility of wealth—the more the better. When combined with his riskless intensive measure of utility, this explains why you would refuse to engage in a not-to-be-repeated wager that gains \$21,000 or loses 20,000, each with probability $\frac{1}{2}$, even though the wager has an expected gain of \$500. Moreover, it also answers a question raised by Daniel's cousin Nicholas Bernoulli in 1713: Why would most people in possession of an option that pays $\$2^n$ if the first head in a sequence of tosses of a fair coin occurs at the nth toss gladly sell the option for perhaps \$25 despite its infinite expected return? This is Nicholas's famous St. Petersburg paradox (Menger, 1967; Samuelson, 1977).

In a postscript to his 1738 paper, Daniel says that he has learned from Nicholas that another Swiss mathematician, Gabriel Cramer, provided an answer to Nicholas's question that is remarkably similar to his own. The postscript quotes extensively from the letter to Nicholas in which Cramer describes his resolution of the matter, including

> You asked for an explanation of the discrepancy between the mathematical calculation and the vulgar evaluation. I believe that it results from the fact that, *in their theory,* mathematicians evaluate money in proportion to its quantity while, *in practice,* people with common sense evaluate money in proportions to the utility they can obtain from it. (Translated from the French by L. Sommer, *Econometrica,* 1954, p. 33.)

Unlike Daniel Bernoulli, Cramer pays little attention to total wealth, suggests $\sqrt{x}$ as a reasonable utility for *gain* x, and evaluates the minimum selling price of Nicholas's option at about \$6.

More than two centuries after Daniel's paper was published, von Neumann and Morgenstern (1944) axiomatized a version of expected utility that has the same mathematical form as Bernoulli's version but differs radically in its interpretation. Rather than beginning with a riskless intensive measure of utility and then combining this with probability by the expected value operation, von Neumann and Morgenstern start with a binary relation $>$ on a convex set P. Although $>$ is an undefined primitive in their theory and P can be any abstract convex set, we interpret members of P as probability measures on a Boolean algebra of subsets of a set X of outcomes, monetary or otherwise, and read $p > q$ as "p is preferred to q." For $0 \leqslant \lambda \leqslant 1$ and $p, q \in P$, $\lambda p + (1 - \lambda)q$ is defined by $(\lambda p + (1 - \lambda)q)(A) = \lambda p(A) + (1 - \lambda)q(A)$ for each A in the algebra. In addition, we denote the symmetric complement of $>$ by $\sim$, so $p \sim q$ if neither $p > q$ nor $q > p$, and read $p \sim q$ as "p is indifferent to q."

The axioms of von Neumann and Morgenstern deal solely with $>$ and $\sim$ on P. They are, for all $p, q, r \in P$ and all $0 < \lambda < 1$:

A1 (order)	$>$ is asymmetric; $>$ and $\sim$ are transitive
A2 (independence)	$p > q \Rightarrow \lambda p + (1 - \lambda)r > \lambda q + (1 - \lambda)r$
A3 (continuity)	$p > q > r \Rightarrow \alpha p + (1 - \alpha)r > q$ and $q > \beta p + (1 - \beta)r$ for some $\alpha, \beta \in (0,1)$

A1 implies that $\sim$ on P is an equivalence and the classes in the quotient set $P/\sim$ are linearly ordered by the natural extension of $>$. A2 says that preference is preserved under similar convex combinations. Continuity, A3, prohibits infinitely undesirable [r such that $\alpha p + (1 - \alpha)r > q$ for no $\alpha < 1$] and infinitely desirable [p such that $q > \beta p + (1 - \beta)r$ for no $\beta > 0$] measures and is needed for real-valued as opposed to vector-valued or nonstandard utilities (Chipman, 1960; Fishburn, 1974; Hausner, 1954; Skala, 1975).

We say that $u: P \to \mathbb{R}$ is *linear* if for all $p, q \in P$ and all $0 \leqslant \lambda \leqslant 1$

$$u(\lambda p + (1 - \lambda)q) = \lambda u(p) + (1 - \lambda)u(q)$$

In Theorem 1 and later, a *functional* is a real-valued function.

