EXISTENCE OF PERIODICALLY INVARIANT CURVES IN 3-DIMENSIONAL MEASURE-PRESERVING MAPPINGS

CHONG-QING CHENG and YI-SUI SUN

Department of Astronomy, Nanjing University, China

(Received: 20 December 1988; accepted: 20 September 1989)

Abstract. With the aid of a normal form of a family of measure-preserving mappings in dimension 3, which is deduced in this paper, we prove that there are periodically invariant curves which survive the nonlinear perturbations in the generic case.

1. Introduction

In recent years, extensive numerical studies of 3-dimensional measure-preserving mappings have revealed a lot of interesting phenomena [2, 5, 6]. However, little qualitative information seems to be available. The aim of this paper is to prove the existence of periodically invariant curves of some family of maps. In Hamilton's systems, the canonical structure makes it convenient to study the persistence of invariant manifolds under nonlinear perturbations. In this paper, simulating the case of Hamilton's systems, we will at first work with the normal form of the maps near a normally elliptic invariant curve. For such normal form, we prove the persistence of periodically invariant curves under nonlinear perturbations.

2. Normal Form

We assume that there is an invariant curve l for the 3-dimensional analytical measure-preserving mapping M. We consider a family of mappings near the curve l which is of the form

$$\begin{cases} X_1 = A(s)X + F_0(s, X), \\ s_1 \;= s + B(s) + G_0(s, X), \qquad (s \bmod 2\pi,\ X \in \mathbb{R}^2) \end{cases} \tag{1}$$

where A, B, F_0, G_0 are all of them 2π-periodic functions in s.

$$F_0(s, X) = O(\|X\|^2),$$
$$G_0(s, X) = O(\|X\|),$$

where $\|\cdot\|$ denotes the Euclid norm.

Because of the absence of a Floquet theory for difference equations as there is for ordinary differential equations with periodic coefficients, we cannot confirm that it is always possible to reduce the mapping (1) to the mapping with A, being independent of the angular variable s. In fact, such reduction is certainly impossible if the rotation

Celestial Mechanics and Dynamical Astronomy **47**: 293–303, 1990.
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number of the map confined to l is a rational number. However, because of the fact introduced in the Appendix of this paper, we know that there exist a kind of mappings which can be reduced to the form as follows:

$$\begin{aligned} z_1 &= \lambda_0 z + F(\varphi, z, \bar{z}), \\ \varphi_1 &= \varphi + \beta_0 + G(\varphi, z, \bar{z}), \qquad (\varphi \bmod 2\pi) \end{aligned} \tag{2}$$

where z is a complex variable; $\lambda_0 = e^{i\alpha_0}$;$\alpha_0$, β, are real numbers and (α_0, β_0) satisfies the inequalities

$$|k_1\alpha_0 + k_2\beta_0 + n| > C_0(|k_1| + |k_2|)^{-\mu}\, \forall (k_1, k_2, n) \in \mathbb{Z}^3 \backslash \{(0,0,0)\} \tag{3}$$

for some positive C_0 and μ. Furthermore we assume $F = O(|z|^2)$, $G = O(|z|)$. We look for a change of variables

$$\begin{cases} z = u\left(1 + \sum_{k+l=1}^{n} a_{kl}(\psi) u^k \bar{u}^l\right) \\ \quad = u\left(1 + \sum_{m \geqslant 1} Z_m(\psi, u, \bar{u})\right) = u + Z, \\ \varphi = \psi + \sum_{k+l=1}^{n} b_{kl}(\psi) u^k \bar{u}^l \\ \quad = \psi + \sum_{m \geqslant 1} \Phi_m(\psi, u, \bar{u}) = \psi + \Phi, \end{cases} \tag{4}$$

where $a_{kl}(\psi)$ and $b_{kl}(\psi)$ are 2π-periodic functions in ψ, Z_m, Φ_m are homogeneous polynomials in u and $\bar{u}$ of degree m, such that (2) will be transformed into the following normal form

$$\begin{aligned} u_1 &= u\left(\sum_{k=0}^{n} \lambda_k (u\bar{u})^k\right) + U(u, \bar{u}, \psi), \\ \psi_1 &= \psi + \sum_{k=0}^{n} \beta_k (u\bar{u})^k + \Psi(u, \bar{u}, \psi), \end{aligned} \tag{5}$$

$$\begin{aligned} U(u, \bar{u}, \psi) &= O(|u|^{2n+2}), \\ \Psi(u, \bar{u}, \psi) &= O(|u|^{2n+1}). \end{aligned}$$

