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The problem of indeterminacy in approval, multiple, and truncated voting systems

DONALD G. SAARI and JILL VAN NEWENHIZEN*
Department of Mathematics, Northwestern University, Evanston, IL 60201

Abstract. It is well known that a plurality election need not reflect the true sentiments of the electorate. Some of the proposed reform procedures, such as approval and cumulative voting, share the characteristics that there are several ways to tally each voter's preferences. Voting systems that permit truncated ballots share this feature. It is shown that the election results for any such procedure can be highly indeterminate; all possible election results can occur with the same choice of sincere voters. This conclusion of indeterminacy holds even when measures of voters' sentiments, such as the existence of a Condorcet winner or even much stronger measures, indicate there is considerable agreement among the voters. Then, multiple systems are compared with all standard tallying procedures. For instance, a corollary asserts it is probable for the plurality voting method to elect the Condorcet winner while approval voting has an indeterminate outcome.

1. Introduction

It is well known that the results of a plurality election need not reflect the true sentiments of the voters. With more than two candidates running for the same office, the winner could be the least preferred. Examples abound, and they probably occur often in many closely contested elections among three or more candidates. Perhaps the best known one is the 1970 senatorial election in New York. Conservative James Buckley benefited from a split vote for his two liberal opponents; he won with 39 percent of the vote even though the rest of the electorate appeared to prefer a liberal. In the 1983 Chicago Democratic Party mayoral primary the black candidate, Harold Washington, won because of a split vote for his two white opponents, Jane Bryne and Richard Daley, even though opinion polls indicated that in a two-candidate election, he would have lost to either person. Indeed, during the summer of 1986, the Chicago papers speculated that Mayor Washington would try to win reelection with another split vote by running as an Independent, rather than as a Democrat. (He did not.) In response, some of his opponents proposed a nonpartisan 'reform' procedure that includes a run-off.

To illustrate the problem, suppose that fifteen voters are considering the

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three candidates A, B, and C and suppose that the voters' preferences are split in this way: six voters have the ranking A > C > B; five have the ranking B > C > A; and four have the ranking C> B > A. The result of a plurality election is A > B > C with a tally 6:5:4. Although A wins, a majority (sixty percent) of these voters prefers B to A. More seriously, a majority (sixty percent) of these voters prefers the last place candidate, C, to A, and 2/3 of them prefer C to B. The preferred candidate appears to be C and one could argue that the 'true' ranking is C > B > A. The election results reflect neither this ordering, nor that B would win a run-off election.

There is little question that this instrument of democracy is in need of reform. But, what should replace it? A final resolution of this question involves a wide spectrum of issues such as whether voters will understand and accept a new system, whether the system can be easily manipulated, and so forth. This essay concentrates on the critical issue of how the choice of a tallying procedure affects the election results. One of us discusses other aspects, in particular, questions about strategic voting, elsewhere. (See, for example, Saari, 1986.)

To counter the difficulties that the plurality method poses, a voter must be able to register in the election system more information than just who is his top choice. One such system, *cumulative voting*, has been used to elect representatives to the Illinois General Assembly (e.g., see Sawyer and MacRae, 1962). Here each voter has three votes; he can cast all of them for his top ranked candidate, he can split them equally by casting 1½ points for each of his two top ranked candidates, he can give two votes to his top ranked candidate and the last one to his second ranked candidate, or he can cast one vote for each of his three top ranked candidates. A simpler version of this system to use with our example would be to let each voter choose how to split two votes. If enough of the voters had sufficiently high regard for C, she would be elected.

A second proposed reform method, independently invented by R. Weber, S. Brams and P. Fishburn, and others (see Brams and Fishburn, 1982), is *approval voting*. In this system, a voter votes for *all* candidates he approves of. As such, with N candidates, the voter has N choices; he can vote approval for his i top candidates, i = 1, . . ., N. If we tally votes under approval voting for our election example, then C again emerges victorious should enough of the voters favor her.

Approval voting enjoys the support of several experts in this field. It was employed for a straw ballot during the Pennsylvania Democratic Party conference in December, 1983 (Nagel, 1984), it was used to select faculty members to the Northwestern University Presidential search committee in November, 1983, and the Mathematical Association of America adopted it for certain elections in 1986. Approval voting also '. . . is now used in academic societies such as the Econometric Society, in the selection of members of the National Academy of Science during final balloting, and by the United Nations Security Council in the election of a Secretary General. Bills to enact this reform are now before the state legislatures of New York and Vermont.' (Brams and Fishburn, 1983)

Much of this support is a consequence of a careful analysis of its properties by two of its foremost advocates, Steven Brams and Peter Fishburn. Most of their conclusions, which highlight several of the desirable properties of this system, are summarized in their book *Approval Voting*. To demonstrate the strength of approval voting, they compare it with systems that distinguish between two sets of candidates – the top k and the rest. These comparisons partially reflect their critical and controversial assumption that the preferences are 'dichotomous.' (See, for example, Niemi, 1984.) Plurality voting is the special case where $k = 1$; it distinguishes between the top ranked candidate and all others. But Brams and Fishburn did not discover all of the properties of approval voting, nor did they compare it with all other voting systems.

We began our analysis of approval voting with the expectation that, in some sense, it is an improvement over most other systems. But we found that it has several disturbing features that seem to make it worse than even the plurality voting system. Indeed, *these properties appear to be sufficiently bad to disqualify approval voting as a viable reform alternative.* These negative features emerge, moreover, in *any* voting system that includes more than one way to tally each voter's ranking of the candidates. Such multiple voting systems include approval voting, cumulative voting, cardinal voting, as well as any system that creates alternative tallying procedures for those voters that cast a truncated ballot.