Theorem 1 (von Neumann-Morgenstern). There exists a linear functional u *on* P such that, for all $p, q \in P$,

$$p > q \Leftrightarrow u(p) > u(q)$$

if and only if A1, A2, and A3 hold. Moreover, u is unique up to a positive affine transformation (i.e., given u as just specified, v is another order-preserving linear functional on P if and only if, for all $p \in P$, $v(p) = au(p) + b$ for reals $a > 0$ and b). (Proofs are given in Fishburn, 1970, 1982a.)

If P contains every one-point measure, hence contains every simple (finite-support) measure by convexity, and if $u(x)$ for outcome x is defined by $u(x) = u(p)$ when $p(x) = 1$, then linearity yields

$$u(p) = \sum_x u(x)\, p(x)$$

for every simple p. Additional axioms (Fishburn, 1970, 1982a) are needed to extend this expected utility form to $u(p) = \int u(x)\, dp(x)$ for nonsimple measures in P.

The difference between Bernoulli and von Neumann and Morgenstern is illustrated by

$$u(\$300) = \tfrac{1}{2}\, u(\$0) + \tfrac{1}{2}\, u(\$800)$$

For von Neumann and Morgenstern, this means that the individual is indifferent between \$300 with certainty and an even-chance gamble that pays \$0 or \$800. For Bernoulli, it means that, quite apart from any consideration of chance, the intensities of the individual's preference for \$300 over \$0 and for \$800 over \$300 are equal, as suggested by $u(\$300) - u(\$0) = u(\$800) - u(\$300)$. Because the von Neumann-Morgenstern axioms refer solely to simple preference comparisons, they neither support nor encourage a Bernoullian notion of riskless comparable preference differences.

The phrases decision under risk and preference under risk are applied to Bernoulli and von Neumann and Morgenstern to denote that their formulations take probabilities as given. In contrast, decision under uncertainty and preference under uncertainty denote formulations in which probabilities are derived from axioms or in some other way, if at all.

The first general theory of expected utility for preference under uncertainty was outlined by Frank P. Ramsey in 1926 and published posthumously (Ramsey, 1931). Although von Neumann and Morgenstern did not mention Ramsey, Savage (1954) drew heavily on both Ramsey and von Neumann-Morgenstern to produce the first complete theory for expected utility with subjective probability. Since space constraints prohibit a full account of Savage's theory, I note only things needed to appreciate its generalizations.

For Savage, uncertainty resides in a set S of states of the world. The individual is presumed to be uncertain about which state is the true state (i.e., which state obtains), but the outcome of decision depends on this state. Subsets of S are called events, and we say that event A obtains if it contains the true state. In Savage's representation, subjective or personal probabilities are assigned to events while utilities to outcomes in X.

Savage's axioms for preference under uncertainty apply $>$ to the set $F = X^S$ of acts $f, g, \ldots$, each of which assigns an outcome to each state in S. His axioms imply the existence of a bounded functional u on X and a finitely additive probability measure π on the set 2^S of all events such that, for all $f, g \in F$

$$f > g \Leftrightarrow \int_S u(f(s))d\pi(s) > \int_S u(g(s))d\pi(s)$$

Moreover, u is unique up to a positive affine transformation and π is unique and satisfies $\{A \subseteq S; \pi(A) > 0; 0 < \lambda < 1\} \Rightarrow \{\pi(B) = \lambda\pi(A)$ for some $B \subset A\}$, which forces S to be infinite. Savage's connection between preference and probability follows the lead of Ramsey (1931) and de Finetti (1937). With $>$ extended from F to X in the natural way by constant acts, and with $S\backslash A$ denoting the complement of A in S, this connection is specified by

$$\pi(A) > \pi(B) \Leftrightarrow f > g \text{ whenever } x > y, f = x \text{ on } A,$$
$$f = y \text{ on } S\backslash A, g = x \text{ on } B, g = y \text{ on } S\backslash B$$

Thus, for Savage, you regard A as more probable than B if you would rather bet on "A obtains" than "B obtains" for a preferred outcome.

An especially important independence axiom used by Savage is his substitution principle: if $f(s) = f'(s)$ and $g(s) = g'(s)$ for all $s \in A$, and if $f(s) = g(s) = x$ and $f'(s) = g'(s) = y$ for all $s \in S\backslash A$, then $f > g \Leftrightarrow f' > g'$. Thus, if $f > g$, and if f and g are changed to another outcome (y) on an event ($S\backslash A$) where they are constant and equal (x), then $f > g$ after the change.

