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Solution of integral equations for plane elastostatical problems with discontinuously prescribed boundary values

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ABSTRACT

Engineers are often confronted with boundary value problems of plane elastostatics where the boundary tractions or displacements or their derivatives have a jump. The discontinuities represent by no means impediments to treating the problems economically with the aid of integral equations. However, it is necessary to know the structure of the solutions before starting numerical calculations.

In this paper singular and regular integral equations of the second and of the first kind are investigated by methods which are based mainly on mechanical ideas. The essential terms of the solutions are determined for boundary values with a jump or a jumping derivative. The solutions contain both discontinuous or discontinuously differentiable terms and also logarithmically diverging terms.

Particular attention is paid to the most frequently applied integral equation of the indirect method for the plane problem with prescribed tractions. The solutions of this equation for elastic slices loaded by concentrated forces and moments are deduced as special cases of the general results.

(For an extensive survey of this paper: see Chapter 1.)

ZUSAMMENFASSUNG

Ingenieure werden oft mit Randwertproblemen der ebenen Elastostatik konfrontiert, bei denen die Randspannungen oder Randverschiebungen oder deren Ableitungen Sprünge aufweisen. Die Unstetigkeiten stellen keine grundsätzlichen Hindernisse dar, die Probleme auf ökonomische Weise mit Hilfe von Integralgleichungen zu behandeln. Jedoch ist es dazu unabdingbar, die Struktur der Lösungen zu kennen, bevor man mit den numerischen Rechnungen anfängt.

In diesem Aufsatz werden singuläre und reguläre Integralgleichungen zweiter und erster Art mit Methoden untersucht, die hauptsächlich auf mechanischen Gesichtspunkten basieren. Die wesentlichen Terme der Lösungen werden für Randwerte mit einem Sprung oder einer unstetigen Ableitung bestimmt. Die Lösungen enthalten sowohl unstetige als auch unstetig differenzierbare Summanden und darüber hinaus auch logarithmisch divergierende Terme.

Besondere Aufmerksamkeit wird der am häufigsten benutzten Integralgleichung der indirekten Methode für das ebene Problem mit vorgeschriebenen Spannungen geschenkt. Die Lösungen dieser Integralgleichung werden für elastische, durch Einzelkräfte und Einzelmomente belastete Scheiben als Spezialfälle der allgemeinen Resultate hergeleitet.

(In Kapitel 1 befindet sich ein ausführlicher Überblick über den Aufsatz.)

1. Introduction, survey

Boundary value problems of plane elastostatics can be formulated in various manners as integral equations. In this paper we consider integral equations representing indirect boundary element method (see [1–12]) statements. The equations are of the first and of the second kind and have regular kernels or kernels with poles of the first order. Integral equations with logarithmically singular kernels are not investigated at all and equations representing direct boundary element method (see [13–18]) statements are only investigated concisely by the example of Rizzo's equation for the problem with given boundary tractions.

Dealing with practically occurring engineering problems the prescribed boundary tractions or boundary displacements or their derivatives are generally not continuous. As a consequence, the solutions of the integral equations or their derivatives contain discontinuous and even singular terms. In this paper the discontinuities and singular summands of the solutions are calculated. The impulse for the investigations originates in a paper by Kompiš [19] in which the solution of an integral equation for the circular elastic disc with discontinuous boundary tractions is given.

The aim of Chapter 2 of this paper is to demonstrate that the continuity properties of the solutions of integral equations for elastostatical boundary value problems are not only of theoretical interest but that to know them in advance can be advantageous for effective numerical treatment of the problems.

Chapter 3 deals with a well-known and frequently used singular integral equation of the second kind for the circular elastic disc with prescribed boundary tractions. Kelvin's solution represents the kernel and the sought function of the equation can be interpreted physically as a layer of forces. Since in this particular case the eigenvalues and eigenfunctions of the integral operator are well-known [2, 20] we succeed in representing the solution as an integral transformation of the boundary tractions.

In Chapter 4 we prescribe tractions with a jump of the n-th derivative (the special case $n = 0$ constitutes a discontinuity of the tractions) at the boundary of the circle and determine the corresponding solution of the integral equation with the aid of the formula deduced in Chapter 3. The solution does not only contain a summand, the n-th derivative of which is discontinuous, but also a term with a logarithmically divergent n-th derivative.

In Chapter 5 we prove that the results obtained in the preceding chapter are not only valid for the circular disc but also for arbitrarily shaped slices with the boundary analytical in the neighbourhood of the point at which the boundary values have a discontinuity. In Chapter 4 the solution of a special integral equation for the statical boundary value problem has been deduced. In Chapter 5 we investigate a whole class of integral equations for the problems with prescribed boundary tractions and with prescribed boundary displacements the kernels of which have a pole of the first order or are regular.

In Chapter 6 the results of Chapter 5 are interpreted extensively. Furthermore the solution of the inverse problem is given, i.e. the formula for the tractions caused by a layer of forces with a prescribed discontinuity of the n-th derivative.

In Chapter 7 the solutions of the integral equations are determined for boundary loads containing concentrated forces and concentrated moments.

Chapter 8 contains a short remark on the treatment of the three-dimensional problem.

In Chapter 9 the solution of Rizzo's integral equation (i.e. of an integral equation of the direct method) for prescribed boundary tractions with a jump of the n-th derivative is discussed.

2. Purposes for application of the results of this paper

For the treatment of elastostatical boundary value problems with the aid of an integral equation it can often be of decisive importance to know the structure of the solution before beginning with numerical calculations. It is true that a finished program almost always yields a solution but by no means in every case is this solution determined in an economic manner, i.e. often the accuracy of the results does not justify the expense of the evaluations.

The most frequently used numerical methods coincide in various aspects with ordinary quadrature procedures. For discretization of the problem the boundary of the elastic body is divided into a number of elements and the sought function – a layer of forces or other singularities – is interpolated piecewise over each element. The individual numerical methods differ by the degree of the interpolating functions. In general the calculatory expense increases with the degree, and programming becomes more complicated. The greater expense can only be justified if the accuracy of the results increases appropriately. However, it is by no means obvious that polynomials of a higher degree automatically entail more precise results, or even that they yield reasonable results at all. (The circumstances are similar as in ordinary integration. For example: the midpoint rule yields less accurate results than Simpson's rule for analytical integrands (error order: $O(h^2)$ and $O(h^4)$ respectively). However, in the case of $\int_0^1 \sqrt{x}\,dx$ the midpoint rule is superior to Simpson's rule (error order $O(h^{3/2})$ for both rules) and Euler's formula even fails completely.) Up to now the degree of interpolation for the numerical treatment of integral equations for elastostatical problems has been chosen mostly at will. In order to settle the question as to which degree of polynomials is to be most economically applied the differentiability properties of the solution must be predicted.

As in ordinary integration the Richardson extrapolation can be applied effectively for the determination of extremely accurate solutions. To that end the integral equation is not only solved once but several times with interpolations of the same degree but with differently fine divisions of the boundary into elements. By suitable linear combination of these approximative solutions one obtains an "improved solution" which is generally far more accurate than the best of the directly calculated solutions. For application of the Richardson extrapolation one has to know which kinds of terms occur in the asymptotic expansion of the error of the numerical solution. In turn, the knowledge of the discontinuous and singular summands of the exact solution is a significant presupposition for establishing the asymptotic expansion.