Our main result is that for any such multiple voting system, the election outcome can be indeterminate rather than decisive. More precisely, if there are N candidates, then there are N! possible ways to rank them without ties. If a voting system is decisive, then a given set of voters' profiles uniquely determines one of these rankings. For any multiple voting system, however, there are a large number of examples in which *all* N! election outcomes can occur for the same profile. Each voter votes according to his sincere ranking of the candidates, but as the voters vary their choice of tallying methods, each of the N! possible rankings emerge.

The preceding example illustrates this phenomenon for approval voting. Let w, y, z denote, respectively, the number of each kind of voter who votes approval for his top two candidates rather than just his top ranked candidate. Then, $0 \leq w \leq 6$, $0 \leq y \leq 5$, $0 \leq z \leq 4$, and the tally for A:B:C is $6:5+z:4+w+y$. It follows immediately that *any election outcome is attainable*. For instance, the result $B > A > C$ occurs when $z \geq 2$ (at least two voters from the last set vote for their two top ranked candidates) and $w + y \leq 1$. Even ties are possible. A deadlocked election of $A = B = C$

results from $z = 1$ and $w + y = 2$, while $B = C > A$ results from $w + y - 1 = z \geq 2$.

The plurality election ranking, no matter what bad features it may have, always is one of the possible approval voting outcomes. But our example shows that the set of approval voting outcomes can include a host of other kinds of pathologies. Our general results go beyond this claim; they assert that even if other measures indicate that there is a remarkable agreement among the voters concerning the candidates, if these same voters use a multiple voting method, then the result can be indeterminate. So, even though a multiple system may have other 'desirable' features, just the fact its outcome can be completely indeterminate, even in 'ideal' situations, raises serious questions of whether these methods offer appropriate reform.[1] For instance, one argument for certain multiple systems is that they account for the intensity of preference or distaste for certain candidates. Is this feature worth the accompanying cost of indeterminacy? We require answers to this kind of question to assure us that the proposed cure is not worse than the disease.

Our results do not mean that we must accept and live with the failings of plurality voting. There are other ways to tally a ballot to reflect a voter's first, second, ..., last ranked candidates, such as a Borda Count. (The Borda ranking for our example is the desired one of $C > B > A$ with a tally of 34:29:27.) Out of all possible ways there are to tally the ballots, the problem is to isolate the ones that best capture the wishes of the electorate in the sense that the ranking of the N candidates has some consistency with its rankings of all other subsets of the same candidates. It turns out that the unique solution is the Borda Count (Saari, 1985).

Although our results demonstrate other favorable properties of the Borda Count, we do not argue for it here because, as we asserted earlier, the decision to adopt one system over another involves other issues that we do not address here. So, this essay treats the Borda Count either as a technical condition or as an example. But whatever good features the Borda Count (or any other system) may have, if a 'tolerant' organization modifies it so that a truncated ballot (a voter does not rank all of the candidates) can be tallied, then the modified system admits these indeterminate outcomes.

[1] If other reasons for using a multiple system can be found that are compelling enough to overcome the objections of indeterminacy, then it is of interest to discern which of the indeterminate rankings are most likely. Such an analysis would be based on a probability model, so the comments introducing Theorem 6 are relevant. (Also see the concluding paragraphs in Section 4.) There are certain settings, clearly not all of them, in which some insight into this issue might come from the growing literature on strategic voting. But even in this restricted setting, the issue is by no means settled. For example, for three different starting assumptions that lead to different conclusions, see (Brams and Fishburn, 1982; Niemi, 1984; and Saari, 1986). Indeed, it is now known (Saari, 1986) that we can justify *any* system as being 'strategically the best' just by adopting the appropriate assumptions.

2. The main results

Denote the set of $N \geq 3$ candidates by $\{a_1, \ldots, a_N\}$. Let $\underline{W} = (w_I, \ldots, w_N)$ be a *voting vector* such that its components satisfy the inequalities $w_J \geq w_K$ if and only if $J < K$, and $w_I > w_N$. Furthermore, assume that all of the weights are rational numbers. (This last requirement simplifies the proofs; clearly, it does not impose any practical limitations.) Such a vector defines the tallying process for an election – w_J points are tallied for a voter's J^{TH} ranked candidate. The sum of the points tallied for a candidate determines her final ranking.

The vector (1, 0, ..., 0) corresponds to the plurality vote while $\underline{B}^N = (N, N-1, \ldots, 1)$ defines the usual Borda Count procedure. (More generally, call $\underline{W}$ a *Borda vector* if the differences $w_J - w_{J+1}$ are the same nonzero constant for $j = 1, \ldots, N-1$. We call any election tallied with a Borda vector a Borda Count. It is easy to show that the election ranking is the same independent of which Borda vector is used.) The vector $\underline{E}_N = N^{-1}(1, 1, \ldots, 1)$ is not a voting vector because all of the components are equal; a tally with $\underline{E}_N$ does not distinguish how the candidates are ranked.

Definition. A *simple voting system* uses one and only one voting vector to tally the voters' rankings of the candidates. A *multiple voting system* is where (1) there is a specified set of at least two voting vectors, $\{\underline{W}_J\}$, such that the difference between any two of them is not a scalar multiple of $\underline{E}_N$, and (2) each voter selects any one of the voting vectors to tally his ballot.

Examples. For *cumulative voting* with three votes, the set of voting vectors is {(3, 0, ..., 0), (1.5, 1.5, 0, ..., 0), (2, 1, 0, ..., 0), (1, 1, 1, 0, ..., 0)}.