Another important principle that follows from his representation is based on the measure π_f on 2^X induced by π acting on f that is defined by $\pi_f(Y) = \pi(\{s: f(s) \in Y\})$ for each $Y \subseteq X$. Thus, $\pi_f(Y)$ is the probability of getting an outcome in Y when f is used. The other principle is the reduction principle: $\{p_f = \pi_{f'}; \pi_g = \pi_{g'}\} \Leftrightarrow (f > g \Leftrightarrow f' > g')$. Thus, given π, $>$ on F is characterized by $>$ on the set of measures on outcomes induced by π from the acts in F.

Numerous variations on the themes of the preceding representations that are not considered later are reviewed in Fishburn (1970, 1981, 1982a, 1986b).

2. Rumblings in the foundations

Many people, including Allais (1953, 1979a), Ellsberg (1961), MacCrimmon and Larsson (1976), Kahneman and Tversky (1979), Slovic and Lichtenstein (1983), Machina (1983a), and Tversky and Kahneman (1987) have written eloquently about systematic violations of axioms for preference under risk and uncertainty that support expected utility theory. I therefore confine myself to a few examples that illustrate the problems.

Violations of order and transitivity are usually associated with multidimensional outcomes or alternatives (MacCrimmon and Larsson, 1979; May, 1954) or with the confluence of probability and value (Grether and Plott, 1979; Lichtenstein and Slovic, 1971; Slovic and Lichtenstein, 1983; Tversky, 1969). May (1954) asked 62 college students to make binary comparisons between three hypothetical marriage partners characterized by three criteria—intelligence, looks, and wealth. Seventeen of the 62 had cyclic choices ($x > y, y > z, z > x$), and for each of these 17, "the intransitivity pattern is easily explained as the result of choosing the alternative that is superior in two out of three criteria" (May, 1954, p. 7).

Another source of intransitivity is known as the preference reversal phenomenon (Grether and Plott, 1979; Lichtenstein and Slovic, 1971; Slovic and Lichtenstein, 1983). This occurs when one monetary lottery p is preferred to another monetary lottery q, but an individual in possession of one or the other would sell p for less than q. For example: $p(\$30) = 0.9, p(\$0) = 0.1; q(\$100) = 0.3, q(\$0) = 0.7; p > q$, p's minimum selling price is \$25, and q's is \$27. Under the usual assumptions that more money is preferred to less, and that the minimum selling price of a lottery (its certainty equivalent) is indifferent to the lottery, preference reversals contradict the von Neumann-Morgenstern order axiom.

The von Neumann-Morgenstern independence axiom A2, $p > q \Rightarrow \lambda p + (1 - \lambda)r > \lambda q + (1 - \lambda)r$, is vulnerable to Allais's certainty effect (Allais, 1953, 1979a) and related phenomena in which violations arise from different foci in two comparisons. Many people prefer \$1,000,000 with certainty to a lottery that pays \$3,000,000 with probability 0.98 (nothing otherwise) because of the certainty of the first payoff. At the same time, they also prefer \$3,000,000 with probability 0.049 (nothing otherwise) to \$1,000,000 with probability 0.050 (nothing otherwise) because the 1 in 1000 difference between their payoff probabilities is overwhelmed by the much higher payoff of the first lottery. A similar example in Kahneman and Tversky (1979) has more modest payoffs:

p: \$3,000 with probability 1

q: \$4,000 with probability 0.8, nothing otherwise

p': \$3,000 with probability 0.25, nothing otherwise

q': \$4,000 with probability 0.20, nothing otherwise

Here $r(\$0) = 1, \lambda = \frac{1}{4}, p' = \lambda p + (1 - \lambda)r$ and $q' = \lambda q + (1 - \lambda)r$. In the Kahneman-Tversky experiment with these lotteries, a majority of 95 respondents had $p > q$ and $q' > p'$.