If the differentiability properties of the solution are well-known another promising approach to the problem suggests itself. The solution can be split up into a well-known part containing the discontinuities and singularities and an unknown part. The numerical determination of the unknown part does not involve particular difficulties because of the smoothness of the interesting functions.

In the preceding has been revealed the great importance of knowing the differentiability properties of the exact solution of the integral equation. In the following are described in brief the causes of the discontinuous and singular terms of it. The solution is determined firstly by the shape of the boundary of the elastic body and secondly by the boundary tractions and displacements. If the boundary and the boundary values are analytical the solution is too arbitrarily often differentiable. However, in engineering problems the boundary generally does not represent an analytic curve. It may have corners. At points at which two circular arcs with different radii join each other tangentially the curvature of the boundary is discontinuous. Also the boundary values or its derivatives are generally not prescribed continuously. Often the boundary tractions have a jump or even the boundary is charged by concentrated forces or moments. Consequently, it must be settled in which manner discontinuities of the boundary and of the boundary values influence the exact solution of the integral equation. In anticipation of the results of the following chapters we already mention here that a discontinuity of a certain derivative of the boundary or of the boundary values not only entails a discontinuity of the corresponding derivative of the solution but also a singular term. In this paper we investigate only the effects of discontinuities of the boundary tractions and displacements and of their derivatives on the solution and presuppose that the boundary is analytical at, and in the neighbourhood of, the point at which the boundary values are discontinuous.

At the end of this chapter we warn about a fairly frequently used trick to seemingly circumvent the difficulties caused by discontinuities of the boundary or the boundary values. The essence of this trick consists of, for example, replacing a corner of the boundary by a circular arc with a small radius or a jump of the boundary tractions by tractions with a large but finite gradient. The solution of the so constructed substitute problem has finite values where that of the original one diverges. Now, it is true that the difference between the stress fields of the substitute problem and the original problem can actually be tolerated in many cases. However, the artificially finitely made values are generally still much larger than those of the solution at other parts of the boundary. Therefore the numerical determination of the solution of the substitute problem often turns out to be problematic.

In some cases it may even be advantageous to apply the trick the other way round, i.e. to replace, for example, boundary tractions with a large gradient by tractions with a jump.

3. Solution of an integral equation for a circular elastic disc with prescribed boundary tractions by an expansion into eigenfunctions

We consider the following singular integral equation of the second kind for the statical boundary value problem of plane elastostatics (plane stress)

$$\tfrac{1}{2}\delta_{jk}R_k(s)+|\oint|\,(\Pi R)_{j.k}(x_t(s),\bar{x}_t(\bar{s}))R_k(\bar{s})\,\mathrm{d}\bar{s}=\Pi_j(s) \tag{3.1}$$

which has been explained extensively in [10] (see [20] Eqs. (3.45), (3.16)). The symbol $|\oint|$ means that the integral is defined as the Cauchy principal value. On the right hand side of Eq. (3.1) the given boundary traction Π_j is prescribed as a function of the arc length s of the boundary. The sought function R_k can be interpreted as a layer of forces. The tensorial kernel $(\Pi R)_{j.k}$ describes the stress vector caused by a concentrated force acting at a point of the infinite elastic medium (Kelvin's solution). It depends on the difference of the position vectors x_t and $\bar{x}_t$ of the field and of the source point:

$$(\Pi R)_{j.k}(x_t,\bar{x}_t)=-\{(n_iq_i/\rho)[(m-1)\delta_{jk}+2(m+1)q_jq_k] \\ +(e_{ti}n_iq_t/\rho)(m-1)e_{jk}\}/(4\pi m), \tag{3.2}$$

$$\tilde{x}_i=x_i-\bar{x}_i,\qquad \rho^2=\tilde{x}_i\tilde{x}_i,\qquad q_i=\tilde{x}_i/\rho. \tag{3.3), (3.4), (3.5}$$

δ_{ij} is the identity tensor and e_{ij} the permutation tensor

$$\delta_{11}=\delta_{22}=e_{12}=-e_{21}=1,\qquad \delta_{12}=\delta_{21}=e_{11}=e_{22}=0 \tag{3.6), (3.7}$$

and n_i is the normal vector of the boundary at the field point and m is Poisson's ratio.

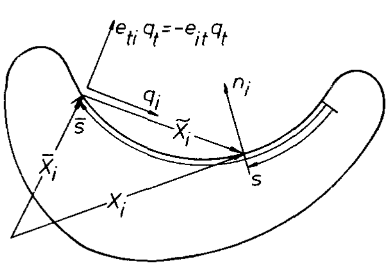


Figure 1. Problem geometry.

In the special case of a circular boundary Eqs. (3.1) and (3.2) change into (see [20] Eqs. (3.63), (3.54))

$$\tfrac{1}{2}\boldsymbol{R}_1(\phi)-\frac{1}{8\pi m}\oint\left\{(3m-1)\cos(\bar{\phi}-\phi)-2\right\}\boldsymbol{R}_1(\bar{\phi})\,\mathrm{d}\bar{\phi} \\ -\frac{1}{8\pi m}|\oint|\left\{-(3m-1)\sin(\bar{\phi}-\phi)+(m-1)\operatorname{ctg}\frac{\bar{\phi}-\phi}{2}\right\}\boldsymbol{R}_2(\bar{\phi})\,\mathrm{d}\bar{\phi}=\boldsymbol{\Pi}_1(\phi) \tag{3.8a}$$

$$\tfrac{1}{2}\boldsymbol{R}_2(\phi)-\frac{1}{8\pi m}|\oint|\left\{(3m-1)\sin(\bar{\phi}-\phi)-(m-1)\operatorname{ctg}\frac{\bar{\phi}-\phi}{2}\right\}\boldsymbol{R}_1(\bar{\phi})\,d\bar{\phi}$$
$$-\frac{1}{8\pi m}\oint\left\{(3m-1)\cos(\bar{\phi}-\phi)+2m\right\}\boldsymbol{R}_2(\bar{\phi})\,d\bar{\phi}=\boldsymbol{\Pi}_2(\phi) \quad (3.8b)$$

where ϕ is the central angle. The bold-face types $\boldsymbol{R}_i$ and $\boldsymbol{\Pi}_i$ shall indicate that the vector components are not related here to a space-fixed Cartesian co-ordinate system but to a local basis. For example, $\boldsymbol{\Pi}_1$ is the normal component and $\boldsymbol{\Pi}_2$ is the tangential component of the stress vector (see [20], Fig. 3). If $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ are functions of the form $\sin(\mu\phi)$ or $\cos(\mu\phi)$, ($\mu>1$ integer) some integrals in Eqs. (3.8a), (3.8b) vanish (see [20], Eqs. (4.11)–(4.13)) and we obtain:

$$\tfrac{1}{2}\boldsymbol{R}_1(\phi)-\frac{m-1}{8\pi m}|\oint|\operatorname{ctg}\frac{\bar{\phi}-\phi}{2}\boldsymbol{R}_2(\bar{\phi})\,d\bar{\phi}=\boldsymbol{\Pi}_1(\phi), \quad (3.9a)$$

$$\tfrac{1}{2}\boldsymbol{R}_2(\phi)+\frac{m-1}{8\pi m}|\oint|\operatorname{ctg}\frac{\bar{\phi}-\phi}{2}\boldsymbol{R}_1(\bar{\phi})\,d\bar{\phi}=\boldsymbol{\Pi}_2(\phi). \quad (3.9b)$$

