EFFECT OF OBLATENESS ON THE NON-LINEAR STABILITY OF L_4 IN THE RESTRICTED THREE-BODY PROBLEM

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Abstract. Non-linear stability of the libration point L_4 of the restricted three-body problem is studied when the more massive primary is an oblate spheroid with its equatorial plane coincident with the plane of motion. Moser's conditions are utilised in this study by employing the iterative scheme of Henrard for transforming the Hamiltonian to the Birkhoff's normal form with the help of double D'Alembert's series. It is found that L_4 is stable for all mass ratios in the range of linear stability except for the three mass ratios:

$$\mu_{c1} = 0.0242\cdots - 0.1790\cdots A_1,$$
$$\mu_{c2} = 0.0135\cdots - 0.0993\cdots A_1,$$
$$\mu_{c3} = 0.0109\cdots - 0.0294\cdots A_1.$$

Key words: restricted 3-body problem, more massive primary oblate, non-linear stability, triangular point L_4, double D'Alembert's series method

1. Introduction

The perturbed restricted three-body problem, when the secular effects of oblateness of the more massive primary are taken into consideration, can be modelled in terms of two parameters: the mass parameter (μ) and the oblateness coefficient (A_1). Of the five equilibrium solutions for the problem, the three collinear equilibria L_1, L_2 and L_3 are unstable, while the two triangular solutions L_4 and L_5 are stable, in the linear analysis, for only a certain range of μ which decreases with A_1 (cf. Subba Rao and Sharma, 1975, 1976; Sharma and Subba Rao, 1978; 1986). The linear analysis, however, is inconclusive as the second-order part of the Hamiltonian (H_2) is indefinite. The present endevour is to provide an account of the investigation carried out relating to the non-linear aspects of the stability of L_4/L_5, making use of Moser's conditions by employing the iterative scheme of Henrard (Deprit and Deprit-Bartholme, 1967), for transforming the involved Hamiltonian to the Birkhoff's normal form with the help of double D'Alembert's series. It is found that the triangular equilibria are stable in the non-linear sense too for $0 < \mu < \mu_0$ except when $\mu = \mu_{ci}$ $(i = 1, 2, 3)$ which decrease with A_1.

It is to be pointed out that a study of the non-linear aspects of the stability of L_4 was attempted by Bhatnagar, Gupta and Bhardwaj (1994) for perturbed potentials, of which the results of the present investigation can be derived as a particular case.

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However, it is noticed that some expressions in Bhatnagar, Gupta and Bhardwaj (1994) are erroneus leading to results contrary to those reported herein. While some of the corrections for Bhatnagar, Gupta and Bhardwaj (1994) are pointed out in the text, the complete details will be addressed to separately. Another noteworthy difference in the present approach is that the transformations, utilised for reduction of the Hamiltonian to the normal form and for subsequent analysis, are dependent on the perturbed frequencies (directly available from the characteristic equation) rather than the unperturbed ones as in Bhatnagar, Gupta and Bhardwaj (1994) and Bhatnagar and Hallan (1983) and this symplifies the derivation significantly.

2. Equations of Motion and Linear Stability

Using dimensionless variables and a synodic coordinate system (x, y), the equations of motion are Szebehely (1967) and Sharma and Subba Rao (1976):

$$\ddot{x} - 2n\dot{y} = \Omega_x, \qquad \ddot{y} + 2n\dot{x} = \Omega_y, \tag{1}$$

where

$$\Omega = \frac{n^2}{2}[(1-\mu)R_1^2 + \mu R_2^2] + \frac{(1-\mu)}{R_1} + \frac{\mu}{R_2} + \frac{(1-\mu)A_1}{2R_1^3},$$

$$R_1^2 = (x-\mu)^2 + y^2, \qquad R_2^2 = (x+1-\mu)^2 + y^2, \quad \mu = \frac{m_2}{(m_1+m_2)} \leqslant \frac{1}{2},$$

$m_1, m_2 (m_1 \geqslant m_2)$ being the masses of the primaries. Overhead dots indicate differentiation with respect to time, and the perturbed mean motion of the primaries n, is given by

$$n^2 = 1 + \frac{3A_1}{2}, \qquad A_1 = \frac{(R_e^2 - R_p^2)}{5R^2},$$

R_e and R_p being, respectively, the equatorial and polar radii of the more massive primary, while R is the distance between the primaries.

The locations $(a, \pm b)$ of the triangular libration points L_4 and L_5 are given by

$$R_1 = 1, \qquad R_2^3 = \frac{1}{n^2},$$

resulting in

$$a = -\tfrac{1}{2}(\gamma + A_1), \qquad b = \frac{\sqrt{3}}{2}\left(1 - \frac{A_1}{3}\right), \quad \text{with } \gamma = 1 - 2\mu,$$

to first-order terms in the oblateness coefficient A_1.

The Lagrangian of the system (1) is

$$\mathcal{L} = \frac{(\dot{x}^2+\dot{y}^2)}{2} + \frac{n^2(x^2+y^2)}{2}$$
$$+n(x\dot{y}-\dot{x}y) + \frac{(1-\mu)}{R_1} + \frac{\mu}{R_2} + \frac{(1-\mu)\cdot A_1}{2R_1^3}.$$

Shifting the origin to L_4 and expanding in power series of x and y, we note that

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \cdots, \tag{2}$$

where

$$\mathcal{L}_0 = \frac{(11+\gamma^2)}{8} + \frac{(13+4\gamma+3\gamma^2)}{16}A_1,$$

$$\mathcal{L}_1 = -\frac{\sqrt{3}}{24}(12+5A_1)\dot{x} - \tfrac{1}{8}[4\gamma+(4+3\gamma)A_1]\dot{y},$$

$$\mathcal{L}_2 = \tfrac{1}{2}(\dot{x}^2+\dot{y}^2) + \tfrac{1}{4}(4+3A_1)(x\dot{y}-\dot{x}y) + \tfrac{3}{16}[2+(5+4\gamma)A_1]x^2$$
$$+\tfrac{3}{16}(6+11A_1)y^2 - \frac{\sqrt{3}}{8}[6\gamma+(6+13\gamma)A_1]xy,$$

$$\mathcal{L}_3 = -\tfrac{1}{32}[14\gamma+(-6+25\gamma)A_1]x^3 - \frac{\sqrt{3}}{32}[6+(43+60\gamma)A_1]x^2y$$
$$+\tfrac{3}{32}[22\gamma+(22+65\gamma)A_1]xy^2 - \frac{\sqrt{3}}{32}(6+23A_1)y^3,$$

$$\mathcal{L}_4 = -\tfrac{1}{256}[74+(285+200\gamma)A_1]x^4 + \frac{5\sqrt{3}}{192}[30\gamma+(-54+53\gamma)A_1]x^3y$$
$$+\tfrac{3}{128}[82+(405+340\gamma)A_1]x^2y^2$$
$$-\frac{5\sqrt{3}}{64}[18\gamma+(18+71\gamma)A_1]xy^3 + \tfrac{3}{256}(2+65A_1)y^4.$$

The second-order part H_2 of the corresponding Hamiltonian H takes the form

$$H_2 + \tfrac{1}{2}(p_x^2+p_y^2) + n(yp_x - xp_y) + Ex^2 + Fy^2 + Gxy, \tag{3}$$

where

$$E = \tfrac{1}{16}[2-3(1+4\gamma)A_1],$$
$$F = -\tfrac{1}{16}(10+21A_1),$$
$$G = \frac{\sqrt{3}}{8}[6\gamma+(6+13\gamma)A_1],$$

and p_x, p_y defined as

$$p_x = \dot{x} - ny, \qquad p_y = \dot{y} + nx,$$

are the momenta, conjugate to x and y, respectively.