Inserting (4), (5) into (2), we obtain

$$\begin{aligned} &u\left(\sum_{k=0}^{n} \lambda_k (u\bar{u})^k\right) + U(u, \bar{u}, \psi) + \sum_{k+l \geqslant 1} a_{kl}(\psi_1)\, u_1^{k+1} \bar{u}_1^l \\ &\quad = \lambda_0\left(u + \sum_{k+l \geqslant 1} a_{kl}(\psi) u^{k+1} \bar{u}^l\right) + F(\psi + \Phi, u + Z, \bar{u} + \bar{Z}), \end{aligned}$$

$$\psi + \sum_{k=0}^{k} \beta_k(u\bar{u})^k + \Psi(u, \bar{u}, \psi) + \sum_{k+1 \geqslant 1} b_{kl}(\psi_1)u_1^k \bar{u}_1^l$$

$$= \psi + \beta_0 + \sum_{k+l \geqslant 1} b_{kl}(\psi)u^k\bar{u}^l + G(\psi + \Phi, u + Z, \bar{u} + \bar{Z}). \tag{6}$$

It is easy to see that the terms of degree 0 or 1 in $u, \bar{u}$ on both sides of (6) agree if b_{10} and b_{01} are taken equal to zero. Suppose now that for some $m > 1$, the condition that coefficients of all terms of degree less than m in (6) agree uniquely determines the polynomials Z_l, Φ_l $(l = 2, 3, \ldots m)$. For $m = 2$, this is true, and we will prove the assertion for $m + 1$ in place of m. Comparing the terms of degree m in (6), we are led to the conditions

$$\begin{aligned} \lambda_0^{k-l}a_{kl}(\psi + \beta_0) - a_{kl}(\psi) &= -\delta_k^l\lambda_k + \tilde{F}_{kl}(\psi), \\ \lambda_0^{k-l}b_{kl}(\psi + \beta_0) - b_{kl}(\psi) &= -\delta_k^l\beta_k + \tilde{G}_{kl}(\psi), \end{aligned} \tag{7}$$

where δ_k^l is the Kronecker function, $\tilde{F}_{kl}, \tilde{G}_{kl}$ only depend on those $a_{pq}b_{pq}$ $(p + q < m)$ and those $\lambda_r, \beta_r (2r < m)$.

Equation (7) can be solved by means of a Fourier expansion. Setting

$$\tilde{F}_{kl} = \sum_{n \in Z'} \tilde{F}_{kl}^n e^{in\psi}, \qquad a_{kl} = \sum_{n \in Z'} a_{kl}^n e^{in\psi},$$

$$\tilde{G}_{kl} = \sum_{n \in Z'} \tilde{G}_{kl}^n e^{in\psi}, \qquad b_{kl} = \sum_{n \in Z'} b_{kl}^n e^{in\psi},$$

we obtain from (7) the relations

$$\left.\begin{aligned} a_{kl}^n(\psi) &= \frac{\tilde{F}_{kl}^n}{e^{i(n\beta - (k-l)\alpha_0)} - 1}, \\ b_{kl}^n(\psi) &= \frac{\tilde{G}_{kl}^n}{e^{i(n\beta - (k-l)\alpha_0)} - 1}. \end{aligned}\right\} \quad (k \neq 1)$$

If $k = 1$, because of the inequality (3), the following series are convergent

$$a_{kl}(\psi) = \sum_{n \in Z^1} a_{kl}^n(\psi), \qquad b_{kl}(\psi) = \sum_{n \in Z^1} b_{kl}^n(\psi).$$

If $k = 1$, let $\lambda_k = \tilde{F}_{kl}^0$, $\beta_k = \tilde{G}_{kl}^0$, and

$$a_{kl}(\psi) = \sum_{\substack{n \in Z^1 \\ n \neq 0}} \frac{\tilde{F}_{kl}^n e^{in\psi}}{e^{in\beta_0} - 1}$$

$$b_{kl}(\psi) = \sum_{\substack{n \in Z^1 \\ n \neq 0}} \frac{\tilde{G}_{kl}^n e^{in\psi}}{e^{in\beta_0} - 1}$$

which preserves the convergence.

