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ON THE COMPOUNDED BIVARIATE POISSON DISTRIBUTION: A UNIFIED TREATMENT

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Abstract. A unified treatment is presented here of compounding with the bivariate Poisson distribution. Exploiting the exponential nature of its probability generating function, it is shown that the pgf of the compound distribution is the moment generating function of the compounding random variable. This relationship leads to rather interesting general results. Particularly, the development of the conditional distribution is simplified. Four cases are presented in detail.

Key words and phrases: Bivariate Poisson distribution, probability generating function, bivariate-Hermite, -negative binomial, -Poisson-Inverse Gaussian, -Neyman Type A, conditional distributions.

1. Introduction

The bivariate Poisson distribution is defined by the probability generating function

$$\Pi(z_1, z_2) = \exp\{w_1(z_1 - 1) + w_2(z_2 - 1) + w_3(z_1z_2 - 1)\},$$

where w_1, w_2, w_3 are all positive. (See Feller (1957), p. 261.) This distribution has been widely applied in the literature to describe several real life models. On the other hand, several authors have examined the problem where $w_i = \tau\lambda_i$, $i = 1, 2, 3$ with $\lambda_1, \lambda_2, \lambda_3$ being constants and τ a random variable characterizing an 'individual' in the population. Such models have been considered among others by Holgate (1966), Subrahmaniam (1966), Gillings (1974) and Kemp and Papageorgiou (1982). In each case τ is assumed to have a distribution of the discrete or continuous type.

In the present paper, we present a unified development of the distributions associated with such compounding.

2. Preliminaries

Let X, Y be jointly distributed with the bivariate Poisson distribution defined by the (joint) probability generating function

$$(2.1) \qquad \Pi(z_1, z_2 \mid \tau) = \exp[\tau\{\lambda_1(z_1 - 1) + \lambda_2(z_2 - 1) + \lambda_3(z_1 z_2 - 1)\}] .$$

Here τ is a 'random' parameter characterizing an individual in the population, while $\lambda_1, \lambda_2, \lambda_3$ are constants. For example, X and Y could be the number of accidents observed over two consecutive periods of time. While for a given individual it is realistic to assume that X and Y are jointly distributed with the pgf (2.1), their distribution over the whole population will have to take into consideration the variability in τ. Let $g(t)$ be the probability distribution of τ with the moment generating function $M(\theta)$. Then the joint distribution of X and Y has the pgf

$$(2.2a) \qquad \Pi(z_1, z_2) = \sum_\tau g(\tau)\Pi(z_1, z_2 \mid \tau) \qquad \text{(discrete case)} ,$$

$$(2.2b) \qquad = \int g(\tau)\Pi(z_1, z_2 \mid \tau)d\tau \qquad \text{(continuous case)} .$$

In either case, upon substituting for $\Pi(z_1, z_2 \mid \tau)$ from (2.1) in (2.2a) or (2.2b) we have

$$(2.3) \qquad \Pi(z_1, z_2) = M[\lambda_1(z_1 - 1) + \lambda_2(z_2 - 1) + \lambda_3(z_1 z_2 - 1)] .$$

If $\lambda_3 = 0$, then (2.1) shows that X and Y are, conditional on τ, independent. However, the joint pgf of X and Y is

$$(2.4) \qquad \Pi(z_1, z_2) = M[\lambda_1(z_1 - 1) + \lambda_2(z_2 - 1)] ,$$

which shows that they are not independent.

3. Probability function and moments: Joint distribution

The joint distribution of (X, Y) is defined by the probability function

$$(3.1) \qquad f_{r,s} = P\{X = r, Y = s\} = \frac{1}{r!s!} \frac{\partial^{r+s}}{\partial z_1^r \partial z_2^s} \Pi(z_1, z_2) \Big|_{z_1 = z_2 = 0} ,$$

while the joint factorial moment, around zero,

$$\mu_{[r,s]} = E[X^{[r]} Y^{[s]}] ,$$

is given by

$$\mu'_{[r,s]} = \frac{\partial^{r+s}}{\partial z_1^r \partial z_2^s} \Pi(z_1, z_2) \Big|_{z_1=z_2=1} . \tag{3.2}$$

Here $X^{[r]} = X(X-1)\cdots(X-r+1)$.