To investigate the stability of the motion, as in Whittaker (1965: § 84), we consider the following set of linear equations in the variables x, y:

$$\begin{aligned} -\lambda p_x &= \frac{\partial H_2}{\partial x} = 2Ex + Gy - np_y, & \lambda x &= \frac{\partial H_2}{\partial p_x} = p_x + ny, \\ -\lambda p_y &= \frac{\partial H_2}{\partial y} = 2Fy + Gx + np_x, & \lambda y &= \frac{\partial H_2}{\partial p_y} = p_y - nx, \end{aligned} \tag{4}$$

i.e., $\mathcal{A}X = 0$,
where

$$X = \begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix} \quad \text{and} \quad \mathcal{A} = \begin{pmatrix} 2E & G & \lambda & -n \\ G & 2F & n & \lambda \\ -\lambda & n & 1 & 0 \\ -n & -\lambda & 0 & 1 \end{pmatrix}.$$

Evidently, *det.* $\mathcal{A} = 0$ implies that

$$16\lambda^4 + 8(2 - 3\gamma A_1)\lambda^2 + 9(1 - \gamma^2)(3 + 13A_1) = 0. \tag{5}$$

Equation (5) is the characteristic equation whose descriminant is

$$\mathcal{D} = 64[(27 + 117A_1)\gamma^2 - 12A_1\gamma + 117A_1 - 23]. \tag{6}$$

Stability is assured only when $\mathcal{D} > 0$, implying that (Subba Rao and Sharma, 1975)

$$\mu < \mu_{c0} = \mu_0 - \frac{1}{9}\left(1 + \frac{13}{\sqrt{69}}\right) A_1 = \mu_0 - 0.2850017877\cdots A_1, \tag{7}$$

where

$$\mu_0 = \frac{1}{2}\left(1 - \frac{\sqrt{69}}{9}\right) = 0.0385208965\cdots.$$

When $\mathcal{D} > 0$, the roots $\pm i\omega_1$ and $\pm i\omega_2$ (ω_1, ω_2 being the long-/short-period frequencies) are related to each other as

$$\omega_1^2 + \omega_2^2 = \tfrac{1}{2}(2 - 3A_1\gamma), \tag{8}$$

$$\omega_1^2\omega_2^2 = \tfrac{9}{16}(1-\gamma^2)(3+13A_1), \qquad \left(0<\omega_1<\omega_2<\frac{1}{\sqrt{2}}\right). \tag{9}$$

From (4), it may be noted that ω_j $(j = 1,2)$ satisfy (see Subba Rao and Sharma, 1988):

$$81\gamma^2 = 3(16\omega_j^4 - 16\omega_j^2 + 27) + 8A_1\omega_j^2(9\gamma - 26\omega_j^2 + 26),$$

$$\text{i.e., } \gamma^2 = \gamma_1 + \gamma_2 A_1 + \gamma_3 A_1\gamma, \tag{10}$$

where

$$\gamma_1 = \frac{(16\omega_j^4 - 16\omega_j^2 + 27)}{27}, \qquad \gamma_2 = \tfrac{208}{81}\omega_j^2(1-\omega_j^2), \qquad \gamma_3 = \tfrac{8}{9}\omega_j^2.$$

Alternatively, it can also be seen that if $u = \omega_1\omega_2$, then

$$\gamma^2 = (1 - \tfrac{16}{27}u^2) + \tfrac{208}{81}u^2A_1 \equiv \overline{\gamma_1} + \overline{\gamma_2}A_1. \tag{11}$$

It may be noted (Subba Rao and Sharma, 1988) that the perturbed frequencies ω_j are related to the unperturbed ones $(\omega_{j,0})$ as

$$\omega_j^2 = \omega_{j,0}^2 + \frac{\omega_{j,0}^2}{12(1-2\omega_{j,0}^2)}[26(1-\omega_{j,0}^2) + 9\gamma]A_1.$$

3. Determination of the Normal Coordinates

For expressing H_2 in a simpler form, we consider the set of linear equations (4), the solution of which can be obtained as

$$\frac{x}{(2n\lambda - G)} = \frac{y}{(\lambda^2 - n^2 + 2E)} = \frac{p_x}{(n\lambda^2 - G\lambda - 2nE + n^3)}$$
$$= \frac{p_y}{(\lambda^3 + n^2\lambda + 2E\lambda - nG)}. \tag{12}$$

Substituting $\lambda = \pm i\omega_1$ and $\pm i\omega_2$, we obtain the solution sets as

$$\begin{aligned}
x_j &= K_j(2ni\omega_j - G),\\
p_{x,j} &= K_j(-n\omega_j^2 - iG\omega_j - 2En + n^3),\\
y_j &= K_j(-\omega_j^2 - n^2 + 2E),\\
p_{y,j} &= K_j[-i\omega_j^3 + i\omega_j(n^2+2E) - Gn],\\
x_{j+2} &= K_{j+2}(-2in\omega_j - G),\\
p_{x,j+2} &= K_{j+2}(-n\omega_j^2 + iG\omega_j - 2En + n^3),\\
y_{j+2} &= K_{j+2}(-\omega_j^2 - n^2 + 2E),\\
p_{y,j+2} &= K_{j+2}[i\omega_j^3 - i\omega_j(n^2+2E) - Gn],
\end{aligned} \tag{13}$$

where $j = 1, 2$ and K_j, K_{j+2} are constants of proportionality.

As in Whittaker (1965: § 192), we effect the transformation

$$X = J \begin{pmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{pmatrix},$$

where

$$J = \begin{pmatrix} x_1 - i\frac{\omega_1 x_3}{2} & -x_2 - i\frac{\omega_2 x_4}{2} & -i\frac{x_1}{\omega_1} + \frac{x_3}{2} & -i\frac{x_2}{\omega_2} - \frac{x_4}{2} \\ y_1 - i\frac{\omega_1 y_3}{2} & -y_2 - i\frac{\omega_2 y_4}{2} & -i\frac{y_1}{\omega_1} + \frac{y_3}{2} & -i\frac{y_2}{\omega_2} - \frac{y_4}{2} \\ p_{x,1} - i\frac{\omega_1 p_{x,3}}{2} & -p_{x,2} - i\frac{\omega_2 p_{x,4}}{2} & -i\frac{p_{x,1}}{\omega_1} + \frac{p_{x,3}}{2} & -i\frac{p_{x,2}}{\omega_2} - \frac{p_{x,4}}{2} \\ p_{y,1} - i\frac{\omega_1 p_{y,3}}{2} & -p_{y,2} - i\frac{\omega_2 p_{y,4}}{2} & -i\frac{p_{y,1}}{\omega_1} + \frac{p_{y,3}}{2} & -i\frac{p_{y,2}}{\omega_2} - \frac{p_{y,4}}{2} \end{pmatrix}, \quad (14)$$

under the normality conditions:

$$x_1 p_{x,3} - x_3 p_{x,1} + y_1 p_{y,3} - y_3 p_{y,1} = 1,$$
$$x_2 p_{x,4} - x_4 p_{x,2} + y_2 p_{y,4} - y_4 p_{y,2} = 1.$$

Equivalently,

$$\begin{aligned} &-4i\omega_1 K_1 K_2[\omega_1^2(F - E + 2n^2) + G^2 + 2E^2 + 3n^2 E \\ &\quad +n^2 F - 2EF - 2n^4] = 1, \\ &-4i\omega_2 K_2 K_4[\omega_2^2(F - E + 2n^2) + G^2 + 2E^2 + 3n^2 E \\ &\quad +n^2 F - 2EF - 2n^4] = 1, \end{aligned} \quad (15)$$

K_j's being arbitrary, we follow the approach of Breakwell and Pringle (1966) and choose $J_{1,1} = J_{1,2} = 0$, implying that

$$K_1(2in\omega_1 - G) = \frac{\omega_1 K_3}{2}(2n\omega_1 - iG),$$
$$K_2(G - 2in\omega_2) = \frac{\omega_2 K_4}{2}(-iG + 2n\omega_2),$$

i.e.,

$$\frac{K_1}{\omega_1(2n\omega_1 - iG)} = \frac{K_3}{2(2ni\omega_1 - G)} = h_1 \text{ (say)}, \quad (16)$$

$$\frac{K_2}{\omega_2(2n\omega_2 - iG)} = \frac{K_4}{2(G - 2ni\omega_2)} = h_2 \text{ (say)}. \quad (17)$$

Substituting (16) and (17) in (15), we observe that

$$h_j = \frac{1}{2\omega_j M_j \overline{M}_j (M_j^*)^2}, \tag{18}$$

where

$$\begin{aligned} M_j &= (\omega_j^2 - 2F + n^2)^{1/2}, \qquad M_j^* = (\omega_j^2 - 2E + n^2)^{1/2}, \\ \overline{M}_j &= \sqrt{2}(\omega_j^2 - E - F - n^2)^{1/2}, \quad (j = 1, 2). \end{aligned} \tag{19}$$

It is now verified that H_2 takes the form:

$$H_2 = \tfrac{1}{2}(P_1^2 - P_2^2 + \omega_1^2 Q_1^2 - \omega_2^2 Q_2^2). \tag{20}$$