From the fact $G(\varphi, z, \bar{z}) = \bar{G}(\varphi, \bar{z}, z)$, $F(\varphi, \bar{z}, z) = \bar{F}(\varphi, z, \bar{z})$, it is clear that all of the β_k are real number.

Introduce polar coordinates $u = re^{i\theta}$. By setting

$$\sum_{k=0}^{n} \lambda_k r^{2k} + \frac{1}{u} U(u, \bar{u}, \varphi) = e^w$$

in view of $|\lambda_0| = |e^{i\alpha_0}| \neq 0$, we determine a unique power series:

$$w = i\left(\sum_{k=0}^{n} \alpha_k r^{2k} + \Theta(r, \theta, \varphi)\right) + \sum_{k=1}^{n} \gamma_k r^{2k} + R(r, \theta, \varphi)$$

with Θ and R being 2π-periodic in both θ and φ, for $r < r^k$, a certain positive number. Therefore (5) is transformed into the following form

$$\begin{cases} r_1 = r \exp\left(\sum_{k=1}^{n} \gamma_k r^{2k}\right) + R(r, \theta, \varphi), \\ \theta_1 = \theta + \sum_{k=0}^{n} \alpha_k r^{2k} + \Theta(r, \theta, \varphi), \\ \psi_1 = \psi + \sum_{k=0}^{n} \beta_k r^{2k} + \Psi(r, \theta, \varphi). \end{cases} \tag{8}$$

We assert that all of the $\gamma_k (k = 1, 2, \dots, n)$ must be of order zero. In fact, if there is a $\gamma_k < 0$ for some k, then for sufficiently small positive r_0, the image torus $r_1(r_0, \theta, \psi)$ under (8) would be in the preimage torus $r = r_0$, which contradicts to the fact that the map is measure-preserving. The same argument holds for some $\gamma_k > 0$. Therefore the normal form of the map (2) is of the form

$$\begin{aligned} r_1 &= r + R(r, \theta, \psi), \\ \theta_1 &= \theta + \sum_{k=0}^{n} \alpha_k r^{2k} + \Theta(r, \theta, \psi), \\ \psi_1 &= \psi + \sum_{k=0}^{n} \beta_k r^{2k} + \Psi(r, \theta, \psi), \end{aligned} \tag{9}$$

where

$$R(r, \cdot, \cdot) = 0(r^{2n+3})$$
$$\Theta(r, \cdot, \cdot) = 0(r^{2n+2})$$
$$\Psi(r, \cdot, \cdot) = 0(r^{2n+2})$$

By inserting εr in the place of r, we can make $|R| + |\Theta| + |\Psi|$ be sufficiently small.

Remark. The map defined by (9) would not certainly be measure-preserving. But each torus $r = r(\theta, \psi)$ intersects its image under the map (9).

3. Existence of Periodically Invariant Curves

By the discussion of the last section, we assume that the map takes the form

$$M:\begin{cases}\psi_1 = \psi + f(r) + \Psi(r, \theta, \psi),\\ \theta_1 = \theta + g(r) + \Theta(r, \theta, \psi),\\ \gamma_1 = \gamma + R(r, \theta, \psi),\end{cases} \tag{10}$$

where $|\Psi| + |\Theta| + |R| \leqslant d$

Ψ, Θ, R are analytical functions on the domain

$$|\mathrm{Im}\,\psi| \leqslant \rho, \quad |\mathrm{Im}\,\theta| \leqslant \rho, \quad |r - r_0| \leqslant s \quad \forall r_0 \in [a, b]$$

and f, g are differentiable functions on the interval $[a, b]$ and there is an $r_0 \in [a, b]$ so that $g(r_0) = 2\pi(m/n)$ is a rational number and $f(r_0)$ satisfies inequalities

$$|k \cdot f(r_0) + 2n\pi| > C_0|k|^{-\mu} \quad (C_0 > 0, \mu > 1) \quad \forall (k, n) \in Z^2 \backslash (0, 0) \tag{11}$$

and $f'(r_0) \neq 0$, $g'(r_0) \neq 0$.