In determining (3.1) and (3.2), we find the following theorem of great use:

THEOREM 3.1. *For the joint pgf in* (2.3)

$$\frac{\partial^{r+s}}{\partial z_1^r \partial z_2^s} \Pi(z_1, z_2) = \sum_{k=0}^{\min(r,s)} \binom{r}{k}\binom{s}{k} k!\, M^{(t-k)}(u) T_1^{r-k} T_2^{s-k} \lambda_3^k , \tag{3.3}$$

where $M^{(j)}$ *is the j-th derivative of M, and*

$$u = \lambda_1(z_1 - 1) + \lambda_2(z_2 - 1) + \lambda_3(z_1 z_2 - 1) ,$$
$$t = r + s, \quad T_1 = \lambda_1 + \lambda_3 z_2, \quad T_2 = \lambda_2 + \lambda_3 z_1 .$$

PROOF. Without loss of generality let $r \geq s$. Let $\Pi^{(r,s)}(z_1, z_2)$ represent the (r, s)-th partial derivative of $\Pi(z_1, z_2)$. Then, differentiating (3.3) with respect to z_1 yields

$$\Pi^{(r+1,s)}(z_1, z_2) = \sum_{k=0}^{s} \binom{r}{k}\binom{s}{k} k!\, M^{(t-k+1)}(u) T_1^{r-k+1} T_2^{s-k} \lambda_3^k$$
$$+ \sum_{k=0}^{s-1} \binom{r}{k}\binom{s}{k} k!\, M^{(t-k)}(u) T_1^{r-k} (s-k) T_2^{s-k-1} \lambda_3^{k+1} ,$$

which reduces to

$$\sum_{k=0}^{s} \binom{r+1}{k}\binom{s}{k} k!\, M^{(t+1-k)}(u) T_1^{r+1-k} T_2^{s-k} \lambda_3^k . \tag{3.4}$$

Differentiating (3.4) with respect to z_2, we have

$$\Pi^{(r+1,s+1)}(z_1, z_2)$$
$$= \sum_{k=0}^{s} \binom{r+1}{k}\binom{s}{k} k!\, M^{(t+2-k)}(u) T_1^{r-k+1} T_2^{s-k+1} \lambda_3^k$$
$$+ \sum_{k=0}^{s} \binom{r+1}{k}\binom{s}{k} k!\, M^{(t+1-k)}(u) (r+1-k) T_1^{r-k} T_2^{s-k} \lambda_3^{k+1} .$$

Upon rearranging the terms and changing the order of summation, we have on the right hand side

$$(3.5)\qquad \sum_{k=0}^{s+1} \binom{r+1}{k}\binom{s+1}{k} k!\, M^{(t+2-k)}(u) T_1^{r+1-k} T_2^{s+1-k} \lambda_3^k ,$$

as was to be proved.

From Theorem 3.1 we have the expressions for the joint probability function and the factorial moment:

(i) Putting $z_1 = z_2 = 0$ in $\Pi^{(r,s)}(z_1, z_2)$,

$$(3.6)\qquad f_{r,s} = \frac{1}{r!\,s!} \Pi^{(r,s)}(0, 0) = \frac{\lambda_1^r \lambda_2^s}{r!\,s!} \sum_{k=0}^{\min(r,s)} \binom{r}{k}\binom{s}{k} k!\, M^{(t-k)}(\gamma)\delta^k ,$$

where $\gamma = -(\lambda_1 + \lambda_2 + \lambda_3)$, $\delta = \lambda_3 / \lambda_1 \lambda_2$.

If $\lambda_3 = 0$, then

$$(3.7)\qquad f_{r,s} = \frac{\lambda_1^r \lambda_2^s}{r!\,s!} M^{(t)}(-\lambda_1 - \lambda_2).$$

(ii) Putting $z_1 = z_2 = 1$ in $\Pi^{(r,s)}(z_1, z_2)$,

$$(3.8)\qquad \mu'_{[r,s]} = (\lambda_1 + \lambda_3)^r (\lambda_2 + \lambda_3)^s \sum_{k=0}^{\min(r,s)} \frac{r!\,s!\,\delta^k \mu'_{r+s-k}}{(r-k)!(s-k)!k!} ,$$

where $\delta = \lambda_3 / (\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)$ and μ'_t is the t-th moment of the distribution $g(t)$ around zero.

If $\lambda_3 = 0$,

$$\mu'_{[r,s]} = \lambda_1^r \lambda_2^s \mu'_{r+s} .$$

From (3.8), the correlation between X and Y can be found to be

$$\rho_{XY} = \frac{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)\sigma^2 + \mu'_1 \lambda_3}{\{[(\lambda_1 + \lambda_3)^2 \sigma^2 + \mu'_1(\lambda_1 + \lambda_3)][(\lambda_2 + \lambda_3)^2 \sigma^2 + \mu'_1(\lambda_2 + \lambda_3)]\}^{1/2}} .$$

It should be noted that the intrinsic correlation ρ_I defined by Subrahmaniam (1966) is

$$\rho_I = \lambda_3 / \{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)\}^{1/2} .$$

3.1 *Examples*

Four examples are considered. Some have appeared in the literature. Appropriate references are given.