We observe that

$$J = \begin{pmatrix} 0 & 0 & \frac{M_1}{\omega_1 \overline{M}_1} & \frac{iM_2}{\omega_2 \overline{M}_2} \\ \frac{-2n\omega_1}{M_1 \overline{M}_1} & \frac{2in\omega_2}{M_2 \overline{M}_2} & \frac{G}{\omega_1 M_1 \overline{M}_1} & \frac{iG}{\omega_2 M_2 \overline{M}_2} \\ \frac{-\omega_1(M_1^2 - 2n^2)}{M_1 \overline{M}_1} & \frac{i\omega_2(M_2^2 - 2n^2)}{M_2 \overline{M}_2} & \frac{-nG}{\omega_1 M_1 \overline{M}_1} & \frac{-niG}{\omega_2 M_2 \overline{M}_2} \\ \frac{-\omega_1 G}{M_1 \overline{M}_1} & \frac{i\omega_2 G}{M_2 \overline{M}_2} & \frac{-n(2\omega_1^2 - M_1^2)}{\omega_1 M_1 \overline{M}_1} & \frac{ni(M_2^2 - 2\omega_2^2)}{\omega_2 M_2 \overline{M}_2} \end{pmatrix}. \tag{21}$$

In particular, we have

$$\begin{aligned} J_{1,3} &= \frac{l_1}{2\omega_1 k_1} + \frac{33}{8\omega_1 l_1 k_1} A_1 - \frac{3l_1}{8\omega_1 k_1^3} A_1 \gamma, \\ J_{1,4} &= \frac{l_2}{2\omega_2 k_2} + \frac{33}{8\omega_2 l_2 k_2} A_1 + \frac{3l_2}{8\omega_2 k_2^3} A_1 \gamma, \\ J_{2,1} &= -\frac{4\omega_1}{l_1 k_1} - \frac{6\omega_1 k_1}{l_1^3} A_1 + \frac{3\omega_1}{l_1 k_1^3} A_1 \gamma, \\ J_{2,2} &= \frac{4\omega_2}{l_2 k_2} - \frac{6\omega_2 k_2}{l_2^3} A_1 + \frac{3\omega_2}{l_2 k_2^3} A_1 \gamma, \\ J_{2,3} &= \frac{3\sqrt{3}}{2\omega_1 l_1 k_1}\gamma + \frac{\sqrt{3}(-16\omega_1^4 + 88\omega_1^2 - 63)}{24\omega_1 l_1 k_1^3} A_1 \\ &\quad + \frac{\sqrt{3}(104\omega_1^2 + 135)}{8\omega_1 l_1^3 k_1} A_1 \gamma, \\ J_{2,4} &= \frac{3\sqrt{3}}{2\omega_2 l_2 k_2}\gamma + \frac{\sqrt{3}(16\omega_2^4 - 88\omega_2^2 + 63)}{24\omega_2 l_2 k_2^3} A_1 \\ &\quad + \frac{\sqrt{3}(104\omega_2^2 + 135)}{8\omega_2 l_2^3 k_2} A_1 \gamma, \end{aligned} \tag{22}$$

where

$$l_j^2 = 4\omega_j^2 + 9, \qquad k_j^2 = \pm(2\omega_j^2 - 1), \tag{23}$$

and it is understood that the upper sign corresponds to the case $j = 1$ while the lower sign corresponds to the case $j = 2$.

The Hamiltonian H_2 is transformed further by applying a contact transformation from Q_1, Q_2, P_1, P_2 to Q'_1, Q'_2, P'_1, P'_2 defined by Whittaker (1965: § 193)

$$P'_j = \frac{\partial W}{\partial Q_j}, \qquad Q_j = \frac{\partial W}{\partial P_j}, \quad (j = 1, 2)$$

and

$$W = \sum_{j=1}^{2} \left[Q'_j \sin^{-1}\left(\frac{P_j}{\sqrt{2\omega_j Q'_j}} \right) + \frac{P_j}{2\omega_j}\sqrt{2\omega_j Q'_j - P_j^2} \right].$$

i.e.,

$$Q_j = \sqrt{\frac{2Q'_j}{\omega_j}} \cos P'_j, \qquad P_j = \sqrt{2\omega_j Q'_j} \sin P'_j, \quad (j = 1, 2)$$

to the form

$$H_2 = \omega_1 Q'_1 - \omega_2 Q'_2.$$

Denoting the angular variables P'_1 and P'_2 by ϕ_1 and ϕ_2 and the actions Q'_1 and Q'_2 by I_1, I_2, we note that

$$H_2 = \omega_1 I_1 - \omega_2 I_2. \tag{24}$$

The general solution of the corresponding equations of motion is

$$I_j = \text{const.}, \qquad \phi_j = \pm\omega_j t + \text{const.}, \quad (j = 1, 2) \tag{25}$$

If the oscillations about L_4 are exactly linear, the equations (25) represent the integrals of motion and the corresponding orbits will be given by

$$\begin{aligned} x &= p_1 \cos\phi_1 \sqrt{I_1} + p_2 \cos\phi_2 \sqrt{I_2}, \\ y &= (q_1 \sin\phi_1 + q_3 \cos\phi_1)\sqrt{I_1} + (q_2 \sin\phi_2 + q_4 \cos\phi_2)\sqrt{I_2}, \end{aligned} \tag{26}$$

where $p_1, p_2, q_1, q_2, q_3, q_4$ are as in Appendix I.

4. Stability of Higher-Order

It is known that if H_2 is positive definite, then the equilibrium point L_4 is stable, by virtue of the Liapunov's (1956) theorem, for all orders. However, if H_2 is indefinite, then the stability can be investigated by means of Arnold's theorem (1961), and subsequent extensions by Leontovic (1962) and Moser (1953) which state that if

(1) the basic frequencies ω_1 and ω_2 for the linear dynamical system satisfy the inequalities

$$\alpha\omega_1 + \beta\omega_2 \neq 0, \qquad |\alpha| + |\beta| \leqslant 4, \tag{27}$$

for all rational integer pairs α, β

(2) $\mathrm{D} = \det(d_{i,j}) \neq 0$, where

$$\begin{aligned} d_{i,j} &= \frac{\partial^2 \overline{H}}{\partial I_i \partial I_j}, \quad I_i = 0, \quad I_j = 0, \quad (i, j = 1, 2) \\ d_{i,3} &= d_{3,i} = \frac{\partial \overline{H}}{\partial I_i}, \quad (i = 1, 2), \quad d_{3,3} = 0. \end{aligned} \tag{28}$$

and

$$\overline{H} = \omega_1 I_1 - \omega_2 I_2 + \tfrac{1}{2}(AI_1^2 + 2BI_1I_2 + CI_2^2) + \cdots, \tag{29}$$

the normalized Hamiltonian with I_1 and I_2 as the action momenta coordinates, then the equilibrium L_4 is stable for all orders. We refer the above two conditions as Moser's first and second conditions.

5. Moser's (1953) First Condition

We note that the inequalities (27) are violated when $\omega_1 = 2\omega_2$ and $\omega_1 = 3\omega_2$.

If $\omega_1 = 2\omega_2$, then it follows from (8) and (9) that

$$(675 + 2925A_1)\mu^2 - (675 + 2829A_1)\mu + 16(1 - 3A_1) = 0,$$

implying that

$$\begin{aligned} \mu = \mu_{c1} &= \tfrac{1}{90}(45 - \sqrt{1833}) - \frac{16}{225\sqrt{1833}}(65 + \sqrt{1833})A_1, \\ &= 0.0242938971\cdots - 0.1790727798\cdots A_1. \end{aligned} \tag{30}$$

It is further observed that when $\omega_1 = 3\omega_2$, we get

$$25(3 + 13A_1)\mu^2 - (75 + 319A_1)\mu + 1 - 3A_1 = 0,$$

which gives

$$\begin{aligned} \mu = \mu_{c2} &= \tfrac{1}{30}(15 - \sqrt{213}) - \frac{1}{75\sqrt{213}}(65 + 3\sqrt{213})A_1, \\ &= 0.0135160160\cdots - 0.0993830215\cdots A_1. \end{aligned} \tag{31}$$

Thus, it is inferred that in the interval $0 < \mu < \mu_{c0}$, the mass ratio μ should not assume the values μ_{c1} and μ_{c2} for higher-order stability.