If we neglect the terms R, Θ, Ψ, (10) would have infinite periodically invariant curves which together form an invariant torus

$$\begin{cases}\psi_1 = \psi + f(r),\\ \theta_1 = \theta + 2m\pi/n,\\ r_1 = r_0.\end{cases}$$

We now intend to prove that there are at least two groups of such curves which survive the nonlinear perturbations in the generic case.

By introducing the coordinate transformation

$$T:\begin{cases}\psi = \xi + U_1(\xi, \zeta, \eta)\\ \theta = \zeta + U_2(\xi, \zeta, \eta)\\ \gamma = \eta + V(\xi, \zeta, \eta)\end{cases} \tag{12}$$

we wish to find the map $N = T^{-1}MT$ of the form

$$N:\begin{cases}\xi_1 = \zeta + f(\eta) + W_1(\xi, \zeta, \eta)\\ \zeta_1 = \zeta + g(\eta) + G(\zeta, \eta) + W_2(\xi, \zeta, \eta)\\ \eta_1 = \eta + H(\zeta, \eta) + W_3(\xi, \zeta, \eta)\end{cases} \tag{13}$$

If η is close to r_0. Where G and H are $2\pi/n$ periodic functions in ζ and W_1, W_2, W_3 are much smaller than G, H. The linearized transformation equation is

$$\begin{aligned}&U_1(\xi + f(r_0), \zeta + g(r_0), \eta) - U_1(\xi, \zeta, \eta) = f'(\eta)V(\xi, \zeta, \eta) + \Psi(\xi, \zeta, \eta)\\ &U_2(\xi + f(r_0), \zeta + g(r_0), \eta) - U_2(\xi, \zeta, \eta) = g'(\eta)V(\xi, \zeta, \eta) + \Theta(\xi, \zeta, \eta) - \\ &\qquad - G(\zeta, \eta)\\ &V(\xi + f(r_0), \zeta + g(r_0), \eta) - V(\xi, \zeta, \eta) = R(\xi, \zeta, \eta) - H(\zeta, \eta) - R^*(\eta)\end{aligned} \tag{14}$$

The terms W_1, W_2 and W_3 of (12) satisfy the following equation

$$\begin{aligned} W_1 &= U_1(\xi + f(r_0), \zeta + g(r_0), \eta) - U_1(\xi_1, \zeta_1, \eta_1) \\ &\quad + \Psi(\xi + U_1, \zeta + U_2, \eta + V) - \Psi(\xi, \zeta, \eta) + f''(r_2)V^2, \\ W_2 &= U_2(\xi + f(r_0), \zeta + g(r_0), \eta) - U_2(\xi_1, \zeta_1, \eta_1) \\ &\quad + \Theta(\xi + U_1, \zeta + U_2, \eta + V) - \Theta(\xi, \zeta, \eta) + g''(r_3)V^2, \\ W_3 &= V(\xi + f(r_0), \zeta + g(r_0), \eta) - V(\xi_1, \zeta_1, \eta_1) \\ &\quad + R(\xi + U, \zeta + U, \eta + V) - R(\xi, \zeta, \eta) + R^*(\eta), \end{aligned} \tag{15}$$

with ξ_1, ζ_1, η_1 being shown in (12); r_2 and r_3 are some numbers in $[a, b]$, $R^*(\eta)$ is the mean value of $R(\xi, \zeta, \eta)$.

$$R^*(\eta) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} R(\xi, \zeta, \eta)\, d\xi\, d\zeta.$$

Expressing $R(\xi, \zeta, \eta)$ by means of its Fourier series

$$R = \sum_{k \in Z^2} R_k(\eta) e^{i(k \cdot \phi)} \qquad ((k \cdot \phi) = k_1 \xi + k_2 \zeta),$$

doing the same for $\Psi, \Theta, U_1, U_2, V$ and setting

$$H(\zeta, \eta) = \sum_{\substack{l \in Z^1 \\ l \neq 0}} R_{(0, nl)}(\eta) e^{inl\zeta}, \tag{16}$$

$$G(\zeta, \eta) = \sum_{l \in Z^1} \left(H_{(0, nl)}(\eta) - \frac{g'(\eta)}{f'(\eta)} \Psi_{(0, nl)}(\eta)\right) e^{inl\zeta}, \tag{17}$$