The eigenvalue problem corresponding with Eq. (3.1) for the circular disc with radius a (i.e. the problem corresponding with Eq. (3.8)) reads as follows:

$$\tfrac{1}{2}\boldsymbol{R}_j(\phi)+|\oint|(\boldsymbol{\Pi}R)_{j,k}(\phi,\bar{\phi})\boldsymbol{R}_k(\bar{\phi})\cdot a\,d\bar{\phi}=\Lambda\boldsymbol{R}_j(\phi) \quad \text{i.e.,} \quad (3.10)$$

$$\boldsymbol{\Pi}_j(\phi)=\Lambda\boldsymbol{R}_j(\phi), \qquad \boldsymbol{R}_j(\phi)=\boldsymbol{\Pi}_j(\phi)/\Lambda. \quad (3.11a),(3.11b)$$

In [20] the eigenvalues Λ and the vectorial eigenfunctions $\boldsymbol{R}_j$ have been calculated (see [21] Eq. (13)). The vector components of the eigenfunctions are trigonometric functions of the argument $\mu\phi$ where μ is a non-negative integer. There exist four eigenfunctions for each number $\mu\geq 1$ and two eigenfunctions for $\mu=0$. The eigenvalues are compiled beneath the corresponding eigenfunctions in (3.12). The eigenvalues do not depend explicitly on the parameter μ and apart from that most of them coincide.

$$\left.\begin{array}{ccccl}
\begin{bmatrix}\sin\mu\phi\\ \cos\mu\phi\end{bmatrix} & \begin{bmatrix}\cos\mu\phi\\ \sin\mu\phi\end{bmatrix} & \begin{bmatrix}\sin\mu\phi\\ -\cos\mu\phi\end{bmatrix} & \begin{bmatrix}\cos\mu\phi\\ -\sin\mu\phi\end{bmatrix} & \\
\dfrac{3m-1}{4m} & \dfrac{m+1}{4m} & \dfrac{m+1}{4m} & \dfrac{3m-1}{4m} & \text{for } \mu>1\\
0 & \dfrac{m+1}{4m} & \dfrac{m+1}{4m} & 0 & \text{for } \mu=1\\
\begin{bmatrix}1\\ 0\end{bmatrix} & \begin{bmatrix}0\\ 1\end{bmatrix} & & & \\
\dfrac{m+1}{2m} & 0 & & &
\end{array}\right\} \quad (3.12)$$

We expand the boundary tractions $\boldsymbol{\Pi}_j$ prescribed on the right-hand side of Eq. (3.8) into a Fourier series and rearrange the terms into summands representing eigenfunctions:

$$\boldsymbol{\Pi}_j(\phi) \triangleq \begin{bmatrix} a_0 \\ c_0 \end{bmatrix} + \begin{bmatrix} a_1 \cos \phi + b_1 \sin \phi \\ c_1 \cos \phi + d_1 \sin \phi \end{bmatrix} + \sum_{\mu=2}^{\infty} \begin{bmatrix} a_\mu \cos \mu\phi + b_\mu \sin \mu\phi \\ c_\mu \cos \mu\phi + d_\mu \sin \mu\phi \end{bmatrix}$$

$$= a_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{1}{2}\left\{ (b_1 + c_1) \begin{bmatrix} \sin \phi \\ \cos \phi \end{bmatrix} + (a_1 + d_1) \begin{bmatrix} \cos \phi \\ \sin \phi \end{bmatrix} \right.$$

$$\left. + (b_1 - c_1) \begin{bmatrix} \sin \phi \\ -\cos \phi \end{bmatrix} + (a_1 - d_1) \begin{bmatrix} \cos \phi \\ -\sin \phi \end{bmatrix} \right\}$$

$$+ \sum_{\mu=2}^{\infty} \frac{1}{2} \left\{ (b_\mu + c_\mu) \begin{bmatrix} \sin \mu\phi \\ \cos \mu\phi \end{bmatrix} + (a_\mu + d_\mu) \begin{bmatrix} \cos \mu\phi \\ \sin \mu\phi \end{bmatrix} \right.$$

$$\left. + (b_\mu - c_\mu) \begin{bmatrix} \sin \mu\phi \\ -\cos \mu\phi \end{bmatrix} + (a_\mu - d_\mu) \begin{bmatrix} \cos \mu\phi \\ -\sin \mu\phi \end{bmatrix} \right\}. \tag{3.13}$$

The eigenfunctions corresponding with zero eigenvalues vanish in (3.13) if the resultant and the resultant moment of the boundary tractions $\boldsymbol{\Pi}_j$ vanish (see [20]), i.e.:

$$c_0 = b_1 + c_1 = a_1 - d_1 = 0. \tag{3.14}$$

For determining the solution $\boldsymbol{R}_j$ of integral equation (3.8) we have to divide – according to Eq. (3.11b) – the eigenfunctions in the expansion (3.13) of the boundary tractions $\boldsymbol{\Pi}_j$ by the corresponding eigenvalues:

$$\boldsymbol{R}_j(\phi) \triangleq \frac{2m}{m+1} \left\{ a_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (a_1 + d_1) \begin{bmatrix} \cos \phi \\ \sin \phi \end{bmatrix} + (b_1 - c_1) \begin{bmatrix} \sin \phi \\ -\cos \phi \end{bmatrix} \right\}$$

$$+ \sum_{\mu=2}^{\infty} \left\{ \frac{2m}{3m-1} (b_\mu + c_\mu) \begin{bmatrix} \sin \mu\phi \\ \cos \mu\phi \end{bmatrix} + \frac{2m}{m+1} (a_\mu + d_\mu) \begin{bmatrix} \cos \mu\phi \\ \sin \mu\phi \end{bmatrix} \right.$$

$$\left. + \frac{2m}{m+1} (b_\mu - c_\mu) \begin{bmatrix} \sin \mu\phi \\ -\cos \mu\phi \end{bmatrix} + \frac{2m}{3m-1} (a_\mu - d_\mu) \begin{bmatrix} \cos \mu\phi \\ -\sin \mu\phi \end{bmatrix} \right\}. \tag{3.15}$$

We obtain from Eq. (3.15) by rearranging the summands and by comparison with Eq. (3.13):

$$\boldsymbol{R}_j = A \cdot \boldsymbol{\Pi}_j - \pi \cdot B \sum_{\mu=1}^{\infty} \begin{bmatrix} -c_\mu \sin \mu\phi + d_\mu \cos \mu\phi \\ a_\mu \sin \mu\phi - b_\mu \cos \mu\phi \end{bmatrix} + 2\pi V a_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{3.17}$$

$$V = -\frac{1}{\pi} \frac{m}{3m-1}, \tag{3.16}$$

$$A = \frac{8m^2}{(3m-1)(m+1)}, \qquad B = -\frac{1}{2\pi} \frac{m-1}{m} A. \qquad (3.18), (3.19)$$