Equations (30) and (31) are in conflict with the expressions presented in Bhatnagar *et al.* (1994: 362). This may be attributed to the error in defining the term p_2 in its Appendix. The correct expression for p_2 should be $p_2 = 3\mu_1 a_1' + 4(\mu_1 - \mu_2)a_1$ instead of $p_2 = 3\mu_1 a_1' + (\mu_1 - \mu_2)a_1$ and consequently the correct expression for q_2 should be $q_2 = 3\mu_2 a_2' + 4(2\mu_2 - \mu_1)a_2$. Equations (4) and (5) in Bhatnagar, Gupta and Bhardwaj (1994: 360) stand corrected as

$$\begin{aligned}\mu_1' &= \frac{1}{2}\left(1 - \frac{\sqrt{1833}}{45}\right) + \frac{64}{135\sqrt{1833}}(36\varepsilon - 19\varepsilon') \\ &\quad - \frac{1}{675\sqrt{1833}}[(2080 + 32\sqrt{1833})a_1 + 16(60 + \sqrt{1833})a_1']\varepsilon_1 \\ &\quad - \frac{1}{675\sqrt{1833}}[(2080 - 32\sqrt{1833})a_2 + 16(60 - \sqrt{1833})a_2']\varepsilon_2, \end{aligned} \tag{32}$$

$$\begin{aligned}\mu_2' &= \frac{1}{2}\left(1 - \frac{\sqrt{213}}{15}\right) + \frac{4}{45\sqrt{213}}[36\varepsilon - 19\varepsilon'] \\ &\quad - \frac{1}{225\sqrt{213}}[(130 + 6\sqrt{213})a_1 + 3(20 + \sqrt{213})a_1']\varepsilon_1 \\ &\quad - \frac{1}{225\sqrt{213}}[(130 - 6\sqrt{213})a_2 + 3(20 - \sqrt{213})a_2']\varepsilon_2. \end{aligned} \tag{33}$$

(30) and (31) above can be derived easily from these by setting $\varepsilon_1 = A_1, \varepsilon_2 = 0$, $\varepsilon = \frac{3}{4}A_1, a_1 = -\frac{3}{2}, a_1' = 6, a_2 = 0, a_2' = 0$.

It may further be noted that the co-ordinate systems adopted by Bhatnagar and Hallan (1983) and Bhatnagar *et al.* (1994) are not the same and we notice that the procedure outlined therein for converting some of the intermediary expressions of Bhatnagar and Hallan (1983) to those of Bhatnagar *et al.* (1994) does not provide the appropriate changes. For example, the sign between the last two terms of p_{12} (and also of q_{12}) should be positive instead of being given as negative. These changes bring in error in the subsequent expressions too and also in L_4, and hence the expression for μ_3' in Bhatnagar *et al.* (1994: 361) also needs correction. This will be addressed in detail, in a separate communication.

6. Second-Order Normalization of Hamiltonian

For reducing the Hamiltonian H to the normalized form (29), we utilise Henrard's method (Deprit and Deprit-Bartholme, 1967) in which the coordinates x, y are to be expanded in double D'Alembert's series:

$$x = \sum_{j\geq 1} B_j^{1,0}, \qquad y = \sum_{j\geq 1} B_j^{0,1}, \tag{34}$$

where the homogeneous components $B_j^{1,0}$ and $B_j^{0,1}$ of degree j are of the form

$$\sum_{0\leq n\leq m} I_1^{(n-m)/2} I_2^{m/2} \sum_{p,q}$$
$$\times[C_{n-m,m,p,q}\cos(p\phi_1+q\phi_2)+S_{n-m,m,p,q}\sin(p\phi_1+q\phi_2)], \tag{35}$$

with the condition that p runs over those integers in the interval $0 \leq p \leq n-m$ that have the same parity as $n-m$; and q runs in the interval $-m \leq q \leq m$ having the same parity as m.

As indicated in Deprit and Deprit-Bartholme (1967), in the development of (34) the quantities I_1 and I_2 are to be taken as constants of integration, while ϕ_1 and ϕ_2 are to be determined as linear functions of time in such a way that

$$\dot{\phi}_1 = \omega_1 + \sum_{n\geq 1} f_{2n}(I_1, I_2), \qquad \dot{\phi}_2 = -\omega_2 + \sum_{n\geq 1} g_{2n}(I_1, I_2), \tag{36}$$

where f_{2n} and g_{2n} are homogeneous polynomials of degree n in I_1 and I_2. To ensure the canonical character of the transformation, we need to insist that the D'Alembert's series satisfy the following relations:

$$\begin{aligned} &[x,y]=0, \qquad [x,\dot{x}]=1, \qquad [y,\dot{x}]=0,\\ &[x,\dot{y}]=0, \qquad [y,\dot{y}]=1, \qquad [\dot{x},\dot{y}]=0, \end{aligned}$$

where the Poisson bracket

$$[\xi,\eta] = \frac{\partial\xi}{\partial\phi_1}\frac{\partial\eta}{\partial I_1} - \frac{\partial\xi}{\partial I_1}\frac{\partial\eta}{\partial\phi_1} + \frac{\partial\xi}{\partial\phi_2}\frac{\partial\eta}{\partial I_2} - \frac{\partial\xi}{\partial I_2}\frac{\partial\eta}{\partial\phi_2},$$

and

$$\dot{x} = \dot{\phi}_1\frac{\partial x}{\partial\phi_1} + \dot{\phi}_2\frac{\partial x}{\partial\phi_2}, \qquad \dot{y} = \dot{\phi}_1\frac{\partial y}{\partial\phi_1} + \dot{\phi}_2\frac{\partial y}{\partial\phi_2}. \tag{37}$$

We note, however, that the first-order components $B_1^{1,0}$ and $B_1^{0,1}$ are x and y of (26) and we easily verify that

$$\begin{aligned} (B_1^{1,0})^2 &= a_1 + a_2\cos 2\phi_1 + a_3\cos 2\phi_2 + a_4\cos(\phi_1-\phi_2) + a_5\cos(\phi_1+\phi_2),\\ (B_1^{0,1})^2 &= b_1 + b_2\cos 2\phi_1 + b_3\cos 2\phi_2 + b_4\cos(\phi_1-\phi_2) + b_5\cos(\phi_1+\phi_2)\\ &\quad + b_6\sin 2\phi_1 + b_7\sin 2\phi_2 + b_8\sin(\phi_1-\phi_2) + b_9\sin(\phi_1+\phi_2),\\ B_1^{1,0}B_1^{0,1} &= c_1 + c_2\cos 2\phi_1 + c_3\cos 2\phi_2 + c_4\cos(\phi_1-\phi_2) + c_5\cos(\phi_1+\phi_2)\\ &\quad + c_6\sin 2\phi_1 + c_7\sin 2\phi_2 + c_8\sin(\phi_1-\phi_2) + c_9\sin(\phi_1+\phi_2), \end{aligned} \tag{38}$$

where a_j, b_j and c_j are as in Appendix II.

Proceeding as in Deprit and Deprit-Bartholme (1967), we note that the second-order components $B_2^{1,0}$ and $B_2^{0,1}$ can be obtained as solutions of the partial differential equations

$$\Delta_1\Delta_2 B_2^{1,0} = \Phi_2, \qquad \Delta_1\Delta_2 B_2^{0,1} = -\Psi_2, \tag{39}$$

where

$$\begin{aligned}
\Delta_1 &= \left(\omega_1\frac{\partial}{\partial\phi_1} - \omega_2\frac{\partial}{\partial\phi_2}\right)^2 + \omega_1^2 = \mathcal{D}^2 + \omega_1^2,\\
\Delta_2 &= \mathcal{D}^2 + \omega_2^2,\\
\Phi_2 &= (\mathcal{D}^2 + S_1)X_2 + (S_2\mathcal{D} + S_3)Y_2,\\
\Psi_2 &= (S_2\mathcal{D} - S_3)X_2 - (\mathcal{D}^2 - S_4)Y_2,\\
S_1 &= \frac{-(18+33A_1)}{8}, \qquad S_2 = \frac{(4+3A_1)}{2},\\
S_3 &= \frac{-\sqrt{3}[6\gamma + (6+13\gamma)A_1]}{8}, \qquad S_4 = \frac{3(2+5A_1+4A_1\gamma)}{8},\\
X_2 &= \frac{\partial\mathcal{L}_3}{\partial x}, \quad \text{evaluated for } x = B_1^{1,0}, \quad y = B_1^{0,1},\\
Y_2 &= \frac{\partial\mathcal{L}_3}{\partial y}, \quad \text{evaluated for } x = B_1^{1,0}, \quad y = B_1^{0,1}.
\end{aligned}$$