Approval voting is defined by the set of $N-1$ vectors {(1, 0, ..., 0), (1, 1, 0, ..., 0), ..., (1, 1, ..., 1, 0)}. (We do not include (1, 1, ..., 1) because it is not a voting vector; casting such a ballot does not affect the election results).

The system {(2, 0, ..., 0), (1, 0, ..., 0)} defines a multiple voting system even though the voting vectors are scalar multiples of each other.

For *cardinal* voting, each voter is free to select the values of the weights, w_J, subject to certain constraints. To standardize the choices, we might require the weights to sum to a specified value, or to be bounded above and below by specified constants.

A common way to create a multiple system is to modify a simple voting system to tally a *truncated ballot*. With the sole exception of the plurality method, this modification always defines a multiple method. For the Borda Count, one modification defines the set of N vectors $\{\underline{B}_N, (N, N-1, \ldots, 2, 0), \ldots, (N, 0, \ldots, 0)\}$.

Definition. A multiple voting system for N candidates is *completely indeterminate* if there exist profiles of voters for which *all* N! possible rankings (without ties) of the candidates result from the same profile as the voters vary their choices of how their ballots are to be tallied.

The definition captures the extreme indecisiveness of the election procedure. For these examples of voters' profiles, the different outcomes reflect the voters' fluctuations in their choices of a tallying procedure rather than their rankings of the candidates. This feature should be avoided.

Theorem 1. Assume that there are $N \geq 3$ candidates. All multiple voting systems are completely indeterminate.

This statement serves not only as an argument against using multiple methods such as approval, cardinal, and cumulative voting, but also as an argument against admitting truncated ballots. Recall, we consider only sincere voting, so this conclusion is not the result of any voters' manipulation of the system to misrepresent their true rankings of the candidates. It is a flaw of the system. On the other hand, our statement extends the nice results of Brams and Fishburn (1984) about how 'sincere voters' can manipulate the system by truncating their ballots. Here, the voter remains true to his rankings of the candidates because the manipulation strategy rests on the various ways there are to truncate a ballot. In this same spirit and as an immediate consequence of Theorem 1, *voters can use any multiple voting system in exactly this same manipulative way* because as the voters change their choice of a tallying method any election outcome can occur. So, to manipulate a multiple voting system such as a truncated ballot, cumulative, or approval voting, the voter need not misrepresent his beliefs; the system itself provides the manipulative tools!

Can a multiple system shed this indeterminacy if there is a conformity of opinion about the candidates? To quantify this question, we need an indicator of the voters' sentiments. The 'Condorcet winner' provides a commonly used measure.

Definition. Assume that the $N \geq 3$ candidates are $(a_1, a_2, \ldots, a_N)$. Candidate a_K is called a (strong) Condorcet winner if, in all possible pairwise comparisons, a_K always wins by a majority vote. A Condorcet loser is a candidate who always loses by a majority vote in all possible pairwise comparisons with the other candidates.

A Condorcet winner measures the voters' sentiments in the sense that she is the choice of a majority of the electorate whenever voters compare her with any other candidate. Our earlier New York senatorial and Chicago mayoral examples show that a Condorcet loser can be a plurality election winner, while our constructed example proves that the plurality vote can rank a Condorcet winner in last place and a Condorcet loser in first place. Saari (1985) shows that this kind of result is characteristic of all simple voting systems with the sole exception of the Borda Count.

Theorem 2. (Saari, 1985) Suppose that there are $N \geq 3$ candidates. For any simple voting system other than a Borda Count, there exist examples of voters' preferences for which the Condorcet winner is ranked in last place and the Condorcet loser is ranked in first place. The Borda Count is the unique method that never ranks a Condorcet winner in last place and never ranks a Condorcet loser in first place.

How does a multiple voting system fare? It turns out that it can be much worse. To state a version of our result that is easier to prove, we introduce this definition.

Definition. A multiple voting system $\{\underline{W}_J\}$ is 'plurality like' if there are non-negative scalars (b_J) such that when we compute the differences between successive components of $\Sigma b_J \underline{W}_J$, all but one are zero.

The summation defines a voting vector that distinguishes between only two sets of candidates. We satisfy this condition automatically if the multiple voting method includes a voting vector of this kind, for example, if it includes $(1, 0, \ldots, 0)$ (plurality vector), $(3, 0, 0, \ldots, 0)$, $(1, 1, 1, 0, \ldots, 0)$, and so forth. Thus, approval and cumulative voting are plurality like. Also, this condition is satisfied if $N-1$ of the voting vectors and $\underline{E}_N$ form a linearly independent set. This conclusion occurs because the vectors span R^N. Consequently, cardinal voting and the modification of the Borda Count to tally truncated ballots both satisfy this condition.

Theorem 3. Suppose that there are $N \geq 3$ candidates. Choose a ranking for each of the $N(N-1)/2$ pairs of candidates in any manner desired. (We may do so randomly; the rankings need not be transitive.) Assume that we use a plurality like, multiple voting method to rank all subsets of more than two candidates. Then there exist examples of voters' profiles so that

1) for each of the pairs of candidates, a majority of the voters have the indicated preference, and
2) for each subset of three or more candidates, the outcome is completely indeterminate.

The constraint that a multiple voting method is 'plurality like' is not necessary for this result to hold; the conclusion also holds for almost all multiple voting systems. We impose this assumption only because it significantly simplifies the proof while still including all of the multiple methods that people have seriously considered or used.