(i) *Bivariate Negative Binomial*

Edwards and Gurland (1961) and Subrahmaniam (1966) have independently developed the generalized version of the Bates-Neyman model. Here τ is taken to have the two parameter gamma distribution with pdf

$$g(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \left(\frac{\alpha}{m}\right)^{\alpha} \exp\left(-\frac{\alpha}{m} t\right), \quad t > 0 ,$$

and the mgf

$$M(t) = \left(1 - \frac{m}{\alpha} t\right)^{-\alpha} .$$

The joint pgf of X and Y under this model is

$$\Pi(z_1, z_2) = \left[1 - \frac{m}{\alpha}\{\lambda_1(z_1 - 1) + \lambda_2(z_2 - 1) + \lambda_3(z_1 z_2 - 1)\}\right]^{-\alpha} . \tag{3.9}$$

Since

$$M^{(r)}(t) = \left(\frac{m}{\alpha}\right)^r \frac{\Gamma(\alpha + r)}{\Gamma(\alpha)} \left(1 - \frac{m}{\alpha} t\right)^{-(\alpha+r)} ,$$

we have

$$f_{r,s} = \frac{\lambda_1^r \lambda_2^s \Gamma(\alpha + r + s)}{\Gamma(\alpha) r! s!} \left(\frac{m}{\alpha}\right)^{r+s} \left(1 - \frac{m\gamma}{\alpha}\right)^{-(\alpha+r+s)} \cdot \sum_{k=0}^{\min(r,s)} \frac{\binom{r}{k}\binom{s}{k}}{\binom{r+s+\alpha-1}{k}} \delta^k , \tag{3.10}$$

where $\delta = (\alpha/m - \gamma)\, \lambda_3/\lambda_1\lambda_2$.

For $\lambda_1 = 1$, $\lambda_2 = \alpha_1$, $\lambda_3 = \alpha_2$, this model reduces to Subrahmaniam (1966, Equation 2.3).

(ii) *Bivariate Neyman Type A*

Holgate (1966) and Gillings (1974) have considered a bivariate generalization of the Neyman Type A distribution. In this case

$$g(t) = \lambda^t e^{-\lambda}/t! , \quad t = 0, 1, 2, \ldots ,$$

with

$$M(t) = \exp\{\lambda(e^t - 1)\} .$$

Hence

$$\Pi(z_1, z_2) = \exp\lambda[\exp\{\lambda_1(z_1 - 1) + \lambda_2(z_2 - 1) + \lambda_3(z_1z_2 - 1)\} - 1] . \tag{3.11}$$

Also, since for $r \geq 1$,

$$M^{(r)}(t) = [\exp\{\lambda(e^t - 1)\}][r\text{-th moment of } P(\lambda e^t)] ,$$

the joint probability function is seen to be

$$f_{r,s} = \frac{\lambda_1^r \lambda_2^s}{r!s!} \exp\{\lambda(e^\gamma - 1)\} \sum_{k=0}^{\min(r,s)} \binom{r}{k}\binom{s}{k} k!\tau^k \eta_{t-k} , \tag{3.12}$$

where $\tau = \lambda_3/\lambda_1\lambda_2$, η_r=the r-th raw moment of $P(\lambda e^\gamma)$ and $t = r+s$. This agrees with Gillings (1974).

We note here that the result

$$M^{(r)}(\gamma) = M(\gamma)\eta_r ,$$

where η_r is the r-th raw moment of $P(\lambda e^\gamma)$, seems to characterize the Poisson distribution.

(iii) *Bivariate Hermite*

Kemp and Papageorgiou (1982) have considered

$$g(t) = \frac{1}{\sqrt{2\pi}\sigma} \exp - \frac{(t-\mu)^2}{2\sigma^2} ,$$

with the mgf

$$M(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right) .$$

Since

$$\frac{d^r M(t)}{dt^r} = M(t) P_r(t) ,$$

where $P_r(t)$ is a polynomial of degree r in t, we can write the joint probability function of X and Y as

$$f_{r,s} = \frac{\lambda_1^r \lambda_2^s}{r!s!} M(\gamma) \sum_{k=0}^{\min(r,s)} \binom{r}{k}\binom{s}{k} k!\, P_{r+s-k}(\gamma)\delta^k , \tag{3.13}$$

where $\delta = \lambda_3/\lambda_1\lambda_2$. The pgf corresponding to (3.13) is

$$(3.14) \quad \Pi(z_1, z_2) = \exp[\mu\{\lambda_1(z_1 - 1) + \lambda_2(z_2 - 1) + \lambda_3(z_1z_2 - 1)\} + \frac{\sigma^2}{2}\{\lambda_1(z_1 - 1) + \lambda_2(z_2 - 1) + \lambda_3(z_1z_2 - 1)\}^2] .$$

The pgf (3.14) has not been derived by Kemp and Papageorgiou (1982) except as a natural generalization of the univariate case. Here too, as in their paper, we would obtain the two forms of the bivariate Hermite.