If the tangential component $\boldsymbol{\Pi}_2$ of the boundary traction vanishes, i.e. if $c_\mu = d_\mu =$

0 (see Eq. (3.13)) we obtain from Eqs. (3.17), (3.9) and (3.8):

$$\boldsymbol{R}_1(\phi) = A \cdot \boldsymbol{\Pi}_1(\phi) + V \oint \boldsymbol{\Pi}_1(\bar{\phi})\, d\bar{\phi}$$

$$\boldsymbol{R}_2(\phi) = -\frac{m-1}{4\pi m} \left| \oint \right| \operatorname{ctg} \frac{\bar{\phi}-\phi}{2} A \cdot \boldsymbol{\Pi}_1(\bar{\phi})\, d\bar{\phi}. \quad (3.20)$$

Analogously we have for $\boldsymbol{\Pi}_1 = 0$:

$$\boldsymbol{R}_2(\phi) = A \cdot \boldsymbol{\Pi}_2(\phi), \qquad \boldsymbol{R}_1(\phi) = \frac{m-1}{4\pi m} \left| \oint \right| \operatorname{ctg} \frac{\bar{\phi}-\phi}{2} A \cdot \boldsymbol{\Pi}_2(\bar{\phi})\, d\bar{\phi}. \quad (3.21)$$

Superposition of Eqs. (3.20) and (3.21) yields the solution for the case that both components $\boldsymbol{\Pi}_1$ and $\boldsymbol{\Pi}_2$ are different from zero:

$$\boldsymbol{R}_j(\phi) = A \cdot \boldsymbol{\Pi}_j(\phi) - \frac{B}{2} e_{jk} \left| \oint \right| \operatorname{ctg} \frac{\bar{\phi}-\phi}{2} \boldsymbol{\Pi}_k(\bar{\phi})\, d\bar{\phi} + V \oint \boldsymbol{\Pi}_j(\bar{\phi})\, d\bar{\phi}. \quad (3.22)$$

Using Eq. (3.22) instead of Eq. (3.17) we do not need to expand the boundary values $\boldsymbol{\Pi}_j$ into a Fourier series. The determination of the solution $\boldsymbol{R}_j$ of integral equation (3.9) is reduced to ordinary quadrature.

4. Solution of the integral equation for the circular disc with discontinuous tractions or tractions with discontinuous derivatives prescribed on the boundary

The n-th derivative of the boundary tractions $\boldsymbol{\Pi}_j$ ($n = 0$ or 1 or 2 or $\cdots$) is presupposed to have a discontinuity of magnitude $\Delta\boldsymbol{\Pi}_j^{(n)}$ at the point ϕ^*, i.e. we can split it up in the interval $[\phi^* - \varepsilon, \phi^* + \varepsilon]$ ($\varepsilon \ll 2\pi$) into two summands:

$$\boldsymbol{\Pi}_j^{(n)}(\phi) = \tilde{\boldsymbol{\Pi}}_j^{(n)}(\phi) + H(\phi - \phi^*)\Delta\boldsymbol{\Pi}_j^{(n)}, \quad (4.1)$$

$$\boldsymbol{\Pi}_j(\phi) = \tilde{\boldsymbol{\Pi}}_j(\phi) + H(\phi - \phi^*)\Delta\boldsymbol{\Pi}_j^{(n)}(\phi - \phi^*)^n/n!. \quad (4.2)$$

Outside the direct neighbourhood $[\phi^* - \varepsilon, \phi^* + \varepsilon]$ of the critical point ϕ^* we do not need to specify the second summand. The first summand $\tilde{\boldsymbol{\Pi}}_j$ is analytical (infinitely often differentiable) in the neighbourhood of ϕ^*, i.e. it contributes an analytical term to the solution and we may omit it from the beginning from our considerations:

$$\boldsymbol{\Pi}_j(\phi) = H(\phi - \phi^*)\Delta\boldsymbol{\Pi}_j^{(n)}(\phi - \phi^*)^n/n!. \quad (4.3)$$

$\Delta\boldsymbol{\Pi}_1^{(n)}$ is the jump of the n-th derivative of the normal component and $\Delta\boldsymbol{\Pi}_2^{(n)}$ that of the n-th derivative of the tangential component. H is Heaviside's function:

$$H(\phi - \phi^*) = \begin{matrix} 0 & \text{for} & \phi < \phi^* \\ 1 & \text{for} & \phi > \phi^* \end{matrix} \quad (4.4)$$

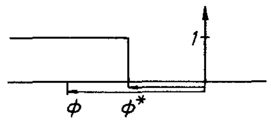


Figure 2. Heaviside's function $H(\phi - \phi^*)$.

i.e. we have

$$\Pi_j(\phi)=\begin{matrix}0\\ \Delta\Pi_j\end{matrix}\quad\text{or}\quad\begin{matrix}0\\ \Delta\Pi_j'\cdot(\phi-\phi^*)\end{matrix}\quad\text{or}\cdots\quad\begin{matrix}\text{for}\quad \phi^*-\varepsilon\le\phi<\phi^*\\ \text{for}\quad \phi^*<\phi\le\phi^*+\varepsilon.\end{matrix}\tag{4.5}$$

Since we are only interested in those summands of the solution which are non-analytical at the point ϕ^* we have only to extend the integration in Eq. (3.22) over the interval $[\phi^*-\varepsilon, \phi^*+\varepsilon]$, and because of the factor $H(\phi-\phi^*)$ in Eq. (4.3), even only over the interval $[\phi^*, \phi^*+\varepsilon]$. The value of the integral at the upper limit $\phi^*+\varepsilon$ of integration is of no importance. Eqs. (3.22) and (4.3) yield:

$$\boldsymbol{R}_j(\phi)=A\cdot\Pi_j(\phi)-\frac{B}{2}e_{jk}\,\Big|\!\!\int_{\phi^*}^{\cdots}\!\!\Big|\,\mathrm{ctg}\frac{\bar\phi-\phi}{2}\,\Delta\Pi_k^{(n)}\frac{(\bar\phi-\phi^*)^n}{n!}\,\mathrm{d}\bar\phi+V\int_{\phi^*}^{\cdots}\Delta\Pi_j^{(n)}\frac{(\bar\phi-\phi^*)^n}{n!}\,\mathrm{d}\bar\phi.\tag{4.6}$$

Inserting the expansion

$$\mathrm{ctg}\frac{\bar\phi-\phi}{2}=\frac{2}{\bar\phi-\phi}-\frac{\bar\phi-\phi}{6}-\frac{(\bar\phi-\phi)^3}{360}-\cdots\tag{4.7}$$

we obtain

$$\Big|\!\!\int_{\phi^*}^{\cdots}\!\!\Big|\,(\bar\phi-\phi^*)^n\,\mathrm{ctg}\frac{\bar\phi-\phi}{2}\,\mathrm{d}\bar\phi=-2(\phi-\phi^*)^n\ln|\phi-\phi^*|+\mathit{analyt.}(\phi).\tag{4.8}$$

In order not to overdo formalism we will denote in the following different functions analytical at the point s^* by the expression *analyt.*(s) and sometimes we will even totally omit these unimportant functions.