A famous example from Ellsberg (1961) suggests failures of Savage's substitution principle for preference under uncertainty due to comparative specificities of events. One ball is to be drawn at random from an urn containing 90 balls, 30 of which are red (R) and 60 of which are black (B) and yellow (Y) in unknown proportion. Consider two pairs of acts:

$\begin{cases} f\text{:} & \text{win \$1,000 if R drawn} \\ g\text{:} & \text{win \$1,000 if B drawn} \end{cases}$

$\begin{cases} f'\text{:} & \text{win \$1,000 if R or Y drawn} \\ g'\text{:} & \text{win \$1,000 if B or Y drawn} \end{cases}$

with \$0 otherwise in each case. Ellsberg claimed, and subsequent experiments have verified, that most people prefer f to g and prefer g' to f' in direct violation of the substitution principle. The specificity of R relative to B apparently motivates $f > g$. Similarly, $g' > f'$ is tied to the fact that exactly 60 balls are B or Y, whereas an unknown number from 30 to 90 are R or Y. The pair $\{f > g; g' > f'\}$ prohibits additive subjective probability by Savage's approach since $f > g$ gives $\pi(\text{R}) > \pi(\text{B})$ and $g' > f'$ gives $\pi(\text{B} \cup \text{Y}) > \pi(\text{R} \cup \text{Y})$, hence $\pi(\text{B}) > \pi(\text{R})$.

Even if additive probabilities are transparent and the substitution principle is adopted, problems can arise with the reduction principle. Consider the following two acts for one roll of a well-balanced die with probability $\frac{1}{6}$ for each face:

	1	2	3	4	5	6
f	\$600	\$700	\$800	\$900	\$1000	\$500
g	\$500	\$600	\$700	\$800	\$900	\$1000

Some people prefer f to g because f pays \$100 more than g for five of the six states; if a 6 comes up, it is merely bad luck. Others dread the thought of choosing f and losing out on the \$500 difference should a 6 come up and therefore prefer g to f. Both $f > g$ and $g > f$ violate reduction, which requires $f \sim g$ since $\pi_f = \pi_g$.

3. Nonlinear preference under risk

All generalizations of the Bernoullian and von Neumann-Morgenstern theories recounted here are designed to accommodate violations of independence. They are therefore nonlinear in the sense that the linearity property or the usual expectation operation is inappropriate for their representations. Some generalizations also accommodate violations of order and allow cyclic preferences.

We begin with three proposals for monetary outcomes that assume the order axiom A1 and agree with first-degree stochastic dominance stated in the decumulative mode with $p^+(x) = p((x,\infty))$ as the

$$\text{FSD principle: } p^+ \neq q^+ \text{ and } p^+(x) \geqslant q^+(x) \text{ for all } x \Rightarrow p > q$$

Here, $p^+(x) = 1 - F(x)$, where F is the usual cumulative distribution function based on p. All three proposals presume continuity conditions sufficient for the existence of a functional V on P for which

$$p > q \Leftrightarrow V(p) > V(q)$$

Allais (1953, 1979a) adopts Bernoulli's riskless intensive approach for outcome utility and assumes (Allais, 1979b; Hagen, 1972) that a constant increase in the utilities of all outcomes increases V by the same amount: $[\delta > 0, p(A) = p'(\{y{:}u(y) = u(x) + \delta, x \in A\})$ for all measurable $A] \Rightarrow V(p') = V(p) + \delta$. This then leads to

$$V(p) = \int_X u(x)dp(x) + \theta(p^*)$$

where θ is a functional and p^* is the measure induced by p on the differences $u(x) - \int u(x)dp(x)$ of utilities from the Bernoullian expected utility of p. This reduces to Bernoulli's form if θ vanishes, but Allais (1953, 1979a) and Hagen (1972) argue that θ ought to depend on at least the second and third moments of p^*.

Machina (1982a) proposes a generalization of von Neumann-Morgenstern expected utility for monetary outcomes in $[0, M]$ that takes V as Fréchet differentiable on P with respect to the norm $\|\lambda(p - q)\| = |\lambda| \int_0^M |p([0,x]) - q([0,x])| dx$. This gives

$$V(p) - V(q) = \int_0^M u(x;q)(dp(x) - dq(x)) + o(\|p - q\|)$$

where $u(x;q)$ is absolutely continuous on $[0,M]$ and the o functional approaches 0 more rapidly than its argument. Consequently, V is nearly linear locally and this allows Machina to derive economically interesting results that are implied by the less general expected utility form (Machina, 1982a, 1982b, 1983a, 1983b). The FSD principle holds if $u(x;q)$ strictly increases in x.