We, obviously, see that

$$\begin{aligned}
X_2 &= (B_1^{1,0})^2T_1 + B_1^{1,0}B_1^{0,1}T_2 + (B_1^{0,1})^2T_3,\\
Y_2 &= (B_1^{0,1})^2T_4 + \tfrac{1}{2}(B_1^{1,0})^2T_2 + 2B_1^{1,0}B_1^{0,1}T_3,
\end{aligned}$$

with

$$\begin{aligned}
T_1 &= \frac{[-42\gamma + (18-75\gamma)A_1]}{32}, \qquad T_2 = \frac{-\sqrt{3}[6+(43+60\gamma)A_1]}{16},\\
T_3 &= \frac{[66\gamma + (66+195\gamma)A_1]}{32}, \qquad T_4 = \frac{-\sqrt{3}(18+69A_1)}{32}.
\end{aligned} \tag{40}$$

Noting that

$$\begin{aligned}
\mathcal{D}\cos(m\phi_1 + n\phi_2) &= (n\omega_2 - m\omega_1)\sin(m\phi_1 + n\phi_2),\\
\mathcal{D}\sin(m\phi_1 + n\phi_2) &= (m\omega_1 - n\omega_2)\cos(m\phi_1 + n\phi_2),\\
\mathcal{D}^2\cos(m\phi_1 + n\phi_2) &= -(m\omega_1 - n\omega_2)^2\cos(m\phi_1 + n\phi_2),\\
\mathcal{D}^2\sin(m\phi_1 + n\phi_2) &= -(m\omega_1 - n\omega_2)^2\sin(m\phi_1 + n\phi_2),
\end{aligned}$$

and

$$B_2^{1,0} = \frac{1}{\Delta_1\Delta_2}\Phi_2, \qquad B_2^{0,1} = \frac{-1}{\Delta_1\Delta_2}\Psi_2,$$

we see that

$$\begin{aligned} B_2^{1,0} &= r_1 I_1 + r_2 I_2 + r_3 \cos 2\phi_1 I_1 + r_4 \cos 2\phi_2 I_2 + r_5 \cos(\phi_1 - \phi_2)\sqrt{I_1 I_2} \\ &\quad + r_6 \cos(\phi_1 + \phi_2)\sqrt{I_1 I_2} + r_7 \sin 2\phi_1 I_1 + r_8 \sin 2\phi_2 I_2 \\ &\quad + r_9 \sin(\phi_1 - \phi_2)\sqrt{I_1 I_2} + r_{10} \sin(\phi_1 + \phi_2)\sqrt{I_1 I_2}, \end{aligned} \tag{41}$$

$$\begin{aligned} B_2^{0,1} &= s_1 I_1 + s_2 I_2 + s_3 \cos 2\phi_1 I_1 + s_4 \cos 2\phi_2 I_2 + s_5 \cos(\phi_1 - \phi_2)\sqrt{I_1 I_2} \\ &\quad + s_6 \cos(\phi_1 + \phi_2)\sqrt{I_1 I_2} + s_7 \sin 2\phi_1 I_1 + s_8 \sin 2\phi_2 I_2 \\ &\quad + s_9 \sin(\phi_1 - \phi_2)\sqrt{I_1 I_2} + s_{10} \sin(\phi_1 + \phi_2)\sqrt{I_1 I_2}. \end{aligned} \tag{42}$$

Non-vanishing of $\Delta_{2,0}; \Delta_{0,2}; \Delta_{1,-1}; \Delta_{1,1}$ where $\Delta_{p,q}$ stands for

$$\Delta_{p,q} = [\omega_1^2 - (p\omega_1 - q\omega_2)^2][\omega_2^2 - (p\omega_1 - q\omega_2)^2],$$

is assumed inherently in the above while solving for $B_2^{1,0}$ and $B_2^{0,1}$. This is assured by excluding $\mu = \mu_{c1}$ and μ_{c2} in $0 < \mu < \mu_{c0}$. The coefficients r_j and s_j, as functions of the angular frequencies ω_1, and ω_2, are provided in Appendix III.

7. Reduction of Third-Order Terms in H

For formally checking the correctness of the expressions presented in Appendices I and III for $B_1^{1,0}, B_1^{0,1}, B_2^{1,0}, B_2^{0,1}$, we need to verify that the third-order coefficients, i.e., coefficients of $I_1^{3/2}, I_2^{3/2}, I_1\sqrt{I_2}$ and $I_2\sqrt{I_1}$ in the transformed Hamiltonian under the transformation

$$x = B_1^{1,0} + B_2^{1,0}, \qquad y = B_1^{0,1} + B_2^{0,1},$$

vanish identically. This can be done by individually considering the coefficients of $I_1^{3/2}\cos\phi_1, I_1^{3/2}\sin\phi_1, \ldots$. For example, the coefficient of $I_1^{3/2}\cos\phi_1$ in the transformed Hamiltonian can be seen to be

$$\begin{aligned} &\omega_1^2(p_1 r_3 + q_1 s_7 + q_3 s_3) - \tfrac{1}{2}S_4 p_1(2r_1 + r_3) + \tfrac{1}{2}S_1[q_3(2s_1 + s_3) + q_1 s_7] \\ &-\tfrac{1}{2}S_3[p_1(2s_1 + s_3) + q_1 r_7 + q_3(2r_1 + r_3)] - \frac{T_1 p_1^3}{4} - \tfrac{3}{8}T_2 p_1^2 q_3 \\ &-\tfrac{1}{4}(q_1^2 + q_3^2)(q_3 T_4 + p_1 T_3). \end{aligned} \tag{43}$$

Utilising the involved expressions from the Appendices I and III, it has been verified that (43) vanishes independently with respect to $A_1, A_1\gamma$ and independent terms. Similarly, the other coefficients of $I_1^{3/2}$ such as $I_1^{3/2}[\sin\phi_1, \cos\phi_1, \sin 3\phi_1]$ are also found to be zero independently. Vanishing of the coefficients of $I_2^{3/2}$ follows from the symmetry of the expressions involved. Again, noting that the coefficient of $I_1\sqrt{I_2}\sin\phi_1$ in the transformed Hamiltonian as

$$\begin{aligned}
&\tfrac{1}{2}[\omega_1^2\{p_1(r_{10}-r_9)+q_1(s_5-s_6)-q_3(s_9-s_{10})\}-\omega_1\omega_2\{p_1(r_9+r_{10})\\
&\quad -q_1(s_5+s_6)+q_3(s_9+s_{10})\}]+\tfrac{1}{2}S_4p_1(r_9-r_{10})+\tfrac{1}{2}S_1[2q_2s_1+q_1(s_5-s_6)\\
&\quad +q_3(s_{10}-s_9)]-\tfrac{1}{2}S_3[2r_1q_2+p_1(s_{10}-s_9)+q_1(r_5-r_6)+q_3(r_{10}-r_9)]\\
&\quad -\tfrac{1}{2}T_4q_2(q_1^2+q_3^2)-\tfrac{1}{4}T_2p_1^2q_2-T_3p_1q_2q_3,
\end{aligned}$$

it has also been verified to be zero. So also the other coefficients involving $I_1\sqrt{I_2}$. Symmetry establishes that the coefficient of $I_2\sqrt{I_1}$ too vanishes independently.