Eqs. (4.6) and (4.8) yield:

$$\boldsymbol{R}_j(\phi)=A\cdot\Pi_j(\phi)+B\cdot e_{jk}\cdot\Delta\Pi_k^{(n)}\frac{1}{n!}(\phi-\phi^*)^n\ln|\phi-\phi^*|.\tag{4.9}$$

5. Proof that the preceding results are valid for arbitrarily shaped slices; treatment of more general types of integral equations

Up to now we have determined the structure of the solution of integral equation (3.1) for a circular disc loaded by discontinuous boundary tractions or tractions with a discontinuous derivative. In this chapter we will generalize the investigations with respect to two aspects. First, we will show that the results are not only valid for the circular disc but also for arbitrarily shaped slices. We presuppose that at the points at which the boundary values are discontinuous the boundary itself is analytical. (Jumps of the derivatives of the boundary necessitate particular investigations.) Secondly, we will explain how to easily predict the structures of the solutions of other integral equations similar to Eq. (3.1) for boundary values with discontinuities. The results can be used for the numerical treatment of the singular and regular integral equations of the first and second kind [10] Eqs. (9.1)–(9.8), [20] Eqs. (3.45)–(3.52). In the preceding chapters we have extended the integration only over an infinitesimally small interval in the neighbourhood of the field point ϕ^*, i.e. the shape of the boundary and the boundary values in those parts of the boundary far away from the field point have no influence on the calculations. Hence we may conclude that the derived formulae are also valid if the boundary values do not fulfill conditions (3.16) and this applies even when the elastic body is not a circular disc. However, we can only generalize the results so simply without further calculations if the field point is situated on a part of the boundary representing a circular arc. The validity of the following considerations is not restricted by such presuppositions. With the abbreviations

$$a=\tfrac{1}{2}, \qquad b=-\frac{1}{4\pi}\frac{m-1}{m} \tag{5.1), (5.2}$$

integral equation (3.1) (see Eq. (3.2)) reads as follows:

$$a \cdot \delta_{jk}R_k(s)+b \cdot e_{jk} \left|\oint\right| \frac{e_{ti}n_i q_t}{\rho} R_k(\bar{s})\, d\bar{s} - \frac{1}{4\pi}\oint \frac{n_i q_i}{\rho}\left[\frac{m-1}{m}\delta_{jk}+2\frac{m+1}{m}q_j q_k\right]R_k(\bar{s})\, d\bar{s}=\Pi_j(s), \tag{5.3a}$$

$$\boxed{a \cdot \delta_{jk}R_k(s)+b \cdot e_{jk} \left|\oint\right| \frac{e_{ti}n_i q_t}{\rho} R_k(\bar{s})\, d\bar{s}+\oint \textit{regular term}_{jk}(s,\bar{s})R_k(\bar{s})\, d\bar{s}=\Pi_j(s)}\ . \tag{5.3b}$$

We presuppose that the given right-hand side of Eq. (5.3) has the structure:

$$\boxed{\Pi_j(s)=\delta_{jk}H(s-s^*)\cdot \Delta\Pi_k^{(n)}\frac{1}{n!}(s-s^*)^n+\textit{analyt.}(s)}\ . \tag{5.4}$$

By analogy with Eq. (4.9) we prescribe the solution R_k in the form:

$$\boxed{R_k(s)=A\cdot \delta_{kr}H(s-s^*)\Delta\Pi_r^{(n)}\frac{1}{n!}(s-s^*)^n +B\cdot e_{kr}\Delta\Pi_r^{(n)}\frac{1}{n!}(s-s^*)^n \ln|s-s^*|+\textit{analyt.}(s)}\ . \tag{5.5}$$

If the solution actually has this structure it should be possible to determine the unknown constants A and B. This will be our task in the following.

The factors of the kernel of integral equation (5.3) can be expanded as follows (see Fig. 1):

$$e_{ti}n_i(s)q_t(s,\bar{s})/\rho(s,\bar{s})=1/(\bar{s}-s)+d_0+d_1(\bar{s}-s)+d_2(\bar{s}-s)^2+\cdots, \tag{5.6}$$

$$n_i(s)q_i(s,\bar{s})/\rho(s,\bar{s}) \quad = \quad e_0+e_1(\bar{s}-s)+e_2(\bar{s}-s)^2+\cdots, \tag{5.7}$$

$$\begin{bmatrix} q_1(s,\bar{s}) \\ q_2(s,\bar{s}) \end{bmatrix}=\mathit{sign}(\bar{s}-s)\cdot\begin{bmatrix} f_0+f_1(\bar{s}-s)+f_2(\bar{s}-s)^2+\cdots \\ g_0+g_1(\bar{s}-s)+g_2(\bar{s}-s)^2+\cdots \end{bmatrix}. \tag{5.8}$$

d_i, e_i, f_i and g_i are analytical functions of the arc length s and are determined in detail by the shape of the boundary. The factor $e_{ti}n_iq_t/\rho$ diverges for $\bar{s} \to s$ whereas n_iq_i/ρ tends to $e_0(s) = \kappa(s)/2$ (κ is the curvature of the boundary). Therefore the summand of the kernel with the factor $e_{ti}n_iq_t/\rho$ is called the singular part of the kernel and that with the factor n_iq_i/ρ the regular part. It is of fundamental importance for the following considerations that the first term of the expansion (5.6) does not depend on the shape of the boundary.

Inserting the expansions (5.6)–(5.8) and the prescribed solution (5.5) into integral equation (5.3a) we see that we have to deal with integrals of the form:

$$\int_{s^*-\varepsilon}^{s^*+\varepsilon} H(\bar{s}-s^*)(\bar{s}-s^*)^n(\bar{s}-s)^p\, d\bar{s} = \int_{s^*}^{s^*+\varepsilon} (\bar{s}-s^*)^n(\bar{s}-s)^p\, d\bar{s} = \mathit{analyt.}(s), \tag{5.9}$$

$$\int_{s^*-\varepsilon}^{s^*+\varepsilon} (\bar{s}-s^*)^n(\ln|\bar{s}-s^*|)\cdot(\bar{s}-s)^p\, d\bar{s} = \mathit{analyt.}(s), \tag{5.10}$$

$$\left|\int\right|_{s^*-\varepsilon}^{s^*+\varepsilon} \frac{H(\bar{s}-s^*)(\bar{s}-s^*)^n}{\bar{s}-s}\, d\bar{s} = \left|\int\right|_{s^*}^{s^*+\varepsilon} \frac{(\bar{s}-s^*)^n}{\bar{s}-s}\, d\bar{s}$$
$$= -(s-s^*)^n \ln|s-s^*| + \mathit{analyt.}(s), \tag{5.11}$$