Our goal to compare the results of multiple voting methods with the Condorcet winner is an immediate corollary of Theorem 3. To see why this is so, just choose the rankings of the pairs so that there is a Condorcet winner. For instance, if $N = 4$, it follows from Theorem 3 that there are examples of voters' profiles so that whenever voters compare a_1 with any other candidate, she always wins a majority vote. Yet, when these same voters use approval voting to rank the candidates, the outcome is completely indeterminate over each set of candidates $\{a_1, a_2, a_3, a_4\}$, $\{a_1, a_2, a_3\}$, $\{a_1, a_2, a_4\}$, $\{a_1, a_3, a_4\}$, and $\{a_2, a_3, a_4\}$. In other words, a Condorcet winner could win an approval election, or any other multiple voting election, only by accident.

Actually, it follows from the theorem that indeterminacy can occur even when the voters are in far more agreement than indicated by the existence of a Condorcet winner. For instance, it follows from this theorem that there are examples of voters' profiles so that for each pair (a_J, a_K), a majority of the voters prefer the candidate with the smaller subscript; that is, by majority votes, the pairwise election outcomes are $a_1 > a_2, a_1 > a_3, \ldots, a_1 > a_N$, $a_2 > a_3, \ldots, a_{N-1} > a_N$. These binary rankings define the transitive ranking $a_1 > a_2 > \ldots > a_N$. Even though this set of binary rankings isolates a situation in which a strong agreement about the candidates prevails, the approval voting result for *any subset of three or more candidates* remains completely indeterminate. A similar example exists for any plurality like multiple voting system.

This kind of an example illustrates that the election outcome for a plurality like multiple voting method can be highly indeterminate even if there is a Condorcet winner. We can relax the restriction of 'plurality like' multiple methods by considering indeterminacy only for one subset of candidates.

Corollary 3.1. Assume that the voters rank the $N \geq 3$ candidates with a given multiple voting method. Assume that the multiple voting system has at least one vector that is not a Borda vector. Then there exist examples of voters' profiles so that even though there is a Condorcet winner, the ranking of the N candidates is completely indeterminate.

We can reconcile these negative statements about approval voting with the more positive ones in the literature. In particular, in certain settings, we know that the approval voting election results can rank a Condorcet winner in first place (Brams and Fishburn, 1982). But, it follows from Corollary 3.1 that such a favorable ranking can occur for *any multiple method* because it is just one of the many indeterminate fluctuations of the election. In other words, if the voters choose to tally their ballots in certain specified ways, then these desirable outcomes will result. If they do not, then anything else can occur.

A different consequence of Theorem 3 concerns 'run-off elections' and the other procedures that use not only the voters' ranking over the set of all of the candidates, but also the election results over certain subsets of the candidates. Consider this standard approach. First rank the N candidates, and then drop the candidate who is in last place. Rerank the remaining set. Continue this elimination procedure until only the required number of candidates remain. Suppose that we use a multiple method, such as approval voting or a simple system modified to count truncated ballots, to rank the candidates at each step of this elimination procedure. It follows from Theorem 3 that the final result can have no relationship whatsoever with how the voters rank the candidates.

So far we have compared multiple voting methods only with the rankings of pairs of candidates. Another test is to compare them with plurality voting and other simple voting methods. For instance, it is reasonable to speculate that approval voting will be indeterminate only in those situations in which the plurality election ranking also has some highly undesired features. But, this is not the case.

Theorem 4. Assume that there are $N > 4$ candidates. Let $\underline{W}_N$ be a voting vector defining a simple voting method, and assume that a multiple voting method is given in which at least two of the vectors and $\underline{E}_N$ are linearly independent. Choose a ranking for each of the pairs of alternatives. There exist examples of voters' profiles so that

a. For each pair of candidates, a majority of the voters have the indicated preference.
b. If $\underline{W}_N$ is not a Borda vector, then the simple voting method has any previously selected ranking of the candidates.
c. The multiple method is completely indeterminate.

If the multiple voting method is either approval or cumulative voting, then the conclusion holds for $N \geq 3$. So, with the exception of the Borda Count, Theorem 4 illustrates that examples exist for which there is a Condorcet winner, yet anything can occur with the simple voting scheme while the multiple method is completely indeterminate. (Theorem 2 serves as an explanation of why we exclude the Borda vector.) In particular, this finding means that there are examples of voters' profiles in which the plurality outcome *does* rank the Condorcet winner in first place while approval voting has a completely indeterminate effect. Indeed, Corollary 4.1 is an immediate consequence.

Corollary 4.1. There are profiles of voters so that the majority votes over pairs of candidates define a binary, transitive relationship, and the plurality election ranking respects this relationship, yet, for these same voters, the approval voting outcome is completely indeterminate.

Consequently, plurality election results *can* reflect the voters' wishes while cumulative, approval voting, truncated ballots, or any other multiple methods does not.

So far, we have compared the outcome of multiple methods with the Condorcet winner. But we can use other possible measures of voters' sentiments to isolate other kinds of agreement among the voters. If such measures rely on the majority vote rankings of pairs of candidates, then it follows from Theorem 3 that for any favorable situation we define in this way, this situation can be accompanied by an indeterminacy conclusion for multiple systems. If we define our measure in terms of election outcomes from a specified simple voting system, then this indeterminacy conclusion for multiple voting methods follows from Theorem 4.

We now introduce a measure that requires far too much agreement among the voters to be of any practical use. In its strictness lies its appeal; we use it to underscore our principal claim that even in a highly ideal situation in which there is an incredibly strong consistency in the voters' beliefs, a multiple system can be indeterminate.