(iv) *Bivariate Poisson-Inverse Gaussian*

Holla (1967) has considered a univariate Poisson-Inverse Gaussian distribution which Sichel (1982) has used for explaining the behaviour of the customer-buying and word frequency. A bivariate extension of this distribution is obtained by taking the inverse Gaussian pdf

$$g(t) = \left\{\frac{\lambda}{2\pi t^3}\right\}^{1/2} \exp\left\{-\frac{\lambda}{2\mu^2 t}(t - \mu)^2\right\}, \quad t > 0 ,$$

with the parameters λ and μ, and mgf (see Tweedie (1957), for details):

$$M(t) = \exp\frac{\lambda}{\mu}\left\{1 - \left[1 - \frac{2\mu^2 t}{\lambda}\right]^{1/2}\right\} .$$

The joint pgf of X and Y is then

$$(3.15) \quad \Pi(z_1, z_2) = \exp\left[\frac{\lambda}{\mu}\left\{1 - \left[1 - \frac{2\mu^2}{\lambda}(\lambda_1[z_1 - 1] + \lambda_2[z_2 - 1] + \lambda_3[z_1z_2 - 1])\right]^{1/2}\right\}\right] .$$

It can be shown that

$$(3.16) \quad M^{(r)}(t) = \exp\left[\frac{\lambda}{\mu}\left\{1 - \left(1 - \frac{2\mu^2}{\lambda}t\right)^{1/2}\right\}\right]\frac{\mu^{2r-1}}{\lambda^{r-1}} \cdot \sum_{k=0}^{r} c_k \left(\frac{\lambda}{\mu}\right)^k \left(1 - \frac{2\mu^2}{\lambda}t\right)^{-(r-(k+1)/2)} ,$$

where c_k is a constant.

The joint probability function can be written as

$$(3.17) \quad f_{r,s} = \frac{\lambda_1^r \lambda_2^s}{r!s!} \exp\left[\frac{\lambda}{\mu}\left\{1 - \left(1 - \frac{2\mu^2}{\lambda}\gamma\right)^{1/2}\right\}\right] \sum_{k=0}^{\min(r,s)} \binom{r}{k}\binom{s}{k} k!\delta^k \cdot \sum_{j=0}^{(r+s-k)} \frac{\mu^{2r+2s-2k-j-1}}{\lambda^{r+s-k-j-1}} c_j \left(1 - \frac{2\mu^2}{\lambda}\gamma\right)^{-\{(r+s-k)-(j+1)/2\}} ,$$

where $\gamma = -(\lambda_1+\lambda_2+\lambda_3)$ and $\delta = \lambda_3/\lambda_1\lambda_2$.

If we consider a univariate Poisson-Inverse Gaussian distribution, the pgf is seen to be

$$\Pi(z) = \exp\left[\frac{\lambda}{\mu}\left\{1 - \left[1 - \frac{2\mu^2}{\lambda}(z-1)\right]^{1/2}\right\}\right].$$

4. Recurrence relations

The joint pgf of (X, Y) can be differentiated with respect to z_1 and z_2 to yield recurrence relations. This is given by:

THEOREM 4.1. *Let $\Pi_{r,s}$ denote the (r, s)-th partial derivative of $\Pi(z_1, z_2)$ with respect to z_1, z_2, respectively. Then for $r \geq 1$, $s \geq 1$, and*

(4.1a) $$r \geq s\,, \quad \Pi_{r,s} = \frac{T_2}{T_1}\Pi_{r+1,s-1} + (r-s+1)\frac{\lambda_3}{T_1}\Pi_{r,s-1}\,,$$

(4.1b) $$r \leq s\,, \quad \Pi_{r,s} = \frac{T_1}{T_2}\Pi_{r-1,s+1} + (s-r+1)\frac{\lambda_3}{T_2}\Pi_{r-1,s}\,.$$

PROOF. The results (4.1a) and (4.1b) follow directly by induction. For example, to verify (4.1a) differentiate both sides of this equation with respect to z_1 to obtain, for $r \geq s$

$$\Pi_{r+1,s} = \frac{T_2}{T_1}\Pi_{r+2,s-1} + (r-s+1)\frac{\lambda_3}{T_1}\Pi_{r+1,s-1} + \frac{\lambda_3}{T_1}\Pi_{r+1,s-1}\,.$$

Rearranging the terms on the right hand side we verify (4.1a).