The third proposal for monetary outcomes that is designed to satisfy the FSD principle takes

$$V(p) = \int_X \tau(p^+(x))du(x)$$

where τ is a transformation of upper-tail (decumulative) probabilities that is continuous and nondecreasing from $[0, 1]$ onto $[0, 1]$, and u is increasing and differentiable. This reduces to the von Neumann-Morgenstern form if τ is the identity function. A first axiomatization was sketched by Quiggin (1982) with $\tau(\frac{1}{2}) = \frac{1}{2}$, a restriction later removed (Chew, 1984; Segal, 1984). Yaari (1987) axiomatizes the special case in which $u(x) = x$, thus turning the von Neumann-Morgenstern approach on its side to yield a representation linear in money instead of linear in probability.

We now turn to proposals for arbitrary outcomes, focusing on theories that retain some vestige of linearity. The earliest of these has a nontransitive representation suggested by Kreweras (1961) and later axiomatized independently by

Fishburn (1982b). It is known as SSB utility theory since it represents preference by a skew-symmetric bilinear functional ϕ on $P \times P$. We recall that ϕ is skew-symmetric if $\phi(q,p) = -\phi(p,q)$ for all p, $q \in P$, and is bilinear if it is linear separately in each argument.

Fishburn's axioms for the SSB theory are, for all $p, q, r \in P$ and all $0 < \lambda < 1$:

B1 (continuity) $p > q > r \Rightarrow q \sim \alpha p + (1 - \alpha)r$ for some $0 < \alpha < 1$

B2 (convexity) $\{p > q; p(> \text{ or } \sim)q\} \Rightarrow p > \lambda q + (1 - \lambda)r$; $\{p \sim q, p \sim r\}$
$\Rightarrow p \sim \lambda q + (1 - \lambda)r$; $\{q > p; r(> \text{ or } \sim)p\}$
$\Rightarrow \lambda q + (1 - \lambda)r > p$

B3 (symmetry) $\{p > q > r; p > r; q \sim \frac{1}{2}p + \frac{1}{2}r\} \Rightarrow [\lambda p + (1 - \lambda)r \sim \frac{1}{2}p + \frac{1}{2}q$
$\Leftrightarrow \lambda r + (1 - \lambda)p \sim \frac{1}{2}r + \frac{1}{2}q]$

Convexity entails the dominance condition that if p is better than (worse than) both q and r, then p is better than (worse than) every convex combination of q and r. The symmetry condition is a special case of the principle that says that if q is a preference midpoint between p and r in the sense that $q \sim \frac{1}{2}p + \frac{1}{2}r$, then every $\sim$ statement confined to the convex hull of $\{p, q, r\}$ remains at $\sim$ when p and r are interchanged throughout.

Without B3, B1 and B2 yield a functional ϕ on $P \times P$ that is linear in its first argument and satisfies $p > q \Leftrightarrow \phi(p, q) > 0$ and $\phi(p, q) > 0 \Leftrightarrow \phi(q, p) < 0$, provided that P has no most-preferred or least-preferred measure. With symmetry, this ϕ can be made skew-symmetric.

Theorem 2. There exists a skew-symmetric bilinear functional ϕ on $P \times P$ such that, for all $p, q \in P$,

$$p > q \Leftrightarrow \phi(p, q) > 0$$

if and only if B1, B2, and B3 hold. Moreover, ϕ is unique up to multiplication by a positive real number.

A proof appears in Fishburn (1982b) along with comments on the effects of B1 and B2 by themselves.

The representation of Theorem 2 implies that each of $\{p: p > q\}$, $\{p: p \sim q\}$, and $\{p: q > p\}$ is a convex subset of P for every q, and it allows preference cycles such as $p > q > r > p$. It reduces to the von Neumann-Morgenstern linear representation if and only if $\phi(p, q) + \phi(q, r) + \phi(r, p) = 0$ for all $p, q, r \in P$, in which case ϕ decomposes as $\phi(p, q) = u(p) = u(q)$.

Assuming that every one-point measure is in P, and defining ϕ on $X \times X$ by $\phi(x, y) = \phi(p, q)$ when $p(x) = q(y) = 1$, bilinearity implies for simple p, $q \in P$ that

$$\phi(p, q) = \sum_{x \in X} \sum_{y \in X} \phi(x, y)p(x)q(y)$$

Hence, $\phi(p, q)$ is the expected value of $\phi(x, y)$ with respect to the product measure $p \times q$. Additional conditions (Fishburn, 1984a) are needed to obtain $\phi(p, q) = \iint \phi(x, y)dp(x)dq(y)$ for arbitrary measures.