Definition. Let $N > 3$ and for each K, $2 \leq K \leq N$, let all subsets of K candidates by ranked by a specified simple voting vector $\underline{W}_K$. If a profile of voters has the property that the election outcomes over all subsets of alternatives are the restrictions of the same transitive ranking of the N alternatives, then we call this profile *completely consistent.*

Trivially, a completely consistent profile admits a Condorcet winner. The consistency condition is much stronger: the majority votes over the pairs define a binary transitive relationship over the set of candidates, and the $\underline{W}_K$ election outcome preserves this relationships for *all* subsets of K candidates, $K = 3, \ldots, N$. This is extreme consistency.

Theorem 5. Let $N > 3$. Let $\underline{W}_K$ be given for each $K = 2, \ldots, N$, and let a multiple voting system be given for the set of N alternatives. Assume that the multiple system has at least one voting vector that is not a Borda vector. Then there exist profiles of voters that are completely consistent (with respect to the simple voting methods), but for which the multiple voting system has a completely indeterminate outcome.

Consequently, there exist examples in which the plurality election outcomes for *all subsets* of alternatives is the appropriate restriction of $a_1 > a_2 > \ldots > a_N$; yet if the same voters use approval voting, the outcome remains completely indeterminate. The same conclusion holds for truncated ballot systems and any other multiple system.

3. Robustness

A final issue concerns the robustness of these assertions. Can we dismiss these statements because the conclusions occur only with specially constructed, highly pathological examples? The answer is no; the results are robust, a conclusion that the results themselves already suggest. We introduced several measures, such as the Condorcet winner, examples for which the rankings over pairs are transitive, completely consistent, and so forth, to identify situations with an increasing conformity of agreement among the voters. Indeed, some of these situations have so much conformity that no viable theory could exclude them. Yet in all cases, the indeterminacy conclusion remained. Theorems 3 and 4, moreover, offer considerable added freedom in choosing the comparative election rankings; yet all of these choices can be accompanied with indeterminacy for a given multiple system.

A technical demonstration that our conclusions are robust turns out to be a simple consequence of our method of proof. Rather than following the somewhat traditional approach of constructing examples to verify the assertions, we use a representation for the space of all possible profiles. (See Saari, 82, 84, 85, 87) If there are N candidates, then there are N! possible rankings. For each ranking A, let n_A be the fraction of the voters with this ranking, and let f be the vector that these N! values define. The sum of the components of f equals unity, so there are N! – 1 degrees of freedom. These numbers define a simplex, Si(N!), in the positive orthant of an N! dimensional space. We can identify the voters' profiles with the (rational) points in Si(N!). If f is a point in Si(N!), then a common denominator of its components corresponds to the total number of voters in an example, while the numerator of n_A determines the number of voters with the ranking A. In the obvious manner, the tally methods can be defined for any point on the simplex whether or not it is rational. However, the image corresponds to a tally of an election only for rational points.

This representation for the space of all profiles allows us to consider questions that would be extremely difficult in other settings. For instance, a natural definition for robustness in a voting system would require a conclusion to hold even for small changes in the supporting profile of voters; for example, changes in the total number of voters or in the number of voters with certain rankings. In our setting this means that a result supported by a profile *f* is robust if the conclusion holds for an open set of profiles about *f*. Moreover, any open set in Si(N!) contains an infinite number of rational points, so it follows that any conclusion supported by an open set in Si(N!) holds for an infinite number of different examples. (Thus, such an assertion cannot be the consequence of just an isolated example.) This means that a useful *measure of robustness is whether a conclusion can be supported by an open set of profiles* from Si(N!). Theorem 6 asserts that all of our conclusions are robust. This has other implications. The standard, continuous probability measures assign a positive probability to open sets. So, Theorem 6 also implies that if we use a probability model over Si(N!), then our results have a positive probability of occurring. (In the concluding paragraphs of the next sextion, we give the interested reader some insight into the large size of these open sets.)

Theorem 6. For each of the preceding theorems, the set of examples defining the described properties contains an open set in Si(N!).

Finally, we remain with the issue of reform. It is difficult to accept a procedure as constituting acceptable reform if it has this robust, complete indeterminacy property, and particularly if it keeps this property even if other measures give evidence of a remarkable conformity in the voters' rankings. This does not mean that we are forced to live in the imperfect world of plurality voting. It does mean that in the search for reform procedures, we must be aware of this theoretical feature of multiple voting methods, and, perhaps, seek a solution elsewhere.

4. Proofs

The statements and proofs of the theorems in this paper are special cases of the general program described in (Saari, 1987). Most of the technical conditions we need are derived in (Saari, 1985).

Proof of Theorem 1. Assume that there are $N \geq 3$ candidates $\{a_1, \ldots, a_N\}$ and that the voting vectors $\{\underline{W}_J\}$, $j = 1, \ldots, s$, $s \geq 2$, define the multiple voting system. Then, each $\underline{W}_J$ is a vector in the N dimensional space R^N. Let A denote the ranking $a_1 > a_2 > \ldots > a_N$, and let P(A) be a generic representation for the N! permutations of A. For voting vector $\underline{W}_J$, any such permutation, P(A), determines how the ballot will be tallied. We can represent this tally by a permutation of the vector $\underline{W}_J$. Denote this permutation by $\underline{W}_{JP(A)}$. For instance, if $\underline{W} = (3, 2, 1)$, then the standard ranking $a_1 > a_2 > a_3$ defines the vector (3, 2, 1). The ranking $a_3 > a_1 > a_2$ defines the permutation of $\underline{W}$, (2, 1, 3), to reflect that for this ranking, two points are tallied for a_1, one for a_2, and three for a_3.