$$\left|\int\right|_{s^*-\varepsilon}^{s^*+\varepsilon} \frac{(\bar{s}-s^*)^n \ln|\bar{s}-s^*|}{\bar{s}-s}\, d\bar{s} = (s-s^*)^n \left|\int\right|_{s^*-\varepsilon}^{s^*+\varepsilon} \frac{\ln|\bar{s}-s^*|}{\bar{s}-s}\, d\bar{s} + \mathit{analyt.}(s)$$
$$= \pi^2(s-s^*)^n H(s-s^*) + \mathit{analyt.}(s). \tag{5.12}$$

where $n, p = 0, 1, 2, \ldots$ (see appendix and [19], eq. (7′)). The integrals (5.9) and (5.10) represent analytical functions of the arc length s which are unimportant for our considerations. Consequently we may totally omit the regular part of the kernel and replace the characteristical factor $e_{ti}n_iq_t/\rho$ of the singular part by the first term of series (5.6). So, equation (5.3) changes into:

$$a\cdot\delta_{jk}R_k(s) + b\cdot e_{jk}\left|\int\right| \frac{R_k(\bar{s})}{\bar{s}-s}\, d\bar{s} = \Pi_j(s). \tag{5.13}$$

Hence, our considerations are not only valid for integral equation (3.1) = (5.3a) but also for other integral equations of the form (5.3b) with an arbitrary regular part of the kernel and the factors a and b of the non-integral term and of the singular part of the kernel being allowed to assume values different from those stated in Eqs. (5.1), (5.2) (see e.g. [10] Eqs. (9.1)–(9.8)). On the right hand sides of these integral equations other physical quantities than boundary tractions may be prescribed and the sought functions need not represent layers of forces as in the case of integral equation (3.1), which has formed the starting point of our considerations.

Inserting the solution (5.5) and the right hand side (5.4) into the reduced equation (5.13) we obtain with the aid of the identity

$$e_{jk}e_{kr} = -\delta_{jr} \tag{5.14}$$

and of the integrals (5.11), (5.12):

$$a\cdot A\cdot H(s-s^*)\Delta\Pi_j^{(n)}\frac{1}{n!}(s-s^*)^n+a\cdot B\cdot e_{jr}\Delta\Pi_r^{(n)}\frac{1}{n!}(s-s^*)^n\ln|s-s^*|$$

$$+b\cdot A\cdot e_{jk}\Delta\Pi_k^{(n)}\frac{1}{n!}\left|\int\right|_{s^*-\varepsilon}^{s^*+\varepsilon} H(\bar s-s^*)\frac{(\bar s-s^*)^n}{\bar s-s}\,d\bar s$$

$$+b\cdot B\cdot e_{jk}e_{kr}\Delta\Pi_r^{(n)}\frac{1}{n!}\left|\int\right|_{s^*-\varepsilon}^{s^*+\varepsilon}\frac{(\bar s-s^*)^n}{\bar s-s}\ln|\bar s-s^*|\,d\bar s$$

$$=a\cdot A\cdot H(s-s^*)\Delta\Pi_j^{(n)}\frac{1}{n!}(s-s^*)^n+a\cdot B\cdot e_{jr}\Delta\Pi_r^{(n)}\frac{1}{n!}(s-s^*)^n\ln|s-s^*|$$

$$-b\cdot A\cdot e_{jr}\Delta\Pi_r^{(n)}\frac{1}{n!}(s-s^*)^n\ln|s-s^*|$$

$$-b\cdot B\cdot\Delta\Pi_j^{(n)}\frac{1}{n!}\pi^2(s-s^*)^nH(s-s^*)+\mathit{analyt.}(s)$$

$$=H(s-s^*)\Delta\Pi_j^{(n)}\frac{1}{n!}(s-s^*)^n+\mathit{analyt.}(s). \tag{5.15}$$

Comparison of the coefficients of the identical functions on the right and on the left hand side of Eq. (5.15) yields the system of equations

$$a\cdot A-b\cdot\pi^2B=1,\qquad -b\cdot A+a\cdot B=0 \tag{5.16}$$

with the solution

$$\boxed{A=a/(a^2-\pi^2b^2),\qquad B=b/(a^2-\pi^2b^2)}\,. \qquad (5.17),(5.18)$$

In order to check the results we insert the particular values (5.1), (5.2) into Eqs. (5.17), (5.18):

$$A=\frac{8m^2}{(3m-1)(m+1)},\qquad B=-\frac{1}{2\pi}\frac{m-1}{m}\frac{8m^2}{(3m-1)(m+1)} \qquad (5.19),(5.20)$$

i.e. the solution coincides with that which has been determined in Chapter 4 (see Eqs. (3.18), (3.19), (4.9)).

To finish this chapter we consider an integral equation which differs from Eq. (5.3b) by one sign and by the fact that the tensors δ_{ij} and e_{ij} are exchanged:

$$\boxed{-a\cdot e_{jk}D_k(s)+b\cdot\delta_{jk}\left|\oint\right|\frac{e_{ti}n_iq_t}{\rho}D_k(\bar s)\,d\bar s+\oint \mathit{regular\ term}_{jk}(s,\bar s)D_k(\bar s)\,d\bar s=E_j(s)}$$

(5.21)

where

$$E_j(s) = H(s-s^*)\Delta E_j^{(n)} \frac{1}{n!}(s-s^*)^n + analyt.(s) \tag{5.22}$$

(see e.g. [20] Eqs. (3.46), (3.48), (3.49), (3.51)). Eq. (5.21) has the solution:

$$D_k(s) = A \cdot e_{kr} H(s-s^*)\Delta E_r^{(n)} \frac{1}{n!}(s-s^*)^n - B \cdot \delta_{kr} \Delta E_r^{(n)} \frac{1}{n!}(s-s^*)^n \ln|s-s^*| + analyt.(s) \tag{5.23}$$

with the constants A and B according to Eqs. (5.17), (5.18).

6. Interpretation of the results

The vector components of the solution have the form (see Eqs. (5.5), (5.6), (5.17), (5.18)):

$$R_1(s) = A \cdot \Pi_1(s) + B \cdot \Delta\Pi_2^{(n)} \frac{1}{n!}(s-s^*)^n \ln|s-s^*|, \tag{6.1}$$

$$R_2(s) = A \cdot \Pi_2(s) - B \cdot \Delta\Pi_1^{(n)} \frac{1}{n!}(s-s^*)^n \ln|s-s^*| \tag{6.2}$$