we can obtain the periodic solutions U_1, U_2 and V. Indeed, a periodic solution of the third equation in (14) exist only if the resonant terms in the right hand side vanish, which is guaranteed by the choice of G as above. Solving the third equation in (14), we get

$$V_{(k_1 k_2)} = \frac{R_{(k_1, k_2)}}{e^{i(k_1 f(\gamma_0) + k_2(2m\pi/n))} - 1}, \qquad (k_1 \neq 0, \text{ or } k_1 = 0, k_2 \neq nl) \tag{18}$$

while we let

$$V_{(0, nl)} = \frac{\Psi_{(0, nl)}}{f'(r_0)} \tag{19}$$

so that a periodic solution of the first equation in (14) exists. The other two equations in (14) give

$$\begin{aligned} U_{1(k_1, k_2)} &= \frac{\Psi_{(k_1 k_2)}}{e^{i(k_1 f(r_0) + k_2(2m\pi/n))} - 1}, \\ U_{2(k_1, k_2)} &= \frac{\Theta_{(k_1, k_2)}}{e^{i(k_1 f(r_0) + k_2(2m\pi/n))} - 1}. \qquad (k_1 \neq 0, \text{ or } k_1 = 0, k_2 \neq n1) \end{aligned} \tag{20}$$

By setting $U_{1(0,nl)}=0$, $U_{2(0,nl)}=0$, we obtain the periodical solutions U_1 U_2 and V.

The estimate of W_1, W_2 and W_3 can be obtained by following the footsteps of [4]. First, we have to find the bound of U_1, U_2 and V. Since the resonant terms in the right hand side in (14) vanish in view of (16) and (17), while the denominators of non-resonant terms can be estimated by

$$|e^{i(k_1 f(r_0)+k_2(2m\pi/n))}-1|=2\left|\sin\frac{k_1 f(r_0)+k_2(2m\pi/n)}{2}\right|$$
$$>4c_0|k|^{-\mu},$$

we find

$$|U_1|+|U_2|+|V|<c_1 d, \tag{21}$$

by trivially imitating the proof in [4] if U_1 U_2 V are defined in narrower strips $|\mathrm{Im}\,\xi|<\rho_1<\rho$, $|\mathrm{Im}\,\zeta|<\rho_1<\rho$. Consequently, we can also imitate the proof to obtain the estimate

$$|W_1|+|W_2|+|W_3|<c_2 d^{4/3}, \tag{22}$$

in the domain

$$|\mathrm{Im}\,\zeta|\leqslant\rho/2,\quad |\mathrm{Im}\,\zeta|\leqslant\rho/2,\quad |\eta-r_0|\leqslant d^{8/9},$$

if we observe the facts that the terms $f''(r_2)V^2$ and $g''(r_3)V^2$ are much smaller than $d^{4/3}$ provided d is chosen sufficiently small, in view of (21), and any torus intersects its image. Here we do not show explicitly the parameters c_2 depends on since we do not need a sequence of coordinate transformations but only one step.

Since the mean value of $H(\zeta,\eta)$ equals to zero and $H(\zeta,\eta)$, $G(\zeta,\eta)$ are $2\pi/n$-periodic in ζ, there are at least $2n$ solutions

$$\zeta_j(\eta)=\begin{cases}\tilde{\zeta}_1(\eta)+\dfrac{j\pi}{n} & j=0,2,4,\ldots,2n-2,\\[2mm] \tilde{\zeta}_2(\eta)+\dfrac{j\pi}{n} & j=1,3,5,\ldots,2n-1,\end{cases}$$

which satisfy the equation $H(\zeta,\eta)=0$

Also, since $f(r_0')\neq 0$, we have $|G(\zeta,\eta)|<cd$, for sufficiently small d, and since $g'(r)\neq 0$, there must be $2n$ values

$$\eta_j=\begin{cases}\tilde{\eta}_1 & j=0,2,4,\ldots,2n-2,\\ \tilde{\eta}_2 & j=1,3,5,\ldots,2n-1,\end{cases}$$

so that (η_j,ζ_j) $(j=0,1,2,\ldots,2n-1)$ are the solutions of the equation

$$\begin{cases}g(\eta)+G(\zeta,\eta)=2m\pi/n,\\ \qquad H(\zeta,\eta)=0.\end{cases} \tag{23}$$