Let $n_{P(A)}$ denote the fraction of the voters with the ranking of the candidates P(A). The tally of a simple election using $\underline{W}_J$ is

$$\sum_{P(A)} n_{P(A)} \underline{W}_{JP(A)} \qquad 4.1$$

such that the summation is over all N! permutations P(A). We determine the outcome of the election by algebraically ranking the components in this vector sum.

There is a geometric representation for this algebraic ranking. Consider the indifference hyperplane in R^N given by $x_J = x_K$. If the vector sum 4.1 is on the $x_K > x_J$ side of this hyperplane, then a_K ranks higher than a_J, and vice versa. In particular, the N(N – 1)/2 possible 'indifference hyperplanes' divide R^N into 'ranking regions,' and the ranking region that contains the vector sum determines the ranking of the candidates.

For a multiple voting system, let $m_{JP(A)}$ denote the fraction of those voters with a P(A) ranking that elect to have their ballots tallied with the j^{TH} voting vector. Then, the fraction of the total number of voters with this tally is $n_{P(A)}m_{JP(A)}$. Consequently, the total tally is given by the double sum

$$\sum_{P(A)} n_{P(A)}[\sum_{J} m_{JP(A)} \underline{W}_{JP(A)}]. \qquad 4.2$$

Again, the ranking region of R^N that contains this vector sum determines the ranking of the candidates,

We represent Equation 4.2 as a mapping. Toward this end, let Si(M) = $\{x_1, \ldots, x_M) \mid x_K \geq 0, \Sigma x_K = 1\}$.

Because each term defines a percentage, the set $\{n_{P(A)}\}$ is a (rational) point in the set Si(N!). For each P(A), the set $\{m_{JP(A)}\}$ is in Si(s) (because the entries define non-negative fractions that sum to unity). This means that a domain point is in the $(N! - 1) + (s - 1)^{N!}$ dimensional space

$$T = Si(N!) \times (Si(s))^{N!}.$$

Any rational point in T corresponds to an example of voters' profiles along with their individual selections of voting vectors to tally the ballots. Thus, we can view Equation 4.2 as a mapping from T to R^N

$$F:T \text{ -------> } R^N, \qquad 4.3$$

such that F is the summation.

Define the 'complete indifference' ranking in R^N to be the line given by all scalar multiples of $\underline{E}_N$. The name reflects that this line corresponds to a complete tie in the rankings of the candidates. Notice that a) the complete indifference ranking is the intersection of all of the indifference hyperplanes, and b) this line is on the boundary of all other ranking regions.

To prove this theorem, we must show the existence of an n* in Si(N!) (a choice of voters' profiles) so that as the variable $m = \{m_{JP(A)}\}$ varies, the image of F((n*, m)) meets all possible ranking regions.

Let n* correspond to where there is an equal number of voters with each possible ranking of the candidates; that is, $n^* = (N!)^{-1}(1, 1, \ldots, 1)$. It follows immediately that if $m_{1P(A)} = 1$ for all choices of P(A) (all voters choose the first voting vector), then Equation 4.2 reduces to Equation 4.1, and the image of F is on the complete indifference line. The same conclusion holds if all of the $m_{JP(A)}$ are equal. This statement holds because the double summation can be interchanged to obtain separate summations of the type given in Equation 4.1, each of which yields a point on the complete indifference line. Denote this domain point by (n*, m*).

The idea is this. Assume that the Jacobian of F at (n*, m*) has rank equal to N, such that in the computation of the Jacobian, we hold the $n_{P(A)}$ variables fixed. (We treat them as parameters.) Hence, there is an open set about the interior point, m*, that is mapped to an open set about the image F((n*, m*)). This open set yields outcomes that occur with the same profile of voters (n*), but for which m (the choice of voting vectors to tally the ballots) varies. Because an open set about any point on the line of complete indifference meets all ranking regions, the conclusion follows. (It is easy to show that there are rational choices of m with this property. For details, see (Saari, 1987). Also, see (Saari, 1982, 1984, 1985).)

Thus, the proof is completed if we can discern certain properties about the Jacobian of F at (n*, m*). There are two cases to consider, and they are based upon the sum of the components of each voting vector. Either at least two of these sums differ, or they are all the same.

Assume that at least two of the sums differ. For each P(A) eliminate the dependency of the components $\{m_{JP(A)}\}$ by setting $m_{1P(A)} = 1 - \sum_{j>1} m_{JP(A)}$. Then, the rank of the Jacobian of F is determined by the maximum number of independent vectors in subsets from

$$\{\underline{W}_{JP(A)} - \underline{W}_{1P(A)}\}, \qquad 4.4$$

for which P(A) ranges over all N! permutations of A and $j = 2, \ldots, s$. There is a choice of j where the sum of the components of $\underline{W}_J$ does not equal the sum of the components of $\underline{W}_1$, say $j = 2$, where we assume that the sum of the components of $\underline{W}_2$ is larger than the sum of the components of $\underline{W}_1$. What we show is that the set of vectors

$$\{\underline{W}_{2P(A)} - \underline{W}_{1P(A)}\} \tag{4.5}$$

spans R^N. This will complete the proof.

It follows immediately that $\sum_{P(A)} \underline{W}_{JP(A)}$ is a nonzero scalar multiple of $\underline{E}_N$ such that the scalar is $(N-1)!$ times the sum of the components of $\underline{W}_J$. Thus, $\sum_{P(A)} (\underline{W}_{2P(A)} - \underline{W}_{1P(A)})$ is a positive scalar multiple of $\underline{E}_N$, so $\underline{E}_N$ is in the space spanned by the vectors in Equation 4.5. The simplex Si(N) has $\underline{E}_N$ as a normal vector, so we prove the theorem if the vectors in Equation 4.5 span the simplex.