8. Moser's (1953) Second Condition

Following the iterative procedure of Henrard, we note that the third-order homogeneous components $B_3^{1,0}$ and $B_3^{0,1}$ in (34) can be obtained by solving the partial differential equations

$$\Delta_1\Delta_2 B_3^{1,0}=\Phi_3, \qquad \Delta_1\Delta_2 B_3^{0,1}=\Psi_3, \tag{44}$$

where

$$\begin{aligned}
\Phi_3 &= X-2f_2P-2g_2Q, \qquad \Psi_3=Y-2f_2U-2g_2V,\\
P &= [\mathcal{D}^2+S_1]\left[\frac{\partial}{\partial\phi_1}\left(\omega_1\frac{\partial B_1^{1,0}}{\partial\phi_1}-\frac{1}{2}S_2B_1^{1,0}\right)\right]\\
&\quad +[S_2\mathcal{D}+S_3]\left[\frac{\partial}{\partial\phi_1}\left(\omega_1\frac{\partial B_1^{0,1}}{\partial\phi_1}+\frac{1}{2}S_2B_1^{1,0}\right)\right],\\
Q &= -[\mathcal{D}^2+S_1]\left[\frac{\partial}{\partial\phi_2}\left(\omega_2\frac{\partial B_1^{1,0}}{\partial\phi_2}+\frac{1}{2}S_2B_1^{0,1}\right)\right]\\
&\quad +[S_2\mathcal{D}+S_3]\left[\frac{\partial}{\partial\phi_2}\left(-\omega_2\frac{\partial B_1^{0,1}}{\partial\phi_2}+\frac{1}{2}S_2B_1^{1,0}\right)\right],\\
U &= [S_2\mathcal{D}-S_3]\left[\frac{\partial}{\partial\phi_1}\left(\omega_1\frac{\partial B_1^{1,0}}{\partial\phi_1}-\frac{1}{2}S_2B_1^{0,1}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
& -[\mathcal{D}^2 - S_4]\left[\frac{\partial}{\partial\phi_1}\left(\omega_1\frac{\partial B_1^{0,1}}{\partial\phi_1} + \frac{1}{2}S_2B_1^{1,0}\right)\right], \\
V &= [S_2\mathcal{D} + S_3]\left[\frac{\partial}{\partial\phi_2}\left(\omega_2\frac{\partial B_1^{1,0}}{\partial\phi_2} + \frac{1}{2}S_2B_1^{0,1}\right)\right] \\
& -[\mathcal{D}^2 - S_4]\left[\frac{\partial}{\partial\phi_2}\left(\omega_2\frac{\partial B_1^{0,1}}{\partial\phi_2} + \frac{1}{2}S_2B_1^{1,0}\right)\right], \\
X &= (\mathcal{D}^2 + S_1)X_3 + (S_2\mathcal{D} + S_3)Y_3, \\
Y &= (-\mathcal{D}^2 + S_4)Y_3 + (S_2\mathcal{D} - S_3)X_3, \\
X_3 &= \frac{\partial}{\partial x}(\mathcal{L}_3 + \mathcal{L}_4) \\
&= 3M_1x^2 + 2M_2xy + M_3y^2 + 4N_1x^3 + 3N_2x^2y + 2N_3xy^2 + N_4y^3, \\
Y_3 &= \frac{\partial}{\partial y}(\mathcal{L}_3 + \mathcal{L}_4) \\
&= M_2x^2 + 2M_3xy + 3M_4y^2 + N_2x^3 + 2N_3x^2y + 3N_4xy^2 + 4N_5y^3,
\end{aligned} \tag{45}
$$

where

$$M_1 = -\tfrac{7}{16}\gamma + (\tfrac{3}{16} - \tfrac{25}{32}\gamma)A_1, \qquad M_2 = -\sqrt{3}[\tfrac{3}{16} + (\tfrac{43}{32} + \tfrac{15}{8}\gamma)A_1],$$

$$M_3 = 3[\tfrac{11}{16}\gamma + (\tfrac{11}{16} + \tfrac{65}{32}\gamma)A_1], \qquad M_4 = -\sqrt{3}[\tfrac{3}{16} + \tfrac{23}{32}A_1],$$

$$N_1 = -[\tfrac{37}{128} + (\tfrac{285}{256} + \tfrac{25}{32}\gamma)A_1], \qquad N_2 = \sqrt{3}[\tfrac{25}{32}\gamma + (-\tfrac{45}{32} + \tfrac{265}{192}\gamma)A_1],$$

$$N_3 = [\tfrac{123}{64} + (\tfrac{1215}{128} + \tfrac{255}{32}\gamma)A_1], \qquad N_4 = -\sqrt{3}[\tfrac{45}{32}\gamma + (\tfrac{45}{32} + \tfrac{355}{64}\gamma)A_1],$$

$$N_5 = 3[\tfrac{1}{128} + \tfrac{65}{256}A_1].$$

The partial derivatives in the last two equations have been obtained (see Deprit and Deprit-Bartholme, 1967) by substituting $x = B_1^{1,0} + B_2^{1,0}$ and $y = B_1^{0,1} + B_2^{0,1}$ in $\mathcal{L}_3$ and $\mathcal{L}_4$. Choosing

$$f_2 = f_{2,0}I_1 + f_{0,2}I_2, \qquad g_2 = g_{2,0}I_1 + g_{0,2}I_2, \tag{46}$$

we find that

$$
\begin{aligned}
f_{2,0} &= \frac{1}{2}\frac{(\text{coefficient of}\cos\phi_1 \text{ in } \Phi_3)}{(\text{coefficient of}\cos\phi_1 \text{ in } P)} = A,\\
f_{0,2} &= g_{2,0} = \frac{1}{2}\frac{(\text{coefficient of}\cos\phi_2 \text{ in } \Phi_3)}{(\text{coefficient of}\cos\phi_2 \text{ in } Q)} = B,\\
g_{0,2} &= \frac{1}{2}\frac{(\text{coefficient of}\cos\phi_2 \text{ in } \Psi_3)}{(\text{coefficient of}\cos\phi_2 \text{ in } Q)} = C,
\end{aligned}
\tag{47}
$$

and that

$$
H = (\omega_1 I_1 - \omega_2 I_2) + \tfrac{1}{2}(AI_1^2 + 2BI_1I_2 + CI_2^2) + \cdots \tag{48}
$$

resulting in

$$
D = -(A\omega_2^2 + 2B\omega_1\omega_2 + C\omega_1^2). \tag{49}
$$

Using MACSYMA software, it is derived that

$$
\begin{aligned}
A = {} & \frac{(\omega_1^2-1)(124\omega_1^4 - 696\omega_1^2 + 81)}{72k_1^4(5\omega_1^2-1)}\\
&+\frac{(1696\omega_1^6 - 20320\omega_1^4 + 14547\omega_1^2 - 1107)}{432k_1^4(5\omega_1^2-1)}A_1\\
&-\frac{(1208\omega_1^8 + 2914\omega_1^6 + 725\omega_1^4 - 624\omega_1^2 + 45)}{48k_1^6(5\omega_1^2-1)^2}A_1\gamma,
\end{aligned}
\tag{50}
$$

$$
\begin{aligned}
B = {} & \frac{u(64u^2+43)}{6k_1^2k_2^2(1-5\omega_1^2)(1-5\omega_2^2)} + \\
&+\frac{u(6719u^2 - 2319)}{36k_1^2k_2^2(1-5\omega_2^2)(5\omega_1^2-1)}A_1 + \\
&+\frac{(1116800u^8 + 15048088u^6 - 10165353u^4 + 1972620u^2 - 93312)}{32ul_1^2l_2^2k_1^2k_2^2(5\omega_1^2-1)^2(5\omega_2^2-1)^2}A_1\gamma,
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
C = {} & \frac{(\omega_2^2-1)(124\omega_2^4 - 696\omega_2^2 + 81)}{72k_2^4(5\omega_2^2-1)}\\
&+\frac{(1696\omega_2^6 - 20320\omega_2^4 + 14547\omega_2^2 - 1107)}{432k_2^4(5\omega_2^2-1)}A_1\\
&-\frac{(1208\omega_2^8 + 2914\omega_2^6 + 725\omega_2^4 - 624\omega_2^2 + 45)}{48k_2^6(5\omega_2^2-1)^2}A_1\gamma,
\end{aligned}
\tag{52}
$$

and hence

$$D = \frac{(644u^4 - 541u^2 + 36)}{8(4u^2 - 1)(25u^2 - 4)}$$
$$+\frac{\begin{array}{l}+3(1593600u^{10} + 21222096u^8 - 13052000u^6 \\ +5408175u^4 - 840076u^2 + 23616)\end{array}}{16(4u^2 - 1)^2(25u^2 - 4)^2(16u^2 + 117)} A_1\gamma$$
$$+\frac{(39176u^4 - 14359u^2 + 492)}{48(4u^2 - 1)(25u^2 - 4)} A_1. \tag{53}$$

Moser's second condition is satisfied if, in the interval $0 < \mu < \mu_{c0}$, the mass parameter does not take the value μ_{c3} which makes $D = 0$.