A consequence of the recurrence relations in Theorem 4.1 is the recurrence relations in the joint probability function $f_{r,s}$ and the joint (raw) factorial moments $\mu_{[r,s]}$.

(i) *Probability function*
Since

$$f_{r,s} = \Pi_{r,s}(0,0)/r!s!\,,$$

we have for $r \geq 1$, $s \geq 1$ and

(4.2a) $$r \geq s\,, \quad f_{r,s} = \frac{\lambda_2}{\lambda_1}\frac{r+1}{s}f_{r+1,s-1} + (r-s+1)\frac{\lambda_3}{\lambda_1}\frac{1}{s}f_{r,s-1}\,,$$

(4.2b) $$r \leq s\ , \quad f_{r,s} = \frac{\lambda_1}{\lambda_2} \frac{s+1}{r} f_{r-1,s+1} + (s-r+1) \frac{\lambda_3}{\lambda_2} \frac{1}{r} f_{r-1,s}\ .$$

These relations facilitate a quick computation of the expected frequencies for the joint distribution. Also,

(4.3) $$f_{r,0} = \frac{\lambda_1^r}{r!} M^{(r)}(\gamma), \quad f_{0,s} = \frac{\lambda_2^s}{s!} M^{(s)}(\gamma)\ .$$

K. Kawamura (1985) has discussed the selection of recurrence relations to get the probability function in the case of the bivariate Poisson distribution.

(ii) *Factorial moments*

Let μ_r' represent the r-th raw moment of the compounding distribution $g(t)$. Let the (r, s)-th joint raw factorial moment of (X, Y) be

$$\mu_{[r,s]}' = E[X^{[r]} Y^{[s]}]\ .$$

Then

$$\mu_{[r,0]}' = \mu_r'(\lambda_1 + \lambda_3)^r\ ,$$

$$\mu_{[0,s]}' = \mu_s'(\lambda_2 + \lambda_3)^s\ .$$

Also

$$\mu_{[r,s]}' = \Pi_{r,s}(1,1)\ .$$

Thus for $r \geq 1$, $s \geq 1$, and

(4.4a) $$r \geq s\ , \quad \mu_{[r,s]}' = \frac{\lambda_2 + \lambda_3}{\lambda_1 + \lambda_3} \mu_{[r+1,s-1]}' + (r-s+1) \frac{\lambda_3}{\lambda_1 + \lambda_3} \mu_{[r,s-1]}'\ ,$$

(4.4b) $$r \leq s\ , \quad \mu_{[r,s]}' = \frac{\lambda_1 + \lambda_3}{\lambda_2 + \lambda_3} \mu_{[r-1,s+1]}' + (s-r+1) \frac{\lambda_3}{\lambda_2 + \lambda_3} \mu_{[r-1,s]}'\ .$$

5. Conditional distribution

Subrahmaniam (1966) has shown that the conditional probability generating function is given by

(5.1) $$\Pi_{y|x}(z) = \Pi_{(x,0)}(0, z) / \Pi_{(x,0)}(0, 1)\ .$$

Here

$$\Pi_{(x,0)} = M^{(x)}(u)\,(\lambda_1 + \lambda_3 z_2)^x\ ,$$

where

$$u = \lambda_1\,(z_1 - 1) + \lambda_2\,(z_2 - 1) + \lambda_3\,(z_1 z_2 - 1)\ .$$

Hence

$$\text{(5.2)} \qquad \Pi_{y|x}\,(z) = \frac{M^{(x)}[\lambda_2 z - (\lambda_1 + \lambda_2 + \lambda_3)]}{M^{(x)}[-\,(\lambda_1 + \lambda_3)]} \left\{ \frac{\lambda_1}{\lambda_1 + \lambda_3} + \frac{\lambda_3 z}{\lambda_1 + \lambda_3} \right\}^x ,$$

or

$$\Pi_{y|x}\,(z) = \Pi_2\,(z)\,\{q + pz\}^x\ .$$

We thus have:

THEOREM 5.1. *Let the conditional pgf of* (X, Y) *for the given r.v.* τ *be*

$$\Pi(z_1,z_2|\tau) = \exp[\tau\{\lambda_1(z_1 - 1) + \lambda_2(z_2 - 1) + \lambda_3(z_1 z_2 - 1)\}]\ ,$$

and let the moment generating function of τ *be* $M(t)$. *Then the conditional distribution of* Y *given* $X=x$ *is the convolution of* (i) $Y_1 \sim B(x, \lambda_3/(\lambda_1+\lambda_3))$ *and* (ii) Y_2, *of which the pgf is given by the ratio*

$$M^{(x)}[\lambda_2(z - 1) - \lambda_1 - \lambda_3]/M^{(x)}[-(\lambda_1 + \lambda_3)]\ .$$

In the particular cases we study below, the r.v. Y_2 will be seen to be a convolution itself. If $\lambda_3=0$, that is, X and Y are conditionally independent, then the conditional distribution of Y given $X=x$ is that of Y_2 with the pgf

$$M^{(x)}[\lambda_2(z - 1) - \lambda_1]/M^{(x)}[-\lambda_1]\ .$$

Using the expansion

$$\text{(5.3)} \qquad M^{(x)}[\lambda_2(z - 1) - \lambda_1 - \lambda_3] = \sum_{j=0}^{\infty} \frac{M^{(x+j)}(-\,\lambda_1 - \lambda_3)}{j!}\,\{\lambda_2(z - 1)\}^j\ ,$$

it is possible to write the pgf of Y_2 in Theorem 5.1 as

$$\text{(5.4)} \qquad \Pi_2(z) = \sum_{j=0}^{\infty} \left\{ \frac{M^{(x+j)}[-\,\lambda_1 - \lambda_3]}{M^{(x)}[-\,\lambda_1 - \lambda_3]} \right\} \frac{\{\lambda_2(z - 1)\}^j}{j!}\ .$$

From (5.4) we see that if $M(t)=\exp(ct)$, c a constant, then $\Pi_2(z)=\exp\theta(z-1)$; that is $Y_2\sim P(\theta)$. On the other hand, if $\Pi_2(z)=\exp\theta(z-1)$, then a necessary condition for this to hold is

$$\frac{M^{(x+j)}(t)}{M^{(x)}(t)} = \text{constant}\ , \quad j = 0, 1, \dots .$$

This implies that $M(t)=\exp(ct)$. These lead to:

THEOREM 5.2. *Let the conditional distribution of* (X, Y) *given* τ *be bivariate Poisson. A necessary and sufficient condition for the conditional distribution of* Y *given* $X=x$ *to be the convolution of* $B(x,p)$ *and* $P(\theta)$ *is that the (compounding) distribution of* τ *be degenerate (at some point* c*).*

The preceding results are exemplified below, for the special cases introduced above. The study is mainly concerned with the examination of $\Pi_2(z)$.

5.1 *Bivariate negative binomial*

From $M(t)=(1-\beta t)^{-\alpha}$ we have

$$M^{(x+j)}(t) = \beta^{x+j}\frac{\Gamma(\alpha+x+j)}{\Gamma(\alpha)}(1-\beta t)^{-(\alpha+x+j)}\ ,$$

or

$$\begin{aligned}\Pi_2(z) &= \sum_{j=0}^{\infty}\frac{\{\lambda_2(z-1)\}^j}{j!}\left\{\frac{\beta^j\Gamma(\alpha+x+j)}{[1+\beta(\lambda_1+\lambda_3)]^j\Gamma(\alpha+x)}\right\}\\ &= [Q-Pz]^{-(\alpha+x)}\ ,\end{aligned}$$

where $Q=1-P$, $P=\beta\lambda_2/(1+\beta\lambda_1+\beta\lambda_3)$. Thus the conditional distribution of Y given $X=x$ is the convolution of

$$Y_1\sim B(x,p)\ , \quad Y_2\sim NB(\alpha+x, P)\ ,$$

where $p=\lambda_3/(\lambda_1+\lambda_3)$, $P=\beta\lambda_2/(1+\beta\lambda_1+\beta\lambda_3)$. This was arrived at by Subrahmaniam (1966).