View each vector in Equation 4.5 as a permutation of the components of $\underline{W} = \underline{W}_2 - \underline{W}_1$. Let vector $\underline{V}$ be the permutation of $\underline{W}$ that has the largest value in the first component, the second largest in the second component, and so forth. (For example, if $\underline{W}_2 = (5, 4, 2, 1)$ and $\underline{W}_1 = (5, 1, 1, 0)$, then $\underline{W} = (0, 3, 1, 1)$ and $\underline{V} = (3, 1, 1, 0)$.) The set of all possible permutations of $\underline{V}$,

$$\{\underline{V}_{P(A)}\}, \tag{4.6}$$

agrees with the set in Equation 4.5. Because $\underline{W}$ is not a multiple of $\underline{E}_N$, we can treat $\underline{V}$ as being a voting vector, and we can interpret the vectors in Equation 4.6 as being the various ways to tally ballots. That this set spans Si(N) follows immediately from the results in (Saari, 1982, 1984, 1985).

Suppose that the sums of the components for each of the voting vectors are the same. We show that the Jacobian of F has rank $N-1$ and its image spans a simplex Si(N). This demonstration requires an adjustment in the proof. First, we map an open set about m* to an open set about F((n*, m*)) *in the simplex*. But, such an open set must meet all ranking regions. (The simplex has codimension one, and the line of complete indifference defines the normal direction.) Thus, all we must show is that the vectors in 4.6 span the subspace orthogonal to $\underline{E}_N$. This is the same argument given above. This completes the proof.

The proofs of Theorems 3 and 4 depend heavily upon the proofs and results in (Saari, 1985). Essentially, the idea of the proofs is to use special ways in which the voters choose their voting vectors to obtain a simple voting systems. Then, modifications of the kind that we used in the proof of Theorem 1 and results from (Saari, 1985) lead to a condition of the sort for which F((n*, m*)) is on the line of complete indifference. The Jacobian condition follows from the analysis in the proofs of (Saari, 1985).

Proof of Theorem 4. The following lemma is a consequence of Theorems 5 and 7 in (Saari, 1985).

Lemma. Suppose that there are two simple voting vectors, $\underline{V}_1$ and $\underline{V}_2$, which, a) form a linearly independent set along with $\underline{E}_N$ and for which b) a Borda vector is not in the span of these three vectors.

Rank the pairs of candidates in any way, and then choose two arbitrary rankings of the N candidates. Then, there exist profiles of voters so that if the same voters consider each pair of candidates, a majority prefers the designated one. If these same voters rank the N candidates by the simple voting system, $\underline{V}_J$, then the outcome is the j^{TH} ranking of the candidates, $j = 1, 2$.

According to the statement of the theorem, the range space containing the tally of the various subsets of candidates is given by $S = (R^N) \times (R^N) \times (R^2)^P$ such that $p = N(N-1)/2$. The first component space is the tally of the simple voting system, the second is the tally of the multiple voting system, and the last p components contain the tally of the binary comparisons. The domain is T. Thus, the obvious summations define the mapping

F*: T ----------> S.

The proof follows much as in that of Theorem 1. We show the existence of a set of profiles, n″, for which the rankings of the simple system and the rankings of the pairs of candidates are as specified. Moreover, we choose n″ so that an accompanying interior point m″ in $Si(s)^N$ can be found with the property that F((n″, m″)) is the ranking of complete indifference. Then, we repeat the previous argument concerning the Jacobian of F*. The main difference is that we evaluate it at (n″, m″), not at at (n*, m*).

First, we find m″. To do so we choose the $m_{JP(A)}$'s to depend on J but not on P(A). Namely, $m_{JP(A)} = m_J$ for all P(A). This procedure defines a continuum of voting vectors where the dimension of the continuum depends upon the number of linearly independent vectors in the multiple voting system. Since we have a continuum of them available, and since at least two vectors in this system define a three dimensional space with $\underline{E}_N$, we can choose the m_J's to obtain a voting vector, $\underline{V}_N$, that, along with $\underline{W}_N$, satisfies the condition of the lemma. This defines m″. To use the lemma, choose the ranking corresponding to $\underline{V}_N$ to be complete indifference, the ranking corresponding to $\underline{W}_N$ to be as specified in the theorem, and the pairwise rankings as specified in the theorem. This defines (n″, m″). The conclusion then follows from these interpretations and the lemma.

Proof of Theorem 3. In this setting the domain and the image of F change drastically from that just given. Here we have $2^N - (N + 1)$ different subsets with at least two candidates. Thus, the range space is the cartesian product over all of these sets of Euclidean spaces of the same dimension as the number of candidates in the subset. The domain also increases significantly. For each subset there is a multiple voting method. Thus, for each ranking of the candidates in each subset, the domain increases by another product of a simplex reflecting the various choices that the voters have to tally their ballots. Let T′ represent the new domain, which is a much larger product space of simplices, and let R′ represent the larger image space. The tally of the ballots still is given by summations of the type found in Equation 4.2. They define a mapping

$$F': T' \text{ -----------> } R'.$$

As in the statement of the theorem, designate for each pair of candidates which one a majority of the voters prefers. We now appeal to Theorem 6 in (Saari, 1985). A consequence of this result is that for 'most' simple voting systems, there exist profiles of voters so that for each of the pairs of the candidates, a majority of them favors the designated candidate. Yet their rankings of all subsets with three or more candidates is complete indifference. 'Most' replaces the linear independence condition in the lemma, and it means that the voting vectors for various subsets do not make a certain determinant vanish. For our purposes it suffices to note that any voting vector that is 'plurality like' satisfies this condition. To use this theorem, we find a special case of the multiple voting system that is a simple voting system satisfying the stated preferences.