For finding μ_{c3}, we first notice that if $A_1 = 0$, then

$$644u^4 - 541u^2 + 36 = 0,$$

implying that

$$u^2 = \frac{(541 - \sqrt{199945})}{1288} = 0.0728632\cdots = u_0 \text{ (say)}.$$

Accordingly, it is seen that Moser's second condition is violated for the unperturbed problem when

$$\mu = \mu_0 = \tfrac{1}{2}(1 - \sqrt{(1 - \tfrac{16}{27}u_0)} = 0.010936677\cdots.$$

When $A_1 \neq 0$ we set $\mu = \mu_0 + \alpha A_1$ and attempt to determine α from $D = 0$. To this extent we observe that

$$\gamma = 1 - 2\mu = 1 - 2\mu_0 - 2\alpha A_1 = \gamma_0 - 2\alpha A_1,$$
$$u^2 = u_0 + (u_1 + \alpha u_2)A_1,$$

with

$$u_1 = \tfrac{117}{16}(1 - \gamma_0^2), \qquad u_2 = \tfrac{27}{4}\gamma_0.$$

It readily follows that

$$\alpha = -\frac{[(1288u_0 - 541)u_1 + 8(D_2^0 + D_3^0\gamma_0)(4u_0 - 1)(25u_0 - 4)]}{u_2(1288u_0 - 541)},$$

where D_2^0 and D_3^0 are D_2 and D_3 as evaluated for the unperturbed problem. Numerical computation yields

$$\alpha = -0.0294\cdots$$

and we then have

$$\mu_{c3} = \mu_0 + \alpha A_1 = 0.010936677\cdots - 0.0294\cdots A_1.$$

As already indicated earlier in Section 5 this is in conflict with Bhatnagar *et al.* (1994) and is explainable. All necessary details of the derivations herein are recorded in Subba Rao and Sharma (1994).

Appendix I

The coefficients occuring in the first-order canonical transformation are:

$$\begin{aligned} p_i &= p_{i,1} + p_{i,2}A_1 + p_{i,3}A_1\gamma, \\ q_i &= q_{i,1} + q_{i,2}A_1 + q_{i,3}A_1\gamma, \quad (i = 1, 2), \\ q_j &= q_{j,1}\gamma + q_{j,2}A_1 + q_{j,3}A_1\gamma, \quad (j = i + 2), \end{aligned}$$

where

$$\begin{aligned} p_{i,1} &= \frac{l_1}{k_i\sqrt{2\omega_i}}, \qquad p_{i,2} = \frac{33}{4l_ik_i\sqrt{2\omega_i}}, \qquad p_{i,3} = \mp\frac{3l_i}{4k_i^3\sqrt{2\omega_i}}, \\ q_{i,1} &= \mp\frac{4\sqrt{2\omega_i}}{l_ik_i}, \qquad q_{i,2} = -\frac{6k_i\sqrt{2\omega_i}}{l_i^3}, \qquad q_{i,3} = \frac{3\sqrt{2\omega_i}}{l_ik_i^3}, \\ q_{j,1} &= \frac{3\sqrt{3}}{k_il_i\sqrt{2\omega_i}}, \qquad q_{j,2} = \pm\frac{\sqrt{3}(-16\omega_i^4 + 88\omega_i^2 - 63)}{12l_ik_i^3\sqrt{2\omega_i}}, \\ q_{i,3} &= \frac{\sqrt{3}(104\omega_i^2 + 135)}{4l_i^3k_i\sqrt{2\omega_i}}. \end{aligned}$$

Appendix II

The coefficients occuring in Eq. (38) are:

$$\begin{aligned} a_1 &= a_2I_1 + a_3I_2, \\ a_{i+1} &= \frac{l_i^2}{4\omega_ik_i^2}\left[1 + \frac{33}{2l_i^2}A_1 \mp \frac{3}{2k_i^2}A_1\gamma\right], \\ a_4 &= \frac{\sqrt{I_1I_2}}{2\Lambda}\left[16u^2 + 117 + \frac{363}{2}A_1\right], \\ b_1 &= \sum_{i=1}^{2}\left[\frac{(4\omega_i^2 + 3)}{4\omega_ik_i^2} + \frac{15}{8\omega_ik_i^2}A_1 \pm \frac{3(4\omega_i^2 - 7)}{8\omega_ik_i^4}A_1\gamma\right] I_i, \end{aligned}$$

$$
\begin{aligned}
b_{i+1} &= \frac{I_i}{8\omega_i l_i^2 k_i^2}\left[2(16\omega_i^4 - 80\omega_i^2 + 27) + \frac{3}{l_i^2}(-432\omega_i^4 + 616\omega_i^2 + 405)A_1\right.\\
&\left.\pm\frac{3}{k_i^2}(16\omega_i^4 + 136\omega_i^2 - 63)A_1\gamma\right],\\
b_{i+3} &= \frac{\sqrt{I_1 I_2}}{\Lambda}\left[\frac{1}{2}(27 \mp 64u - 16u^2)\right.\\
&\left.+\frac{3}{4(16u^2+117)}(\mp 1024u^3 + 3184u^2 \pm 256u + 5859)A_1 + 27A_1\gamma\right];\\
b_{i+5} &= \mp\frac{\sqrt{3}I_i}{l_i^2 k_i^2}\left[12\gamma \pm \frac{2}{3k_i^2}(-16\omega_i^4 + 52\omega_i^2 - 45)A_1 + \frac{(140\omega_i^2+117)}{l_i^2}A_1\gamma\right]\\
b_{i+7} &= -\frac{\sqrt{3}(\omega_1 \pm \omega_2)}{\Lambda}\sqrt{I_1 I_2}\left[12\gamma + 12A_1 + \frac{(560u^2+1917)}{(16u^2+117)}A_1\gamma\right],\\
c_1 &= c_2 + c_3,\\
c_{i+1} &= \frac{\sqrt{3}}{8\omega_i k_i^2}I_i\left[6\gamma \pm \frac{(-16\omega_i^4 + 52\omega_i^2 - 45)}{3k_i^2}A_1 + 13A_1\gamma\right],\\
c_4 &= c_5 = \frac{\sqrt{3}}{4\Lambda}\sqrt{I_1 I_2}\left[66\gamma + \frac{16(2u^2+9)}{3}A_1 + \frac{(3872u^2+16335)}{(16u^2+117)}A_1\gamma\right],\\
c_{i+5} &= \mp\frac{I_i}{2k_i^2}\left[4 + 3A_1 \mp \frac{6}{k_i^2}A_1\gamma\right],\\
c_{i+7} &= \frac{-2(\omega_1 \pm \omega_2)}{\Lambda}\sqrt{I_1 I_2}\left[(\pm 4u + 9) +\right.\\
&\left.+\frac{3}{4(16u^2+117)}(\pm 64u^3 + 496u^2 \mp 500u + 1449)A_1\right],
\end{aligned}
$$

where $\Lambda = l_1 l_2 k_1 k_2 \sqrt{u}$, and it is understood that $i = 1, 2$ and the upper sign ($\pm$ or $\mp$ in some of the terms above) corresponds to $i = 1$ while the lower sign corresponds to $i = 2$.

Appendix III

The coefficients occuring in the second-order canonical transformation are:

$$
\begin{aligned}
r_i &= r_{i,1}\gamma + r_{i,2}A_1 + r_{i,3}A_1\gamma,\\
r_j &= r_{j,1} + r_{j,2}A_1 + r_{j,3}A_1\gamma,
\end{aligned}
$$

$$s_i = s_{i,1} + s_{i,2}A_1 + s_{i,3}A_1\gamma,$$
$$s_j = s_{j,1}\gamma + s_{j,2}A_1 + s_{j,3}A_1\gamma, \quad (i = 1, \ldots, 6 \quad \text{and} \quad j = 7, \ldots, 10).$$

Where for $i = 1, 2$:

$$r_{i,1} = \frac{33}{8\omega_i k_i^2}, \qquad r_{i,2} = \pm\frac{(-416\omega_i^4 + 152\omega_i^2 - 225)}{48\omega_i k_i^4},$$
$$r_{i,3} = \frac{195}{16\omega_i k_i^2},$$
$$s_{i,1} = \frac{\sqrt{3}(8\omega_i^2 + 9)}{24\omega_i k_i^2}, \qquad s_{i,2} = \frac{\sqrt{3}(136\omega_i^2 + 315)}{144\omega_i k_i^2},$$