with

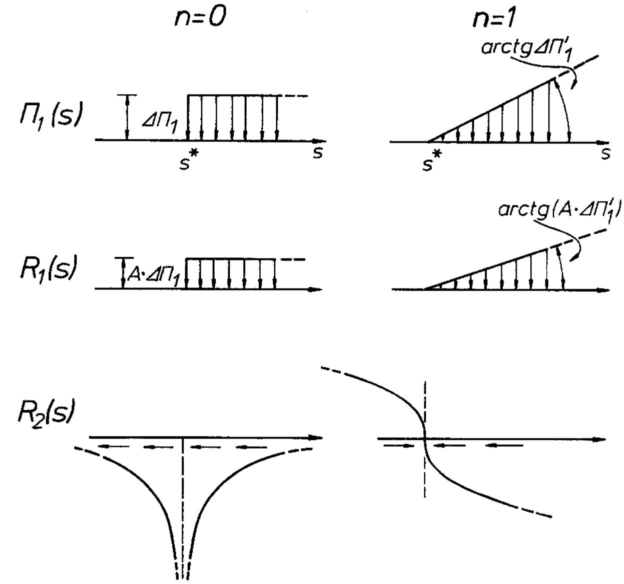
$$\Pi_1(s) = H(s-s^*)\Delta\Pi_1^{(n)} \frac{1}{n!}(s-s^*)^n, \tag{6.3}$$

$$\Pi_2(s) = H(s-s^*)\Delta\Pi_2^{(n)} \frac{1}{n!}(s-s^*)^n, \tag{6.4}$$

$$A = a/(a^2-\pi^2 b^2), \qquad B = b/(a^2-\pi^2 b^2) \qquad (6.5), (6.6)$$

where $n = 0$ or 1 or 2 or $\cdots$.

The solution $R_i(s)$ consists of two summands. The first one differs only by a constant factor A from the given boundary values $\Pi_i(s)$. So the n-th derivative of the solution has a jump of magnitude $A \cdot \Delta\Pi_i^{(n)}$ at the point s^*; $\Delta\Pi_i^{(n)}$ is the jump of the n-th derivative of the boundary values. The second summand represents a logarithmic singularity of the n-th derivative of the solution. To be more precise, the n-th derivative of the component $R_1(s)$ of the solution diverges if the n-th derivative of the component $\Pi_2(s)$ of the boundary values is discontinuous. Analogously a jump of $\Pi_1^{(n)}(s)$ causes a discontinuity of $R_2^{(n)}(s)$ (see Fig. 3).



3a. 3b.

Figure 3. (The elastic body is above the s-axis. B is a negative quantity.)

3a. Top: Boundary tractions with a jump of the normal component and vanishing tangential component. Middle: Normal component of the solution. Bottom: Tangential component of the solution.

3b. Top: Boundary tractions with discontinuous first derivative of the normal component and vanishing tangential component. Middle: Normal component of the solution. Bottom: Tangential component of the solution.

The first summand is essentially determined by the factor a of the non-integral term and the second summand by the factor b of the singular part of the kernel of the integral equation (see Eq. (5.3b)). The kernels of regular integral equations have no singular part i.e. the constants b and B vanish (see e.g. [10] Eqs. (9.2), (9.6), (5.46), (5.51)). Consequently, the solutions of regular equations are discontinuous for discontinuous boundary values, however they do not diverge. On the contrary, the solutions of singular integral equations of the first kind of the type dealt with in this paper (see e.g. [20] Eqs. (3.46), (3.49)) contain only the logarithmically singular term because of $a=0$ and $A=0$.

The curves of Fig. 3 should not be mixed up with those of similar looking diagrams (see e.g. [22]) which, however, have a completely different meaning. In Fig. 3 the solution $R_i(s)$ of the integral equation is presented for discontinuously prescribed boundary values $\Pi_i(s)$. On the contrary, in [22] the inverse problem is considered, i.e. the boundary values $\Pi_i(s)$ are determined for a prescribed solution $R_i(s)$ with discontinuous first derivative. Physically this means calculating the tractions which are caused by a layer of forces acting at a curve (in [22] at a straight line) in the infinite elastic medium. For a layer $R_r(s)$ of forces the n-th derivative of which has a jump $\Delta R_r^{(n)}$ at the point s^* one obtains the following tractions (see Eqs. (5.1), (5.2)):

$$\Pi_j(s) = a \cdot \delta_{jr} R_r(s) - b \frac{1}{n!} e_{jr} \Delta R_r^{(n)} (s - s^*)^n \ln |s - s^*|. \tag{6.7}$$

Formula (6.7) yields the means of understanding an interesting phenomenon which has not to do directly with the subject of this paper – i.e. with discontinuities of the boundary values – but is caused by the artificially enforced structure of the approximate solution. We consider an analytical part of the boundary with analytically prescribed boundary tractions. The exact solution is an unknown analytical layer of forces. Now, for the numerical treatment of the problem the solution is prescribed, for example, as piecewise constant or linear function, i.e. by no means analytically. The node values of this interpolation are determined by fulfilling the boundary conditions at a number of discrete points. The interesting fact is that the tractions caused by the layer of forces which represents the approximate solution are not at all as smooth as the actually prescribed tractions. It is true that they coincide with those at the collocation points, but, however, they or at least their derivatives diverge (see Fig. 4).

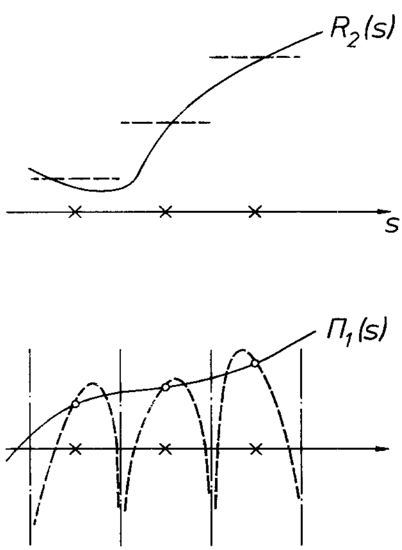


Figure 4a. Top: —— exact solution $R_2(s)$ (layer of forces); --- approximate solution (piecewise constant interpolation). Bottom: —— prescribed boundary tractions $\Pi_1(s)$; --- boundary tractions caused by the approximate solution.

× Collocation points. The vector components $R_1(s)$ and $\Pi_2(s)$ are generally different from zero and behave similarly to $R_2(s)$ and $\Pi_1(s)$ respectively.

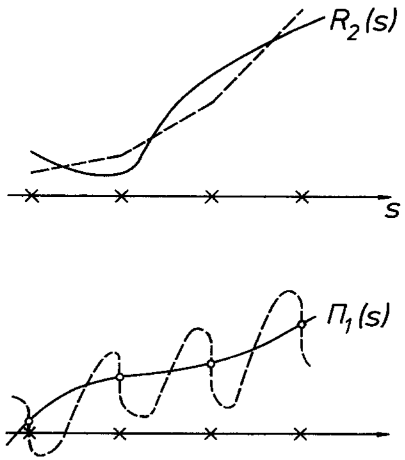


Figure 4b. Top: —— exact solution $R_2(s)$ (layer of forces); --- approximate solution (piecewise linear interpolation). Bottom: —— prescribed boundary tractions $\Pi_1(s)$; --- boundary tractions caused by the approximate solution.

× Collocation points. The vector components $R_1(s)$ and $\Pi_2(s)$ are generally different from zero and behave similarly to $R_2(s)$ and $\Pi_1(s)$ respectively.

7. Treatment of concentrated forces and moments

The vector of a concentrated moment is perpendicular to the plane of the slice, i.e. in the plane theory the moment is a scalar quantity. Since this paper deals with vectorial equations we identify formally the concentrated moment with a vector in the plane of the slice perpendicular to the boundary, the modulus of which is equal to the value M of the moment:

$$M_j = M \cdot n_j \triangleq M \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}. \tag{7.1}$$

We confine ourselves to moments applied at straight parts of the boundary.