For each subset of more than three candidates, choose $m_{JP(A)} = m_J$. This, then, defines a convex combination of the voting vectors that are available to tally the rankings. Now choose the m_J's so that $\Sigma m_J \underline{W}_J$ defines a plurality like vector. Such a vector does not make the determinant condition vanish. There is the possibility that the vector defined by the m_J's is not an interior point. So, if not all of the m_J's are positive, then we can be perturb them so that all are positive and the sum is still a vector that satisfies the non-vanishing of the determinant. (This result reflects that the non-vanishing of a determinant is an open condition.) Thus, for each subset, we can choose the $\{m_J\}$ so that the resulting vectors over all subsets do not satisfy the vanishing determinant condition.

Let m' correspond to these choices of $\{m_{JP(A)}\}$ over all subsets. We have from Theorem 6 in (Saari, 1985) that there exist profiles of voters, n', so that the various components of $F((n', m'))$ are on the line of complete indifference, yet the ranking of the pairs is as designated. What remains for us to show is that the rank of the Jacobian of F', if n is held fixed, is of the rank of the dimension of the range. With the modifications of the kind found in the proof of Theorem 3, this demonstration follows from the proof of Theorem 6 in (Saari, 1985). Indeed, most of the proof of Theorem 6 involves proving this independence condition.

Proof of Corollary 3.1. Theorem 4 in (Saari, 1985) asserts that if the voting vector is not a Borda Vector, then for any rankings of the pairs of alternatives and for any ranking of the N alternatives, a profile of voters exists that realizes all of these outcomes simultaneously. In particular, we can choose the rankings of the pairs to define a Condorcet winner, and the ranking of the N candidates to define the complete indifference ranking. The hypothesis of the corollary asserts that there is at least one voting vector in the multiple methods that is not a Borda Vector. Thus, m′ can be chosen in a manner similar to the construction found in the proof of Theorem 3 so that the resulting voting vector is not Borda. Then, the same type of proof goes through.

Proof of Theorem 5. This proof, essentially, is the same as that for the other theorems. From (Saari, 1985), we get the independence of the vectors defined by the simple voting vectors. The independence of the vectors in Si(s) defined by the multiple voting vectors is the same in the proof for Theorem 3.

Proof of Theorem 6. We demonstrate the basic ideas. Because the ideas extend immediately for the other theorems, we only use the hypothesis of Theorem 1.

To prove the theorem, all that is necessary is to show that there is an open set, U, about n* in Si(N) (the space of voters' profiles) so that if n′ is in U, then there is an interior point, m′, in $(Si(s))^{N!}$ such that F((n′, m′)) is on the line of complete indifference. To show this result, we give a geometric interpretation for Equation 4.2.

For each ranking of the candidates, P(A), the bracketed term in the double summation Equation 4.2 defines the convex hull of the vectors $(\underline{W}_{JP(A)})$. Thus, this means that the double summation yields the convex hull of the N! convex hulls. That the image of F((n*, m)) contains an open set means that this particular convex combination of the convex hulls contains an open set around the point F((n*, m*)) on the line of complete indifference. It now follows from continuity considerations that the conclusion holds. This completes the proofs of the theorems.

Although Theorem 6 proves that these indeterminate results are robust, it does not answer the question about the size of these open sets. In other words, how large, or how small, is the open set of profiles that creates the examples? We can develop some intuition about this question by reviewing the construction used in the proofs of all of the theorems. The basic idea is to find points (n′, m′) so that F((n′, m′)) is on the line of complete indifference. Such a point can be viewed as a boundary point of the region for each of the examples specified in the theorems. Therefore, a crude measure of this likelihood of indeterminancy is the abundance of the points (n′, m′) so that F((n′, m′)) is the rankings of complete indifference. If it is an isolated point, then the accompanying open set probably is small. If it is a line, then the open set probably is larger.

In each of the proofs, an independence argument is used. This argument also serves to show that the implicit function theorem applies. Consequently, the inverse image of the complete indifference line in T is an affine space of codimension $N-1$. To appreciate the large size of this space, let $N = 4$ and let the multiple system be approval voting ($s = 3$). Then, the base points define a 58-dimensional linear subspace in a 61-dimensional space. This very large dimension strongly suggests that the set of profiles leading to any of these examples of indeterminate outcomes is very large.

Incidently, these large dimensions that arise for only 4 candidates indicate (1) why standard methods do not suffice in the analysis of such voting schemes, (2) why we use a convexity argument to prove Theorem 6 instead of an implicit function argument (which would have involved a massive linear independence argument), (3) why the completely indeterminate effect occurs (F is trying to force the input from a 61-dimensional space of profiles and voter types onto a 3-dimensional space; the domain has to be 'squashed' in the image, so we must expect such negative conclusions to result from the overflow.) (4) why simple voting systems do not have the same adverse effects (the subspace is 20-dimensional in a 23-dimensional space, so the corresponding F is only trying to force a 23-dimensional space, rather than a 61-dimensional space, onto a three dimensional space), and (5) that there are many examples other than those specified by the theorems. (For instance, because the 58-dimensional space is affine, it must intersect the boundaries of T. The boundaries correspond to examples of voters' profiles where there are no voters that have certain rankings of the N candidates. This leads to examples of indeterminacy with extreme examples. See (Saari, 1987). Secondly, the added dimension of the domain can be viewed as offering more 'strategies' for a manipulative voter. This suggests, and it is true, that multiple systems are more susceptible to manipulation. See (Saari 1986).

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