$$s_{i,3} = \pm\frac{\sqrt{3}(4\omega_i^2 - 15)}{16\omega_i k_i^4},$$
$$r_{i+2,1} = \frac{(76\omega_i^4 - 321\omega_i^2 - 27)}{8\omega_i l_i^2 k_i^2 (1 - 5\omega_i^2)},$$
$$r_{i+2,2} = \pm\frac{(30592\omega_i^{10} - 52032\omega_i^8 + 40940\omega_i^6 - 87027\omega_i^4 + 28674\omega_i^2 - 1215)}{144\omega_i l_i^2 k_i^4 (1 - 5\omega_i^2)^2}$$
$$r_{i+2,3} = \frac{(2320\omega_i^6 - 6600\omega_i^4 - 11655\omega_i^2 - 2025)}{16\omega_i l_i^4 k_i^2 (1 - 5\omega_i^2)},$$
$$s_{i+2,1} = \frac{\sqrt{3}(736\omega_i^6 - 1444\omega_i^4 + 2733\omega_i^2 - 729)}{72\omega_i l_i^2 k_i^2 (1 - 5\omega_i^2)},$$
$$s_{i+2,2} = \frac{\sqrt{3}(15488\omega_i^8 - 88976\omega_i^6 + 312816\omega_i^4 + 292491\omega_i^2 - 101331)}{432\omega_i l_i^4 k_i^2 (1 - 5\omega_i^2)}$$
$$s_{i+2,3} = \pm\frac{\sqrt{3}(2352\omega_i^8 - 17000\omega_i^6 + 20155\omega_i^4 - 6582\omega_i^2 + 783)}{48\omega_i l_i^2 k_i^4 (1 - 5\omega_i^2)^2},$$
$$r_{i+4,1} = \pm\frac{3(72u^2 \pm 229u + 36)}{4(5u \pm 2) l_1 l_2 k_1 k_2 \sqrt{u}},$$
$$r_{i+4,2} = -\frac{(128u^5 \pm 42624u^4 + 53172u^3 \mp 19806u^2 - 39771u \mp 12150)}{48u(5u \pm 2)^2 l_1 l_2 k_1 k_2 \sqrt{u}},$$
$$r_{i+4,3} = \frac{3(\pm 5760u^4 + 30112u^3 \pm 20976u^2 + 137067u \pm 23436)}{8(5u \pm 2)(16u^2 + 117) l_1 l_2 k_1 k_2 \sqrt{u}},$$
$$s_{i+4,1} = \frac{-\sqrt{3}(144u^3 \pm 160u^2 + 261u \pm 180)}{12(5u \pm 2) l_1 l_2 k_1 k_2 \sqrt{u}},$$

$$
\begin{aligned}
s_{i+4,2} &= \frac{-\sqrt{3}(4608u^5 \pm 34304u^4 + 49104u^3 \pm 198720u^2 + 975159u \pm 696924)}{72(5u \pm 2)(16u^2 + 117)l_1 l_2 k_1 k_2 \sqrt{u}} \\
s_{i+4,3} &= \frac{\sqrt{3}(\pm 64u^4 - 8004u^3 \mp 4842u^2 + 1815u \pm 1134)}{16u(5u \pm 2)^2 l_1 l_2 k_1 k_2 \sqrt{u}}, \\
r_{i+6,1} &= \pm \frac{\sqrt{3}(-44\omega_i^4 + 53\omega_i^2 - 18)}{3l_i^2 k_i^2 (1 - 5\omega_i^2)}, \\
r_{i+6,2} &= \pm \frac{\sqrt{3}(-3248\omega_i^6 + 8072\omega_i^4 + 3573\omega_i^2 + 1458)}{36 l_i^4 k_i^2 (1 - 5\omega_i^2)}, \\
r_{i+6,3} &= \frac{\sqrt{3}\omega_i^2(-468\omega_i^4 + 211\omega_i^2 + 4)}{2l_i^2 k_i^4 (1 - 5\omega_i^2)^2}, \\
s_{i+6,1} &= \pm \frac{(-59\omega_i^2 + 24)}{l_i^2 k_i^2 (1 - 5\omega_i^2)}, \\
s_{i+6,2} &= \frac{(-8528\omega_i^8 + 21468\omega_i^6 - 17005\omega_i^4 + 5226\omega_i^2 - 702)}{18 l_i^2 k_i^4 (1 - 5\omega_i^2)^2} \\
s_{i+6,3} &= \pm \frac{(-3628\omega_i^4 - 2685\omega_i^2 + 1980)}{4 l_i^4 k_i^2 (1 - 5\omega_i^2)}, \\
r_{i+8,1} &= \frac{\sqrt{3}(\mp 44u^2 - 3u \pm 15)\sqrt{(1 \pm 2u)}}{(5u \pm 2) l_1 l_2 k_1 k_2 \sqrt{u}}, \\
r_{i+8,2} &= \frac{\sqrt{3}(\mp 12992u^4 - 4560u^3 \pm 12u^2 - 26811u \mp 38637)\sqrt{(1 \pm 2u)}}{12(5u \pm 2)(16u^2 + 117) l_1 l_2 k_1 k_2 \sqrt{u}}, \\
r_{i+8,3} &= \frac{3\sqrt{3}(\mp 1064u^4 - 3034u^3 \mp 1720u^2 - 121u \pm 54)}{8u(5u \pm 2)^2 l_1 l_2 k_1 k_2 \sqrt{u}\sqrt{(1 \pm 2u)}}, \\
s_{i+8,1} &= \frac{3(-7u \pm 9)\sqrt{(1 \pm 2u)}}{(5u \pm 2) l_1 l_2 k_1 k_2 \sqrt{u}}, \\
s_{i+8,2} &= \frac{(10272u^5 \pm 51664u^4 + 29586u^3 \mp 16368u^2 - 16281u \mp 3402)}{24u(5u \pm 2)^2 l_1 l_2 k_1 k_2 \sqrt{u}\sqrt{(1 \pm 2u)}}, \\
s_{i+8,3} &= \frac{3(-1136u^3 \pm 1488u^2 - 3225u \pm 4347)\sqrt{(1 \pm 2u)}}{4(5u \pm 2)(16u^2 + 117) l_1 l_2 k_1 k_2 \sqrt{u}}.
\end{aligned}
$$

References

1. Arnold, V. I.: 1961, 'On the Stability of Positions of Equilibrium of a Hamiltonian System of Ordinary Differential Equations in the General Elliptic Case', *Dokl. Akad. Nauk. USSR* **37**, 255.
2. Bhatnagar, K. B. and Hallan, P. P.: 1983, *Celest. Mech.* **30**, 97.

3. Bhatnagar, K. B., Gupta, U. and Bhardwaj, Reshmi.: 1994, *Celest. Mech. and Dyn. Astron.* **59**, 345.
4. Breakwell, J. and Pringle, R.: 1966, 'Resonances Affecting Near the Earth–Moon Equilateral Libration Points', in R. L. Duncombe and V. Szebehely (eds.), *Methods in Astrodynamics and Celestial Mechanics*, Vol. 17, 55–74. New York: Acad. Press.
5. Deprit, A. and Deprit-Bartholme, A.: 1967, *Astron. J.* **72**, 173.
6. Leontovic, A. M.: 1962, 'On the Stability of the Lagrange's Periodic Solutions for the Reduced Problem of Three Bodies', *Dokl. Akad. Nauk USSR* **143**, 525.
7. Liapunov, A. M.: 1956, *A General Problem of Stability of Motion*, Acad. Sc. USSR.
8. Moser, J.: 1953, *Math. Ann.* **126**, 325.
9. Sharma, R. K. and Subba Rao, P. V.: 1976, *Celest. Mech.* **13**, 137.
10. Sharma, R. K. and Subba Rao, P. V.: 1978, *Celest. Mech.* **18**, 185.
11. Sharma, R. K. and Subba Rao, P. V.: 1986, 'On Finite Periodic Orbits Around the Equilateral Solutions of the Planar Restricted Three-Body Problem', in K. B. Bhatnagar (ed.), *Space Dynamics and Celestial Mechanics*, pp. 71–85. Dordrecht: D. Reidel Publ. Co.
12. Subba Rao, P. V. and Sharma, R. K.: 1975, *Astron. and Astrophys.* **43**, 381.
13. Subba Rao, P. V. and Sharma, R. K.: 1976, *Astrophys. and Space Sci.* **42**, L17.
14. Subba Rao, P. V. and Sharma, R. K.: 1988, 'Oblateness Effect on Finite Periodic Orbits at L_4'. Paper No. IAF-88-300, 4 pp., 39th Congress of IAF, Bangalore (India).
15. Subba Rao, P. V. and Sharma, R. K.: 1994, *Stability of L_4 in the Restricted Three-Body Problem with Oblateness*, VSSC:TR:346/94, 34 pp., Vikram Sarabhai Space Centre, India.
16. Szebehely, V.: 1967, *Theory of Orbits: The Restricted Problem of Three Bodies*, Acad. Press, New York.
17. Whittaker, E. T.: 1965, *A Treatise on Analytical Dynamics of Particles and Rigid Bodies*, Cambridge Univ. Press, London.