If the von Neumann-Morgenstern order axiom A1 is added to the SSB axioms, then (Fishburn, 1983a) the SSB functional of Theorem 2 decomposes as $\phi(p, q) = u(p)w(q) - u(q)w(p)$.

Theorem 3. There exist linear functionals u and w on P with $w \geqslant 0$ and $w(p) > 0$ for every p in $\{p: q > p > q' \text{ for some } q, q' \in P\}$ such that, for all $p, q \in P$

$$p > q \Leftrightarrow u(p)w(q) > u(q)w(p)$$

if and only if A1, B1, B2, and B3 hold.

This weighted linear representation reduces to the von Neumann-Morgenstern form if the weighting functional w is constant. Its uniqueness specifications are most easily explained by removing the requirements on the sign of w. Then, given that $>$ is not empty, and assuming that the weighted linear representation holds for linear u and w, it also holds for linear u' and w' $[p > q \Leftrightarrow u'(p)w'(q) > u'(q)w'(p)]$ if and only if there are real numbers a, b, c and d such that $u' = au + bw$, $w' = cu + dw$, and $ad > bc$. Moreover, the generalization $p > q \Leftrightarrow u(p)w(q) > u(q)w(p)$ with u and w linear but unrestricted as to sign, is equivalent to the weighted linear representation if and only if $\{(u(p), w(p)): p \in P\}$ does not contain the origin.

Weighted linear utility was first axiomatized by Chew and MacCrimmon (1979). Their axioms were subsequently refined by Chew (1982, 1983) and others (Fishburn, 1983a; Nakamura, 1984, 1985). If w in the representation is positive everywhere, then

$$p > q \Leftrightarrow \frac{u(p)}{w(p)} > \frac{u(q)}{w(q)}$$

thus separating p and q. If, in addition, υ is defined as u/w, then the representation can be expressed by $p > q \Leftrightarrow \upsilon(p) > \upsilon(q)$ along with

$$\upsilon(\lambda p + (1 - \lambda)q) = \frac{\lambda w(p)\upsilon(p + (1 - \lambda)w(q)\upsilon(q)}{\lambda w(p) + (1 - \lambda)w(q)}$$

a form often seen in the literature. Although I know of no precedent to weighted linear utility prior to Chew and MacCrimmon (1979), the quotient form $u(p)/w(p)$ brings to mind Bolker's (1966, 1967) axiomatization for $>$ on $\mathscr{A}\backslash\{\emptyset\}$, where $\mathscr{A}$ is a complete, atom-free Boolean algebra. Bolker's axiom's imply that there are countably additive measures σ and ρ on $\mathscr{A}$ with $\rho > 0$ on $\mathscr{A}\backslash\{\emptyset\}$ such that, for all A, $B \in \mathscr{A}\backslash\{\emptyset\}$,

$$A > B \Leftrightarrow \frac{\sigma(A)}{\rho(A)} > \frac{\sigma(B)}{\rho(B)}$$

This involves quotients of measures rather than quotients of linear functionals, with additivity rather than linearity the key property.

Most of the literature cited in this section, plus Fishburn (1984c), includes economically interesting applications of the proposals sketched here and demonstrates the particular ways that their representations accommodate violations of independence and/or order. The latter aspect is portrayed in Figure 1 by indifference lines through the convex hull of $\{p, q, r\}$ represented barycentrically. We take $p > q > r$ in all cases; arrows show directions of decreasing preference.