Clearly, $(\bar{\zeta}_l + 2k\pi/n, \bar{\eta}_l)$ $(l = 1, 2;\ k = 1, 2, \ldots, n)$ are the periodic points of the map

$$\begin{cases} \zeta_1 = \zeta + g(\eta) + G(\zeta, \eta), \\ \eta_1 = \eta + \qquad H(\zeta, \eta). \end{cases} \tag{24}$$

In the generic case, there is a positive number C_3 so that

$$|H_\zeta(\zeta, \eta)| \geqslant c_3 d > 0 \tag{25}$$

THEOREM. *Assume that the respective terms of the map* (12) *satisfy the conditions* (22) *and* (25). *Then, for sufficiently small d, there are at least two groups of periodically invariant curves which persist under the nonlinear perturbations, each group of which is composed of n closed curves.*

Proof. We introduce the functional space

$$\mathscr{F} = \{\eta(\xi), \zeta(\xi) \mid \eta, \zeta \text{ are } 2\pi\text{-periodic in } \xi; \eta, \zeta \in C^1\}$$

where the norm is defined by

$$\|(\eta_1, \zeta_1) - (\eta_2, \zeta_2)\| = \max(\|\eta_1 - \eta_2\|_{c^1}, \|\zeta_1 - \zeta_2\|_{c^1}),$$

(here $\|\cdot\|_{c^1}$ denotes the usual norm on the function space c^1). $\mathscr{F}$ is a Banach space.

$$\mathscr{F}_1 = \left\{(\eta, \zeta) \in \mathscr{F} \mid \|\eta\|_{c^1} < \frac{1}{4\|f\|_c}\right\}$$

is an open set of $\mathscr{F}$

Inserting the functions $\eta = \eta(\xi), \zeta = \zeta(\xi)$ into (12)

$$\begin{cases} \xi_1 = \xi + f(\eta(\xi)) + W_1(\xi, \zeta(\xi), \eta(\xi)), \\ \zeta_1 = \zeta(\xi) + g(\eta(\xi)) + G(\zeta(\xi), \eta(\xi)) + W_2(\xi, \zeta(\xi), \eta(\xi)), \\ \eta_1 = \eta(\xi) + H(\zeta(\xi), \eta(\xi)) + W_3(\xi, \zeta(\xi), \eta(\xi)), \end{cases} \tag{26}$$

we obtain a map M from $\mathscr{F}_1$ into $\mathscr{F}$ determined by the point map. In fact, for a certain function $(\eta(\xi), \zeta(\xi)) \in \mathscr{F}_1, \xi_1, \xi_1'$, the images of ζ, ξ', respectively, satisfy the following inequality

$$\begin{aligned} |\xi_1 - \xi_1'| \geqslant |\xi - \xi_1'| - \|f\|_{c^1} |\eta(\xi) - \eta(\xi')| \\ - \|W_1\|_{c^1}(|\xi - \xi'| + |\eta(\xi) - \eta(\xi')| + |\zeta(\xi') - \zeta(\xi)|). \end{aligned}$$

In view of (24) and the definition of $\mathscr{F}_1$, this leads to the estimate

$$|\xi_1 - \xi_1'| \geqslant 1/4|\xi - \xi'|$$

If $d < d_1$, a small positive number. This implies that there is an inverse function determined by the first equation of (21)

$$\xi = \Xi(\xi_1)$$

with the property:

$$\Xi^{-1}(\xi_1+2\pi)=\Xi^{-1}(\xi_1)+2\pi.$$

By inserting it into the other two equations of (26), we obtain a map from $\mathscr{F}_1$ into $\mathscr{F}$

$$(\zeta(\xi),\eta(\xi))\to(\zeta_1(\xi_1),\eta_1(\xi_1))$$

with ζ,η being 2π-periodic in ξ.

Clearly, if there is a group of n-periodic points $(\eta(\xi),\zeta(\xi))$ for μ in $\mathscr{F}_1$, it must be the periodically invariant curves we seek for. The rotation number of the map N confined to the curve is determined by the first equation of (26).