5.2 *Bivariate Neyman Type A*

Here

$$M(t) = \exp\lambda(e^t-1)\ ,$$

and

$$M^{(x)}(t) = M(t)\mu_x'(\lambda e^t) ,$$

hence

$$(5.5) \qquad \Pi_2(z) = \{\exp[\lambda e^{-(\lambda_1+\lambda_3)}\{e^{\lambda_2(z-1)} - 1\}]\}\left\{\frac{\mu_x'[\lambda \exp(\lambda_2 z - \lambda_1 - \lambda_2 - \lambda_3)]}{\mu_x'[\lambda \exp(-\lambda_1 - \lambda_3)]}\right\} .$$

The second term on the right hand side of (5.5) can be simplified using the expansion for the k-th moment around zero of $P(\lambda)$, as

$$\mu_k' = \sum_{i=1}^{k} \lambda^i S_{i;k} ,$$

where $S_{i;k}$ is a Stirling number of the second kind. This is defined as

$$S_{i;k} = \frac{1}{i!}\sum_{j=0}^{k}(-1)^{i-j}\binom{i}{j}j^k .$$

This is Riordan's representation for μ_k'. See Haight ((1967), p. 6) for details. Using this representation we can write

$$(5.6) \qquad \mu_x'[\lambda \exp\{-\lambda_1 - \lambda_3 + \lambda_2(z-1)\}]/\mu_x'[\lambda \exp\{-\lambda_1 - \lambda_3\}] = \sum_{i=1}^{x} \omega_i^{(x)} \exp i\lambda_2(z-1) ,$$

where

$$(5.7) \qquad \omega_i^{(x)} = \frac{\lambda^i \exp\{-i(\lambda_1 + \lambda_3)\}S_{i;x}}{\sum_{i=1}^{x} \lambda^i \exp\{-i(\lambda_1 + \lambda_3)\}S_{i;x}} ,$$

and $\sum_{i=1}^{x}\omega_i^{(x)}=1$. Thus the conditional distribution of Y given $X=x$ is the convolution of (i) $Y_1 \sim B(x,p)$, (ii) $Y_2 \sim$ Neyman Type A with the parameters $\lambda \exp(-\lambda_1-\lambda_3)$, λ_2 and (iii) $Y_3 \sim$ mixture of x $P(i\lambda_2)$ with the weights $\omega_i^{(x)}$ given by (5.7).

From this, the conditional moments of Y given $X=x$ are

$$E[Y|X=x] = xp + \lambda_2\lambda \exp(-\lambda_1 - \lambda_3) + \lambda_2 \sum_{i=1}^{x} i\omega_i^{(x)} ,$$

$$V[Y|X=x] = xp(1-p) + \lambda_2\lambda\{\exp(-\lambda_1 - \lambda_3)\}\{1 + \lambda_2\} + \lambda_2 \sum_{i=1}^{x} i\omega_i^{(x)} + \lambda_2^2 \sum_{i=1}^{x} i^2\omega_i^{(x)} - \left\{\lambda_2 \sum_{i=1}^{x} i\omega_i^{(x)}\right\}^2 .$$

5.3 *Bivariate Hermite*

As seen from (3.14)

$$\Pi(z_1, z_2) = \exp[\mu\{\lambda_1(z_1 - 1) + \lambda_2(z_2 - 1) + \lambda_3(z_1 z_2 - 1)\} + \frac{\sigma^2}{2}\{\lambda_1(z_1 - 1) + \lambda_2(z_2 - 1) + \lambda_3(z_1 z_2 - 1)\}^2] . \tag{5.8}$$

Since, for $M(t) = \exp[\mu t + (\sigma^2 t^2/2)]$,

$$M^{(x)}(t) = M(t)\sigma^x H_x^*\left[\sigma t + \frac{\mu}{\sigma}\right],$$

where

$$H_x^*(u) = \sum_{j=0}^{[x/2]} \frac{x!\,u^{x-2j}}{(x-2j)!\,j!\,2^j},$$

a modified Hermite Polynomial defined by Kemp and Kemp ((1965), Equation 13). Hence

$$\Pi_2(z) = \frac{M[\lambda_2(z-1) - \lambda_1 - \lambda_3]}{M[-\lambda_1 - \lambda_3]} \cdot \frac{H_x^*\left[\frac{\mu}{\sigma} + \sigma\lambda_2(z-1) - \sigma(\lambda_1 + \lambda_3)\right]}{H_x^*\left[\frac{\mu}{\sigma} - \sigma(\lambda_1 + \lambda_3)\right]} . \tag{5.9}$$