A concentrated force P_j represents a jump of the integral of the given boundary tractions $\Pi_j(s)$, a moment M_j a jump of the integral of the integral of the tractions. (These facts can be understood easily by recollecting the notions of elementary beam statics as lateral force, longitudinal force and bending moment.) The integral of a function can be interpreted as the minus first derivative of this function. Hence, a concentrated force P_j represents a jump of the -1st derivative and a concentrated moment M_j a jump of the -2nd derivative of the boundary tractions $\Pi_j(s)$, i.e. we are dealing with the cases $n = -1$ and $n = -2$:

$$\Delta\Pi_j^{(-1)} = P_j, \qquad \Delta\Pi_j^{(-2)} = M_j. \tag{7.2}$$

By analogy with Eq. (5.4) we express the boundary values $\Pi_j(s)$ in terms of Dirac's delta functions $\delta^{(1)}$ and $\delta^{(2)}$ of the first and second order:

$$\Pi_j(s) = \delta_{jk}\delta^{(1)}(s-s^*)P_k + analyt.(s), \tag{7.3}$$

$$\Pi_j(s) = \delta_{jk}\delta^{(2)}(s-s^*)M_k + analyt.(s). \tag{7.4}$$

The corresponding solutions of integral equation (5.3) are ascertained by formal generalization of Eq. (5.5):

$$R_k(s) = A \cdot \delta_{kr}\delta^{(1)}(s-s^*)P_r + B \cdot e_{kr}P_r \frac{1}{s-s^*}, \tag{7.5}$$

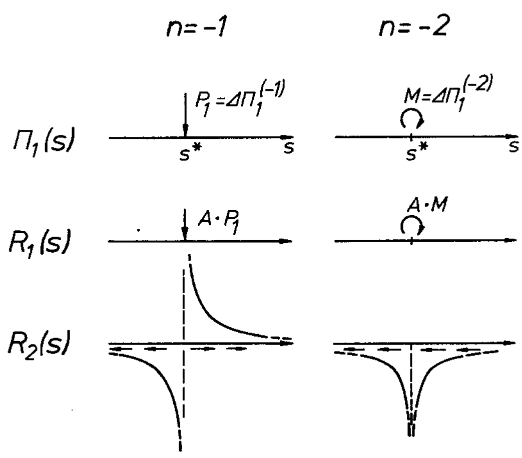
$$R_k(s) = A \cdot \delta_{kr}\delta^{(2)}(s-s^*)M_r - B \cdot e_{kr}M_r \frac{1}{(s-s^*)^2}. \tag{7.6}$$

The solution consists of a concentrated force and a concentrated moment respectively which is A times as large as the force and the moment respectively by which the boundary of the elastic slice is charged. Besides, the solution contains a layer of forces with a pole of the first and second order respectively at the point s^* (see Fig. 5, compare Fig. 5 with Fig. 3).

Seggelke [23] has determined the solution of integral equation (5.3a), (5.1), (5.2) for a circular disc charged by two concentrated forces perpendicular to the boundary. We mention this interesting formula since it has not yet been published:

$$\boldsymbol{R}_1 = \left[-\frac{1}{\pi}\frac{2m}{3m-1} + A\{\delta^{(1)}(\phi) + \delta^{(1)}(\phi-\pi)\}\right]\boldsymbol{P}_1, \qquad \boldsymbol{R}_2 = -B \cdot \text{ctg}(\phi)\boldsymbol{P}_1. \tag{7.7}$$

It contains, except for an analytic function, the two terms given by Eq. (7.5).



5a. 5b.

Figure 5. (The elastic body is above the s-axis. B is a negative quantity.)

5a. Top: Elastic body charged by a concentrated force perpendicular to the boundary. Middle: Normal component of the solution. Bottom: Tangential component of the solution.

5b. Top: Elastic body charged by a concentrated moment. Middle: "Normal-component" of the solution. Bottom: Tangential component of the solution.

8. Three-dimensional problem

The integral equation (3.1) presented in Chapter 3 and investigated extensively in Chapters 4–7 serves for the solution of problems of plane stress. The corresponding equation for the state of plane strain can be obtained by replacing Poisson's ratio m by $m-1$ in the kernel (3.2). Now, the state of plane strain represents a special but genuine three-dimensional state of deformation. Hence the results of this paper should also be applicable to three-dimensional problems.

9. Rizzo's integral equation

In the preceding chapters we have investigated the properties of the solutions of integral equations of the indirect method for discontinuous boundary values. However, effects similar to those described below can also be observed with integral equations of the direct method. As an example we consider here Rizzo's equation for the statical problem [13] (see also [15] Eqs. (6.2), (11.1)). The unknown boundary displacements represent the solution of this equation. If we prescribe the boundary tractions Π_j with a jump $\Delta\Pi_j^{(n)}$ of the n-th derivative ($n=0$ or 1 or 2 or $\cdots$) according to Eq. (5.4) the essential terms of the solution are of the form:

$$u_i = -\frac{1}{2G}\frac{m}{m+1}\frac{1}{(n+1)!}\Delta\Pi_j^{(n)} \times\left\{\frac{2}{\pi}\,\delta_{ij}(s-s^*)^{n+1}\ln|s-s^*|+\frac{m-1}{m}\,e_{ij}(s-s^*)^{n+1}H(s-s^*)\right\}. \quad (9.1)$$

It is noteworthy that not the n-th but the $(n+1)$-th derivative of the solution contains a logarithmically diverging and a discontinuous summand (compare Eqs. (9.1) and (5.5), (5.23)). Hence also application of integral equations of the direct method entails the described problems though in a less difficult form.

10. Appendix

We investigate the integral

$$I(a,c) = -I(-a,c) = [I(a,c)-I(-a,c)]/2 = \left|\int_{-c}^{c}\right| \frac{\ln|x|}{x-a}\,dx$$

$$= \frac{1}{2}\left|\int_{-c}^{c}\right|\left[\frac{\ln|x|}{x-a}-\frac{\ln|x|}{x+a}\right]dx = a\left|\int_{-c}^{c}\right|\frac{\ln|x|}{x^2-a^2}\,dx$$

$$= 2a\left|\int_{0}^{c}\right|\frac{\ln|x|}{x^2-a^2}\,dx \quad (10.1)$$

$$I(a,\infty) = \pi^2/2 \quad \text{for} \quad a>0 \quad (10.2)$$

(see [24] Eq. 4.231.10), i.e.

$$I(a,\infty)=\pi^2[H(a)-\tfrac{1}{2}]. \tag{10.3}$$

Since for $c>|a|$ the integral

$$\left(\int_{-\infty}^{-c}+\int_{c}^{\infty}\right)\frac{\ln|x|}{x-a}\,dx \tag{10.4}$$

is analytical we have

$$I(a,c)=I(a,\infty)+analyt.(a)\quad\text{for}\quad c>|a|. \tag{10.5}$$

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