We examine the map M_0 on $\mathscr{F}_1$, determined by the point map

$$\begin{cases}\xi_1=\xi+f(\eta)+W_1(\xi,\eta,\zeta),\\ \zeta_1=\zeta+g(\eta)+G(\zeta,\eta),\\ \eta_1=\eta \qquad\qquad +H(\zeta,\eta).\end{cases}$$

By a discussion as above, M_0 is well defined. Naturally, it admits two groups of periodically invariant curves

$$(\zeta_0(\xi),\eta_0(\xi))=\left(\xi_j+\frac{2k\pi}{\eta},\eta_j\right)\qquad (j=1,2),$$

the derivative map of μ_0^n-I at its fixed point (ζ_0,η_0) being of the form

$$D=\frac{\partial}{\partial(\zeta\,\eta)}(\mu_0^n-I)=\prod^n\begin{pmatrix}I+G_\zeta & g'+G_\eta\\ H_\eta & I+H_\eta\end{pmatrix}-\begin{pmatrix}I&0\\0&I\end{pmatrix}=A^n-\begin{pmatrix}I&0\\0&I\end{pmatrix}\quad (j=1,2),$$

where I denotes the identity map and we have made use of the fact that GH are $2\pi/n$ periodic in ζ. As the derivatives of G and H with respect to ζ,η are $O(d)$, it is enough to see that 1 is not an eigenvalue of A to ensure that D is regular. But

$$|\det(A-I)|=|g'H_\zeta(1+O(d)|>c_3g'd$$

by condition (25). The nonsingularity of the matrix D is, by the implicit function theorem, sufficient to imply that M also admits two groups of the curves:

$$\begin{cases}\zeta(\xi)=\left(\zeta_j+\frac{2k\pi}{n}\right)(1+O(d^{1/3})),\\ \eta(\xi)=\eta_j(1+O(d^{1/3}))\end{cases}\qquad (j=1,2),$$

if d is sufficiently small, since μ can be treated as the perturbed μ_0 with the perturbation terms of order $d^{4/3}$.

The existence of such curves can also be proved in the case that $f(r_0)$ satisfies the inequalities (11) and $g(r_0)=(m/n)f(r_0)$.

4. Appendix

In the literature [5, 6], a 3-dimensional perturbed extension of area-preserving mappings has been studied

$$M \begin{cases} X_1 = F_0(X) + \varepsilon F(X, y), \\ y_1 = y + R_0 + \varepsilon G(X, y), \end{cases} \qquad (X \in \mathbb{R}^2, y \bmod 2\pi) \tag{27}$$

$$\frac{\partial(X_1, y_1)}{\partial(X, y)} \equiv 1$$

For $\varepsilon = 0$ we assume that $X = 0$ is an elliptic fixed point. Similar to the proof of the theorem in this paper, (27) admits an invariant curve if ε is sufficiently small. It is assumed that the map near the curve has the form

$$\begin{cases} X_1 = (A_0 + \varepsilon A(y))X + O(\|X\|^2), \\ y_1 = y + R_0 + R(y) + O(\|X\|). \end{cases} \tag{28}$$

By the work of Arnold [1] and Herman [3], there is a set S_r with positive Lebesgue measure so that if $R_0 \in S_r$, the map

$$y_1 = y + R_0 + \varepsilon R(y)$$

is conjugate to a rigid rotation

$$\varphi_1 = \varphi + R^*,$$

where R^* satisfies the inequality (11). Trying to find a change of variables

$$\begin{cases} X = T(\varphi)U, \\ T(\varphi) = T(\varphi + 2\pi), \end{cases}$$

leads to a search for a periodical solution of the equation

$$T(\varphi + R^*)^{-1}(A_0 + \varepsilon A(\varphi))T(\varphi) = A^* \tag{29}$$

where A^* is a matrix not depending on φ (but it depends on ε). We put $A^* = A_0 + \varepsilon \tilde{A}^*$, then if we set

$$T(\varphi) = I + \varepsilon T_0(\varphi),$$

its linearized form with respect to ε is

$$T(\varphi + R^*)A_0 - A_0 T_0(\varphi) = A(\varphi) - \tilde{A}^*$$

By requiring that the matrix A_0 be of the form

$$\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}$$

and following the footsteps of Arnold and Herman, we assert that there is a set

$S_\alpha \subset [0, 2\pi]$ with positive Lebesgue measure, such that if $\alpha \in S_\alpha$, for sufficiently small $\varepsilon > 0$, there is a periodic solution of (29). The Lebesgue measure of S_α and $S_r \to 2\pi$ as $\varepsilon \to 0$.

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