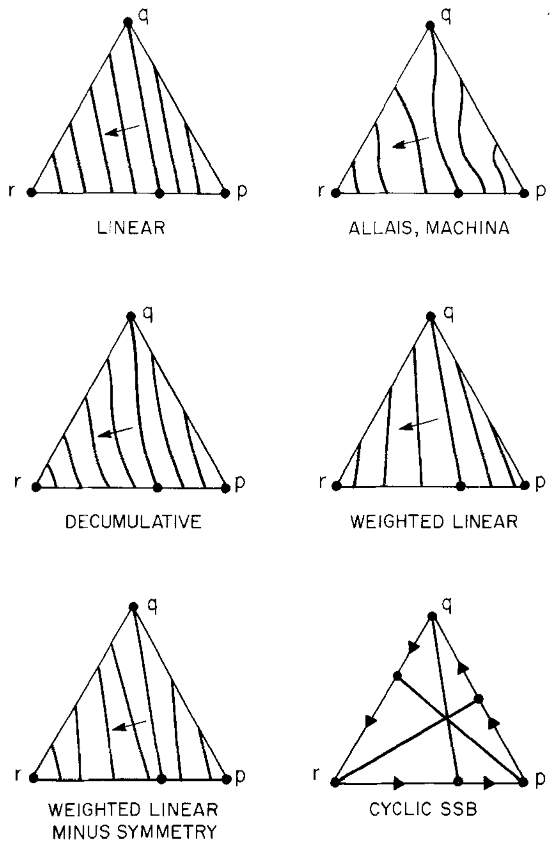

Fig. 1

The indifference lines for the von Neumann-Morgenstern linear representation are parallel and straight. Those for Allais and Machina can be curved. This it true also for the decumulative representation, but its functional form enforces more regularity. The weighted linear lines are straight and intersect at a point outside the triangle if they are not parallel. The lower left diagram shows the effect of dropping symmetry from the weighted linear axioms. The cyclic SSB case, shown in the lower right, has a unique point in the interior of the triangle that is indifferent to every other point.

4. Nonlinear preference under uncertainty

Generalizations of and alternatives to Savage's expected utility representation can be partitioned into those that assume additivity for subjective probability and those that do not. We begin with additive proposals.

Allais (1953, 1979a, 1979b) strongly rejects Savage's theory not only because of its independence assumptions and non-Bernoullian assessment of outcome utility but also because of Savage's connection between subjective probability and preference. Allai's view of subjective probability is based on direct comparisons between uncertain events and events based on the classical notion of equally likely cases, operationalized by drawing balls from an urn. For Allais, probability is to be assessed independently of preference in much the same way that utility is assessed independently of probability. Once additive subjective probabilities have been assessed, Allais uses the reduction principle to reduce acts to measures in P, then applies his approach to preference under risk as described in the preceding section. He therefore accepts the order axiom and the reduction principle, but renounces Savage's substitution principle as well as the von Neumann-Morgenstern independence axiom.

Another theory that adopts additive subjective probability and the Bernoullian riskless intensity notion but differs significantly from Allais in other ways has been proposed by Loomes and Sugden (1982, 1987) and Bell (1982). Unlike Allais, they accept Savage's substitution principle but reject order and the reduction principle. Their general representation is

$$f > g \Leftrightarrow \int_S \phi(f(s),g(s))d\pi(s) > 0$$

where ϕ is a skew-symmetric functional on $X \times X$ that is designed to accommodate a regret/rejoicing factor along with the usual Bernoullian riskless intensity notion. Suppose, for example, that f is chosen from $\{f, g\}$ and s obtains. Then, if $f(s) > g(s)$, one might rejoice at one's good fortune, but experience regret if $g(s) > f(s)$. Depending on ϕ, the representation accommodates either $f > g$ or $g > f$ for the die example at the end of Section 2, and it allows cyclic preference patterns. However, it does not account for the phenomenon illustrated by Ellsberg's example in Section 2.

Although Bell (1982) and Loomes and Sugden (1982, 1987) do not provide an axiomatization for the representation of the preceding prargraph, it is shown in Fishburn (1986c) that this representation for simple (finite image) acts follows from slight changes in Savage's axioms that include weakening of transitivity. However, under the Savage approach, there is no explicit notion of intensity measurement or regret. Hence, the modified Savage approach to $f > g \Leftrightarrow \int \phi(f(s), g(s))\, d\pi(s) > 0$ relates to the Bell and Loomes-Sugden interpretation in much the same way that the von Neumann-Morgenstern approach relates to the Bernoullian interpretation of expected utility.