The right hand side of equation (5.9) can be written as the product of

$$\exp\left[\{\mu\lambda_2 - \sigma^2\lambda_2(\lambda_1 + \lambda_3)\}(z-1) + \frac{\sigma^2\lambda_2^2}{2}(z-1)^2\right], \tag{5.10}$$

and

$$\sum_{j=0}^{[x/2]} \omega_j^{(x)}(Q + Pz)^{x-2j}, \tag{5.11}$$

where

$$P = \frac{\lambda_2\sigma}{\frac{\mu}{\sigma} - \sigma(\lambda_1 + \lambda_3)}, \quad Q = 1 - P,$$

and

$$\omega_j^{(r)} = \frac{\dfrac{r!}{(r-2j)!j!}\left[2\left\{\dfrac{\mu}{\sigma} - \sigma(\lambda_1+\lambda_3)\right\}^2\right]^{-j}}{\sum_{j=0}^{[r/2]}\dfrac{r!}{(r-2j)!j!}\left[2\left\{\dfrac{\mu}{\sigma} - \sigma(\lambda_1+\lambda_3)\right\}^2\right]^{-j}} .$$

Thus the conditional distribution of Y given $X=x$ is the convolution of (i) $Y_1 \sim B(x,p)$, (ii) $Y_2 \sim$ Hermite $([\mu\lambda_2 - \sigma^2\lambda_2(\lambda_1+\lambda_3)], \sigma^2\lambda_2^2/2)$ and (iii) Y_3 which is a mixture of $B(x-2j, P), j=1, 2, \ldots, [x/2]$. In particular, the

$$E[Y|X=x] = xp + \{\mu - \sigma^2(\lambda_1+\lambda_3)\}\lambda_2 + \sum_{j=0}^{[x/2]} \omega_j^{(x)}(x-2j)P ,$$

where $p=\lambda_3/(\lambda_1+\lambda_3)$, $P=\lambda_2\sigma^2/\{\mu-\sigma^2(\lambda_1+\lambda_3)\}$.

The case of $\lambda_3=0$ gives rise to the five parameter Hermite distribution studied by Kemp and Papageorgiou (1982). In this case the conditional distribution of Y given $X=x$ reduces to the convolution of Y_2 and Y_3 alone with the parameters modified appropriately.

5.4 *Bivariate Poisson-Inverse Gaussian*

Here $\Pi_2(z)$ is the ratio of the x-th derivatives of

$$M(t) = \exp\frac{\lambda}{\mu}\left[1 - \left\{1 - \frac{2\mu^2}{\lambda}t\right\}^{1/2}\right], \tag{5.12}$$

evaluated at $t=\lambda_2(z-1)-\lambda_1-\lambda_3$ and $t=-(\lambda_1+\lambda_3)$. We can write

$$M(t) = \left(\exp\frac{\lambda}{\mu}\right)\left(\exp - \frac{\lambda}{\mu}\sqrt{\theta}\right),$$

$\theta=1-2at$, $a=\mu^2/\lambda$. As noted earlier

$$M^{(x)}(t) = e^{\lambda/\mu}e^{(-\lambda/\mu)\sqrt{\theta}}\, a^x\left(\frac{\lambda}{\mu}\right)\sum_{r=0}^{x} c_r\left(\frac{\lambda}{\mu}\right)^r(1-2at)^{-(x-(r+1)/2)} ,$$

where c_r is a constant. Hence

$$\Pi_2(z) = \exp\left[\frac{\lambda}{\mu}(1+2a\lambda_1+2a\lambda_3)^{1/2} \cdot \left\{1 - \left[1 - \frac{2\mu^2\lambda_2(z-1)}{\lambda(1+2a\lambda_1+2a\lambda_3)}\right]^{1/2}\right\}\right] \tag{5.13}$$

$$\cdot \left\{ \sum_{r=0}^{x} d_r \left[1 - \frac{2\mu^2\lambda_2(z-1)}{\lambda(1+2a\lambda_1+2a\lambda_3)} \right]^{-(x-(r+1)/2)} \right\},$$

where

$$d_r = \frac{c_r\left(\frac{\lambda}{\mu}\right)^r (1+2a\lambda_1+2a\lambda_3)^{-(x-(r+1)/2)}}{\sum_{r=0}^{x} c_r\left(\frac{\lambda}{\mu}\right)^r (1+2a\lambda_1+2a\lambda_3)^{-(x-(r+1)/2)}}. \tag{5.14}$$

The first term on the right hand side of (5.13) is the pgf of a univariate Poisson-Inverse Gaussian distribution compounding $P(\lambda_2)$ with IG(μ^*, λ^*) where

$$\mu^* = \mu/(1+2a\lambda_1+2a\lambda_3)^{1/2}, \quad \lambda^* = \lambda .$$

The second term on the right hand side of (5.13) is the mixture of $(x+1)$ negative binomial distributions with the parameters $x-(r+1)/2$ and

$$P = \frac{2\mu^2\lambda_2}{\lambda(1+2a\lambda_1+2a\lambda_3)}, \quad Q = 1 + P .$$

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