Gilboa (1987) makes more drastic revisions in Savage's axioms to accommodate Ellsberg's phenomenon with ordered preferences. His representation is

$$f > g \Leftrightarrow \int_S u(f(s))d\sigma(s) > \int_S u(g(s))d\sigma(s)$$

where u is a functional on X, σ is a monotonic $[A \subseteq B \Rightarrow \sigma(A) \leqslant \sigma(B)]$ but not necessarily additive probability measure on 2^S, and integration is taken in Choquet's sense (Choquet, 1955; Schmeidler, 1986) with

$$\int_S w(s)d\sigma(s) = \int_{c=0}^{\infty} \sigma(\{s\colon w(s) \geqslant c\})dc - \int_{c=-\infty}^{0} [1 - \sigma(\{s\colon w(s) \geqslant c\})]dc$$

The Choquet integral as thus defined is simply a generalization of the expectation operation that is applicable to nonadditive but monotonic probabilities. Gilboa's axioms imply that u is bounded and unique up to a positive affine transformation and that σ is unique. He notes in a sequel (Gilboa, 1985) that equivalence between maximization of $\int u d\sigma$ and minimization of $\int(-u)d\sigma$ forces one to adopt the complementary additivity condition $\sigma(A) + \sigma(S \backslash A) = 1$.

The preceding representations for modifications of Savage's axioms have also been axiomatized in a lottery acts formulation developed by Anscombe and Aumann (1963), Pratt, Raiffa, and Schlaifer (1964), and Fishburn (1970, 1982a). In this formulation, $>$ is applied to the set $\mathbf{F} = P_0^S$ of lottery acts $\mathbf{f}, \mathbf{g}, \ldots$, each of which assigns a lottery in P_0 to each state in S. The set P_0 is viewed as the set of simple probability measures on X whose probabilities are generated by random devices and are not to be confused with subjective probabilities for S. The basic lottery acts representation (Fishburn, 1970, 1982a) that corresponds to Savage's model is

$$\mathbf{f} > \mathbf{g} \Leftrightarrow \int_S u(\mathbf{f}(s))d\pi(s) > \int_S u(\mathbf{g}(s))d\pi(s)$$

with u a linear functional on P_0 and with π additive. Axioms for the skew-symmetric bilinear generalization

$$\mathbf{f} > \mathbf{g} \Leftrightarrow \int_S \sigma(\mathbf{f}(s), \mathbf{g}(s))d\pi(s) > 0$$

are presented in Fishburn (1984b) and Fishburn and LaValle (1987a), with additional discussion in Fishburn and LaValle (1987b). Axioms for

$$\mathbf{f} > \mathbf{g} \Leftrightarrow \int_S u(\mathbf{f}(s))d\sigma(s) > \int_S u(\mathbf{g}(s))d\sigma(s)$$

with u linear and σ monotonic are given by Schmeidler (1984).

Other proposals with nonadditive subjective probability are described in Davidson and Suppes (1956), Fishburn (1983b, 1986a), Luce and Narens (1985), and Wakker (1986).

5. Conclusions

During the past generation, there has been intense activity in the foundations of the theory of rational preference and choice under risk and uncertainty. The expected utility theories of von Neumann and Morgenstern in 1944 and Savage in 1954 ignited a host of theoretical and applied studies along with empirical research to assess both the descriptive validity and normative acceptability of their theories. Much of the experimental work has confirmed what a lot of people, including the founders, suspected all along—that expected utility is not a terribly good predictor of actual choice behavior, but in the process it has given rise to exciting new descriptive theories, exemplary among which is the prospect theory of Daniel Kahneman and Amos Tversky (Kahneman and Tversky, 1979; Tversky and Kahneman, 1987).

The empirical studies have also had an impact on principles regarded as acceptable criteria for carefully reasoned preferences and choices. As the evidence accumulated throughout the 1970s, some people found it necessary to reexamine what they once held to be unassailable normative standards. This reexamination has led to an abundance of normative alternatives to expected utility, and it is these proposals that I have spoken about in the paper. Every topic discussed here is analyzed in much greater depth in Fishburn (1988).

Finally, where does this leave us? I submit two guesses. First, many of the main steps toward viable normative alternatives to expected utility have already been taken. More proposals will come along, but the general outlines of the territory are visible.

Second, I would guess that the next two decades or so will see numerous refinements, applications, and experimental analyses of the types of representations described here. This will be a time of shakedown and sifting. There are some people, including John Pratt (1986) and Howard Raiffa (1968), who have maintained all along that expected utility theory is the only adequate normative theory for preference under risk or uncertainty and would surely hope that the rest of us will come to our senses in the years ahead. We shall have to wait and see.

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